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On the C^0 compactness of the set of the solutions of the Yamabe equation

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Abstract

On a compact Riemannian manifold, we talk again on the C^0 compactness of the set of the solutions of the Yamabe equation. Among other results, we give here a very simple proof of the compactness of this set when the conformal Laplacian L is invertible, except on the standard sphere of course.

Keywords: Yamabe equation; Co-compactness of the set of solutions; Hebey–Vaugon's equality

1. Introduction

On a compact C^∞ Riemannian manifold of dimension n and scalar curvature R, we suppose that the conformal Laplacian L is invertible. The Yamabe equation is [1,9]:

$$L\varphi = \Delta\varphi + \frac{n-2}{4(n-1)} R\varphi = n(n-2)\varphi^{(n+2)/(n-2)}, \quad \varphi > 0. \tag{$*$}$$

There are many articles on the study of the C^0 compactness of $\mathcal{F}$ the set of the solutions of this equation. In particular Y.Y. Li [6,7] (with Zhang), ([8] with Zhu), T. Aubin [2–5] and the references inside.

All proofs begin by considering a sequence $\{u_i\}$ of solutions of $(*)$, such that the sup of $u_i = u_i(P_i) = M_i \to \infty$ with $P_i \to P$ a point of the manifold. Since the problem is conformally invariant, we consider g, in the conformal class of the initial metric, g the Cao–Günther metric at P. Let $\{x^j\}$ or $\{r, \theta\}$ be geodesic coordinates in a ball $B_P(\delta)$.

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Let G_L be the Green function of L at P, denote by $G(P,Q)$ the function proportional to G_L which has r^{2-n} as leading singular part at P.

With the usual notation of Y.Y. Li, the entire number ω is defined by $\|\nabla^\alpha W\|(P)=0$ for $\alpha<\omega$, $\|\nabla^\omega W\|(P)\neq 0$, W being the Weyl tensor. In $B_P(\delta)$ (δ is sufficiently small), let $\bar R$ be the leading part in r of R, and μ the order of $\bar R$. μ satisfies $\mu \geqslant \omega$. We suppose in this article $\mu=\omega$, otherwise it is done [2].

We continue the study of the C^0 compactness of $\mathcal{F}$. We suppose known the essential knowledge of [2] and [4], as well as [3] and [5] on the positive mass.

2. First results. Study of the integral *J*

Let $\mathcal{E}$ be the Euclidean metric on R^n, s the restriction of $\mathcal{E}$ to S_{n-1} and ∇ the covariant derivative with respect to s. In the limited expansion in r of g in $B(\delta)$, we are interested by three terms, $\mathcal{E}$, the terms in $r^{\omega+2}$ and $r^{2(\omega+2)}$, the other terms are in $\tilde h$:

$$g=\mathcal{E}+r^{\omega+2}\bar g+r^{2(\omega+2)}\hat g+\tilde h.$$

($\bar g$ and $\hat g$ are two covariant symmetric 2-tensors on S_{n-1}.)

Since $|g|=1$, $\bar g_{ij}s^{ij}=0$, $\bar g_{ij}\bar g^{ij}=Q(\theta)=2\hat g_{ij}s^{ij}$ and $\bar{\int}_{\partial B(r)}\tilde h_{ij}s^{ij}\,d\sigma=o(r^{2(\omega+2)})$. The indices go up with $((s^{ij}))$ the inverse matrix of $((s_{ij}))$, $\bar{\int}$ means the average.

In [2] and [4] we saw that we have to compute the integral of R on $\partial B(r)$ when $\omega\leqslant[n/2-3]$. Now if $2\omega>n-6$, we prove that the set of the solutions of the Yamabe equation is compact in C^k for any $k\in\mathcal{N}$, by using the positive mass theorem [3,5]. In [2] we proved:

Theorem 1.

$$\bar R=\nabla^{jk}\bar g_{jk},$$
$$r^{-2(\omega+1)}\bar{\int}_{\partial B(r)} R\,d\sigma=B/2-C/4-(1+\omega/2)^2Q+o(1),$$

where $A=\bar{\int}_{\partial B(r)}\nabla_i\bar g^{ik}\nabla^j\bar g_{jk}\,d\sigma$, $B=\bar{\int}_{\partial B(r)}\nabla^i\bar g^{jk}\nabla_j\bar g_{ik}\,d\sigma$, $C=\bar{\int}_{\partial B(r)}\nabla^i\bar g^{jk}\nabla_i\bar g_{jk}\,d\sigma$, $Q=\bar{\int}_{\partial B(r)}Q(\theta)\,d\sigma$.

