Fong Characters and Normal Subgroups of $\pi$-Separable Groups

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1. INTRODUCTION

In this article we will answer a question of Isaacs that arose in the following set up.

Let $\pi$ be an arbitrary set of primes and let $G^*$ denote the set of $\pi$-elements of a $\pi$-separable group $G$. We consider the restriction map from $cf(G)$ to $cf(G^*)$, where $cf(G)$ and $cf(G^*)$ denote class functions defined on $G$ and $G^*$, respectively. In [5], Isaacs shows that if $G$ is a $\pi$-separable group, then $cf(G^*)$ has a unique basis $I_{\pi}(G)$ such that:

$\mathbf{(D)} \quad \chi \in \text{Irr}(G) \Rightarrow \chi^* \text{ is a } \mathbb{Z}^{\geq 0}\text{-linear combination of elements in } I_{\pi}(G), \text{ and}$

$\mathbf{(FS)} \quad \phi \in I_{\pi}(G) \Rightarrow \exists \chi \in \text{Irr}(G) \text{ with } \chi^* = \phi.$

This construction generalised an earlier result of Isaacs appearing in [4], where for $\pi = p'$, the set of irreducible Brauer characters, $\text{IBr}(G)$, is such a basis. Notice that in this case $\pi$-separability is in fact $p$-solvability.

The strategy used to prove the linear independence of $I_{\pi}(G)$ is to define in a canonical way a certain subset $B_{\pi}(G) \subseteq \text{Irr}(G)$ and then to show that $\{\chi^* : \chi \in B_{\pi}(G)\}$ is a basis for $cf(G^*)$ which satisfies (D), noticing that (FS) is trivial in this context.

One of the key results about $B_{\pi}(G)$ is the following theorem.

1.1. THEOREM (8.1 in [5]). Let $G$ be $\pi$-separable and let $H$ be a $\text{Hall } \pi$-subgroup of $G$. Suppose $\chi \in B_{\pi}(G)$. Then the following hold:

(a) \quad if $\alpha \in \text{Irr}(H)$, then $\alpha(1) \geq [\chi_H, \alpha] \chi(1)$.

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(b) $\chi_H$ has an irreducible constituent $\alpha$ with $\chi(1)_\pi = \alpha(1)$.

(c) If $\alpha$ is as in (b), then for $\psi \in B_\pi(G)$ we have

$$[\psi, \alpha] = \begin{cases} 1 & \text{if } \psi = \chi \\ 0 & \text{if } \psi \neq \chi. \end{cases}$$

1.2. Definition. Let $G$ be $\pi$-separable and let $H \in \text{Hall}_\pi(G)$. We say that $\alpha \in \text{Irr}(H)$ is a Fong character of $H$ in $G$, if there exists a $\chi \in B_\pi(G)$ such that $\alpha$ is a constituent of $\chi_H$ and $\alpha(1) = \chi(1)_\pi$. We then say that $\alpha$ is associated with $x$.

Notice that by Theorem 1.1 above, the Fong characters of $H$ associated with some $\chi \in B_\pi(G)$ are precisely the irreducible constituents of $\chi_H$ of minimal degree. If $\chi \in B_\pi(G)$, then $\chi_H = \chi^*_H$ and hence the Fong characters associated with $\chi$ are also associated with $\chi^*$. The existence of the Fong characters is the key to proving the linear independence of the set $\{\chi^* : \chi \in B_\pi(G)\}$. In [1], Fong proved that given $\varphi \in \text{IBr}(G)$, there exists an irreducible constituent $\alpha$ of $\varphi_H$ such that $\alpha^G = \Phi_\varphi$, the projective or principal indecomposable-character of $G$ associated with $\varphi$ and also that $\alpha(1) = \varphi(1)_\rho'$. The study of Fong characters is the subject of study of [4], where Isaacs finds an equivalence relation defined on the set of characters (not necessarily irreducible) of a Hall $\pi$-subgroup of a $\pi$-separable group $G$ that uniquely characterizes the Fong characters. In his study of the Fong characters he notices that Fong characters do not in general behave well with respect to normal subgroups. Isaacs provides us with an example (9.1 in [4]) to illustrate that which we look at in detail at the end of this article. He constructs a $\pi$-separable group $G$ and a $x \in B_\pi(G)$ and then he goes on to show that there exists an $\alpha \in \text{Irr}(H)$ and an $N \triangleleft G$ such that $\alpha$ is Fong associated with $\chi \in B_\pi(G)$ but no irreducible constituent of $\alpha_{N \cap H}$ is Fong for $N$, where $H$ is a Hall $\pi$-subgroup of $G$. So he asks the following question:

1.3 Question (9.2 in [4]). Let $G$ be $\pi$-separable and $H \in \text{Hall}_\pi(G)$. If $\varphi \in B_\pi(G)$, does there necessarily exist an associated Fong character $\alpha \in \text{Irr}(H)$ such that for every $N \triangleleft G$, every irreducible constituent of $\alpha_{N \cap H}$ is Fong in $N$?