It is easy to verify that $B=A-(n-1)Q$. As the integral of $\bar R$ on $\partial B(r)$ is zero, there exists a function φ on S_{n-1} such that $\nabla^{ij}\bar g_{ij}=\Delta\varphi$. Let E_k be the eigenspace of Δ on S_{n-1} of eigenvalue $k(n+k-2)$. As $\bar R$ is an homogenous polynome in $\{x^j\}$ of degree ω : $\bar R=r^\omega\bar R(\theta)$, $\bar R(\theta)\in\bigcup_{k_0}^{\omega}E_k$, with $k_0=1$ or 2 according to ω is odd or not.

Set $\bar g_{ij}=b_{ij}+a_{ij}$ with

$$b_{ij}=\sum_{k_0}^{\omega}\big[(n-1)\nabla_{ij}\varphi_k+\lambda_k\varphi_k s_{ij}\big]/(n-2)(\lambda_k+1-n).$$

$\varphi=\sum_{k\leqslant\omega}\varphi_k$ with $\varphi_k\in E_k$.

We have $s^{ij}b_{ij}=0$, $\nabla^i b_{ij}=-\sum_{k\leqslant\omega}\nabla_j\varphi_k$ and $\nabla^{ij}b_{ij}=\Delta\varphi$. Thus $\nabla^{ij}a_{ij}=0$. A simple computation leads to:

$$Q = \bar{\int}_{S_{n-1}} a^{ij} a_{ij}\, d\sigma + (n-1) \sum_{k \leqslant \omega} \left[\lambda_k^2 \bar{\int}_{S_{n-1}} \varphi_k^2\, d\sigma / (n-2)(\lambda_k + 1 - n)^2 \right],$$

$$(n-2)(C-B) = \sum_{k \leqslant \omega} \lambda_k \bar{\int}_{S_{n-1}} \varphi_k^2\, d\sigma + \bar{\int}_{S_{n-1}} (\nabla_l a_{ij} - \nabla_j a_{il})^2\, d\sigma.$$

If we had $\bar{g}_{ij} = a_{ij}$, the scalar curvature R_a would satisfy $\bar{R}_a = 0$, and we know that in this case, $\bar{\int}_{S_{n-1}} R_a\, d\sigma < 0$. Thus we have to study $\bar{\int}_{S_{n-1}} R_b\, d\sigma$ only, with R_b the scalar curvature in case $\bar{g}_{ij} = b_{ij}$. We have $\bar{\int}_{S_{n-1}} R\, d\sigma = \bar{\int}_{S_{n-1}} (R_a + R_b)\, d\sigma + o(r^{2(\omega+1)})$, $A = \sum_{k \leqslant \omega} \lambda_k \bar{\int}_{S_{n-1}} \varphi_k^2\, d\sigma = (n-2)(C-B)$ and $Q = Q_a + Q_b$ with $Q_b = (n-1) \sum_{k \leqslant \omega} [\lambda_k / (n-2)(\lambda_k - n + 1)]$. Let us define,

$$J = \bar{\int}_{\partial B(r)} \left[R + \bar{R}\left(G r^{n-2} - 1 \right) \right] d\sigma.$$

$J = J_a + J_b + o(r^{2(\omega+1)})$. We know that $J_a < 0$. Thus if $J_b < 0$, we are in condition to apply Theorem 8 of [2], and we get the conclusion of Theorem 3. Moreover $G - r^{2-n} = -f(\theta) r^{\omega+4-n} + o(r^{\omega+4-n})$ with $f(\theta)$ satisfying (see [2]): $f(\theta) = \sum_{k \leqslant \omega} a_k \psi_k$ with

$$a_k = (n-2)/4(n-1)\left[\lambda_k + (n-4-\omega)(\omega+2) \right]$$

if $\bar{R}(\theta) = \sum_{k \leqslant \omega} \psi_k$. Thus $\bar{\int}_{S_{n-1}} f(\theta) \bar{R}(\theta)\, d\sigma = \sum_{k \leqslant \omega} [\lambda_k^2 a_k \bar{\int}_{S_{n-1}} \psi_k^2\, d\sigma]$.