We will provide an affirmative answer to Isaacs’ question and we will in fact show in our Theorem 3.23, that given $\chi \in B_\pi(G)$ there exists a Fong character $\alpha$ such that every irreducible constituent of $\alpha_{S \cap H}$ is Fong for all $S \triangleleft \triangleleft G$. 
2. REVIEW OF THE $\pi$-THEORY

In this section we shall state the results about $B_\pi(G)$ that are needed to answer Question 1.3. A very important role in the theory of $B_\pi(G)$ is played by a certain set of irreducible characters of $G$ which were studied extensively by Gajendragadkar in [2], namely the set $X_\pi(G) \subseteq \text{Irr}(G)$ called $\pi$-special characters.

2.1. DEFINITION. Let $G$ be $\pi$-separable. We say $\chi \in \text{Irr}(G)$ is $\pi$-special provide that:

(i) $\chi(1)$ is a $\pi$-number and

(ii) for all $S \triangleleft \triangleleft G$ and all irreducible constituents $\theta$ of $\chi_S$, the determinantal order $o(\theta)$ is a $\pi$-number.

2.2. LEMMA (Proposition 4.1 in [2]). Let $M$ be a subnormal subgroup of $G$. Let $\phi$ be an irreducible constituent of $\chi_M$ for some $\chi \in X_\pi(G)$. Then $\phi$ lies in $X_\pi(M)$.

2.3. THEOREM (Proposition 7.1 in [2]). Let $G$ be $\pi$-separable and assume that $\alpha, \beta \in \text{Irr}(G)$ are $\pi$-special and $\pi'$-special, respectively. Then the product $\alpha\beta$ is irreducible. If also $\alpha\beta = \alpha'\beta'$, where $\alpha'$ and $\beta'$ are $\pi$-special and $\pi'$-special, then $\alpha = \alpha'$ and $\beta = \beta'$.

2.4. DEFINITION. If $\chi \in \text{Irr}(G)$ can be written in the form $\chi = \alpha\beta$, where $\alpha$ is $\pi$-special and $\beta$ is $\pi'$-special, then we say that $\chi$ is $\pi$-factorable.

2.5. PROPOSITION (Lemma 5.4 in [5]). Let $\chi \in \text{Irr}(G)$, where $G$ is $\pi$-separable. Then the following are equivalent:

(i) $\chi$ is $\pi$-special.

(ii) $\chi \in B_\pi(G)$ and $\chi(1)$ is a $\pi$-number.

(iii) $\chi \in B_\pi(G)$ and $\chi$ is $\pi$-factorable.

2.6. THEOREM (Theorem 2.1 in [6]). Let $G$ be $\pi$-separable and let $N \triangleleft \triangleleft G$.

(i) If $\chi \in B_\pi(G)$, then every irreducible constituent of $\chi_N$ lies in $B_\pi(N)$.

(ii) If $G/N$ is a $\pi$-group and $\theta \in B_\pi(N)$, then every irreducible constituent of $\theta^G$ lies in $B_\pi(G)$.

(iii) If $G/N$ is a $\pi'$-group and $\theta \in B_\pi(N)$, then $\theta^G$ has a unique constituent $\chi \in B_\pi(G)$. Also $[\chi_N, \theta] = 1$. 
2.7. Corollary (Corollary 7.5 in [5]). Let $G$ be a $\pi$-separable group and let $\chi \in B_\pi(G)$. Then every irreducible constituent of $\chi_S$ lies in $B_\pi(S)$ for every subnormal subgroup $S$ of $G$.

3. Subnormally Fong Characters

In order to give our answer to Isaacs' question we need to define the lower $\pi\pi'$-series of a $\pi$-separable group $G$. Given a $\pi$-separable group $G$ we define $O^\pi(G)$ as the intersection of all normal subgroups of $G$ with $\pi$-quotient in $G$. Thus $G/O^\pi(G)$ is the maximal $\pi$-factor group of $G$ and $O^\pi(G)$ is a characteristic subgroup of $G$. We define similarly $O^{\pi'}(G)$.