Summarizing, all these inequalities must be satisfied for $k \leqslant \omega$:

$$\frac{n-3}{n-2} - \frac{(n-1)[n-1+(\omega+2)^2]}{(n-2)(\lambda_k + 1 - n)} \leqslant \frac{(n-2)\lambda_k}{(n-1)[\lambda_k + (\omega+2)(n-4-\omega)]}.$$

The worst case is for $k = \omega$. To conclude we have to check for which pair (ω, n), with $\omega \leqslant [n/2 - 3]$ of course, the following inequality is satisfied:

$$\frac{(n-2)^2 (n-4-\omega)(\omega+2)}{2(n-4)(\omega+1)} - \frac{(n-1)^2 [n-1+(2+\omega)^2]}{(n-1+\omega)(\omega-1)} < 1.$$

We can verify that this inequality is satisfied for any n when $\omega = 2$ or if $\omega = [n/2 - 3]$. We can see also for which n this inequality is satisfied for any $\omega \leqslant [n/2 - 3]$.

Theorem 2. *For r small, J is negative for $n < 38$, and for any n if $\omega = 2$ or $\omega = [n/2 - 3]$. When $J < 0$, $\mathcal{F}$ is compact in C^k for any $k \in \mathcal{N}$.*

In $B_P(\delta) \backslash P$, $M_i u_i \to G$ in C^k_{loc} $(k \geqslant 0)$, and the limited expansion in r of G is $G = r^{2-n} - f(r, \theta) r^{4-n} + o(r^{\omega - n + 4})$. As we proved this result in [2], we verify, since u_i satisfies $(*)$, that

$$M_i u_i = \left(r^2 + \epsilon^2 \right)^{1-n/2} + \gamma_i + o\left(r^{\omega - n + 4} \right) + O\left(\epsilon^2 \right)$$

here $\epsilon^2 = M_i^{-4/(n-2)}$, with γ_i the solution of $L\gamma_i = (n-2) R \phi_\epsilon / 4(n-1)$. γ_i tends to $-f(r, \theta) r^{4-n}$ with $\epsilon \to 0$.

With this method we get only partial results.

Below we give a short proof of the C^0 compactness, without the inequality on J.

3. Proof of the C^0 compactness

Considering a unbounded sequence of solutions of the Yamabe equation, after using the Pohozahev identity (see [2,6,7]), we get an equality denoted $B = E$ in [2]. B is an integral on $B(r)$, and E an integral on the boundary $\partial B(r)$. B is infinite or negligible ($O(r)$), E is finite ($O(1)$). Two very different cases arise.

If $n < 2\omega + 6$, B is negligible and according to Proposition 4 of [2],

$$E \leqslant -(n-2)^2 \bar{\int}_{\partial B(r)} \left(G - r^{2-n}\right) d\sigma.$$

As we have proved in all generality the positive mass theorem (under a necessary and sufficient condition) (Aubin [3,5]), E is negative. Thus we arrive to a contradiction, E cannot be finite with $B = O(r)$.

When $n \geqslant 2\omega + 6$ we have to consider two cases. If J is of order $2(\omega+1)$ in r, B is infinite according to Theorem 8 of [2]. We get a contradiction, B cannot be equal to E finite.

Otherwise $B = O(r)$. But in this case, we will see that $E = O(r^k)$ with $k \leqslant 0$. So $E \neq B$, a contradiction.

Henceforth we suppose that J is of order $2(\omega+2)$ in r. Let us study one more time the integral E, as in the proof of Proposition 4 in [2].

If $n > 2\omega + 6$, define $H = G - r^{2-n}$. We have $\bar{\int}_{\partial B(r)} H\, d\sigma = O(1)$, $r^{6-n} \bar{\int}_{\partial B(r)} f^2\, d\sigma = O(r^k)$ with $k < 0$ and $(1/r)\int_{\partial B(r)} |\nabla_\theta G|^2\, d\sigma = O(r^k)$, with $k < 0$. These two last terms cannot be $O(1)$, otherwise $\bar{R}$ vanishes, and we know that in this case $J < 0$ is of order $2(\omega+1)$ in r. We have the wished for contradiction. We get the same result when $n = 2(\omega+3)$, but now the three terms are $O(1)$ and positive (here $\bar{\int}_{\partial B(r)} R\, d\sigma = O(r^{2\omega+4})$ with $2\omega + 4 > n - 4$).

Theorem 3. *When the conformal Laplacian is invertible on a compact Riemannian manifold, not conformal to the canonical sphere, the set of solutions of the Yamabe equation is compact in C^k for any $k \in \mathcal{N}$.*

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