3.1. Definition. Let $G$ be $\pi$-separable, we construct the lower $\pi\pi'$-series of $G$ by repeatedly applying $O^\pi$ and $O^{\pi'}$. This is then the series

$$1 = N_m \triangleleft K_m \triangleleft \cdots \triangleleft N_0 \triangleleft K_0 = G$$

(3.1a)

defined by $N_i = O^\pi(K_i)$ and $K_i = O^{\pi'}(N_{i-1})$ for $i = 0, 1, \ldots, m$.

Thus $N_i/K_{i+1}$ is a $\pi'$-group and $K_i/N_i$ is a $\pi$-group. For the first few terms of the lower $\pi\pi'$-series we shall often use the notation $N_0 = O^\pi(G)$, $K_1 = O^{\pi'}(G)$, $N_1 = O^{\pi\pi'}(G)$. It is also worth noting that the lower $\pi\pi'$-series is a characteristic series of $G$. Notice that the lower $\pi\pi'$-series of $G$ will indeed terminate in 1 since every chief factor of $G$ is either a $\pi$ or $\pi'$-group.

3.2. Definition. Let $G$ be $\pi$-separable, $H \in \text{Hall}_\pi(G)$, $\chi \in B_\pi(G)$, and $\alpha$ a Fong character of $H$ associated with $\chi$. Then $\alpha$ is said to be $\pi$-Fong if given the lower $\pi\pi'$-series of $G$,

$$1 = N_m \triangleleft K_m \triangleleft \cdots \triangleleft N_0 \triangleleft K_0 = G,$$

every irreducible constituent of $\alpha_{N_i \cap H}$ is Fong for $N_i$ for all $i \in \{0, 1, \ldots, m\}$.

3.3 Definition. Let $G$ be $\pi$-separable, $H \in \text{Hall}_\pi(G)$, and $\alpha \in \text{Irr}(H)$ such that $\alpha$ is Fong associated with some $\chi \in B_\pi(G)$. Then $\alpha$ is said to be subnormally Fong if every irreducible constituent of $\alpha_{S \cap H}$ is Fong for $S$ for all $S \triangleleft G$.

Notice that if $\alpha$ is $\pi$-Fong, then every irreducible constituent of $\alpha_{K_i \cap H}$ is Fong for $K_i$ for all $i \in \{0, 1, \ldots, m\}$. Also if $\alpha$ is $\pi$-Fong, then every irreducible constituent of $\alpha_{N_i \cap H}$ is $\pi$-Fong for $N_i$ for all $i \in \{0, 1, \ldots, m\}$ and in fact, every irreducible constituent of $\alpha_{K_i \cap H}$ is $\pi$-Fong for $K_i$ for all $i$. Clearly if $\alpha$ is subnormally Fong, then it is $\pi$-Fong. In particular, it follows
that if $\chi$ is $\pi$-special, then the unique Fong character $\alpha$ associated with $\chi$ is $\pi$-Fong and in fact it is subnormally Fong.

**Theorem A.** Let $G$ be $\pi$-separable and $\chi \in B_{\pi}(G)$. Then there exists a subnormally Fong character $\alpha$ associated with $\chi$.

In proving this theorem we also prove the following two theorems.

**Theorem B.** Let $G$ be $\pi$-separable, $H \in \text{Hall}_{\pi}(G)$, and $\chi \in B_{\pi}(G)$. Then there exists a Fong character $\alpha \in \text{Irr}(H)$ associated with $\chi$ such that

(i) every irreducible constituent of $\alpha_{\text{O}^\pi(G) \cap H}$ is Fong for $O^\pi(G)$, and also

(ii) if $\delta \mid \alpha_{\text{O}^\pi(G) \cap H}$, then $I_H(\phi) = I_H(\delta)$, where $\phi \mid \chi_{\text{O}^\pi(G)}$ such that $\delta$ is Fong for $\phi$.

**Theorem C.** Let $G$ be $\pi$-separable, $H \in \text{Hall}_{\pi}(G)$, and $\chi \in B_{\pi}(G)$. Then some Fong character $\alpha \in \text{Irr}(H)$ associated with $\chi$ is $\pi$-Fong.

We shall prove these theorems after we prove some preliminary lemmas.

**3.4. Lemma.** Let $G$ be $\pi$-separable, $H \in \text{Hall}_{\pi}(G)$, and $N \triangleleft G$ such that $G = NH$. Let $\chi \in B_{\pi}(G)$. If $\alpha \mid \chi_H$ and $\phi \mid \chi_N$ have a common irreducible constituent $\delta$ upon restriction to $N \cap H$ such that $\delta$ is Fong for $\phi$, then $I_H(\phi) \geq I_H(\delta)$.

**Proof.** Since every element of $N$ stabilities $\phi$, the $G$-orbit of $\phi$ is in fact the $H$-orbit of $\phi$. Hence, if $\delta \mid \phi_{N \cap H}$ and $h \in I_H(\delta)$, then $\delta \mid \phi^h_{N \cap H}$. By hypothesis, the character $\delta$ is Fong associated with $\phi$. So $\delta(1) = \phi(1) = \phi^h(1)_{\pi}$, and so $\delta$ is Fong for $\phi^h$. By Theorem 1.1(iii), $\phi$ must equal $\phi^h$; therefore $h \in I_H(\phi)$ and the lemma is proved.

**3.5. Lemma.** Let $G$ be $\pi$-separable, $N \triangleleft G$ such that $G/N$ is a $\pi$-group, and $H \in \text{Hall}_{\pi}(G)$. Let also $\chi \in B_{\pi}(G)$, $\phi \mid \chi_N$, and $\delta$ a Fong character of $N \cap H$ for $\phi$ such that $I_H(\phi) = I_H(\delta)$. Then there exists a unique Fong character $\alpha$ of $H$ associated with $\chi$ and lying over $\delta$; in fact, $\alpha$ is the only irreducible constituent of $\chi_H$ lying over $\delta$.

**Proof.** Since $G/N$ is a $\pi$-group we have that $G = NH$. Let $T = I_G(\phi)$ and $I = I_H(\phi)$ ($= I_H(\delta)$ by hypothesis). To prove the lemma we use induction on $|G|$. Since $G = NH$ and $N \subseteq T$, it is trivial that $T = N(T \cap H) = NI$. Thus $T \cap H \in \text{Hall}_{\pi}(T)$. Assume first that $T = G$, that is to say $I = H$. By Theorem 1.1 the hypotheses of Lemma 4.1 in [5] are satisfied and by that result there exists a unique irreducible constituent $\alpha$ of $\chi_H$ lying over $\delta$ and $\chi(1)/\alpha(1) = \phi(1)/\delta(1)$. Considering the $\pi$-parts of both sides of this equation
we have $\chi(1)/\alpha(1) = \varphi(1)/\delta(1)$ and since $\delta$ is Fong for $\varphi$, we have $\delta(1) = \varphi(1)/\pi$ and hence $\chi(1)/\pi = \alpha(1)$. Thus $\alpha$ is Fong for $\chi$ and the lemma is true in this case.

Therefore from now on we may assume that $T < G$, that is, $I < H$. Thus we have the following diagram:

```
G
\chi

\xi
\nu

N \cap H
\delta

\nu

\chi_N
\varphi

\nu

[Irr(T|\nu) and $\xi^G = \chi$. Let $[\chi_N, \varphi] = e$. By Corollary 7.6 in [5], the character $\xi \in B_\nu(T)$, and so applying our inductive hypothesis to $T$, we can conclude that there exists a unique Fong character $\gamma$ of $\xi$ such that $\gamma$ lies over $\delta$. Since $\delta$ is invariant in $I$ by hypothesis, we have $\gamma_{N \cap H} = [\gamma_{N \cap H}, \delta]$.

By the second part of the inductive hypothesis, the character $\gamma$ is the only irreducible constituent of $\xi$ lying over $\delta$ and furthermore $[\xi, \gamma] = 1$ by Theorem 1.1. Therefore we have $[\xi_{N \cap H}, \delta] = [\gamma_{N \cap H}, \delta]$. Next we prove that $[\gamma_{N \cap H}, \delta] = e$. This is because $[\gamma_{N \cap H}, \delta] = [\xi_{N \cap H}, \delta] = [\xi_N, \varphi] [\varphi_{N \cap H}, \delta] = [\chi_N, \varphi] [\varphi_{N \cap H}, \delta] = e [\varphi_{N \cap H}, \delta] = e$. Let $\alpha = \gamma^H$. Since $\gamma \in Irr(I|\delta)$ and $I = I_H(\delta)$, the character $\alpha$ is irreducible by Clifford’s Theorem and $\gamma(1) |H : I| = \alpha(1)$. Since $\gamma$ is a Fong character for $\xi$, we have $\gamma(1) = \xi(1)/\pi$, so $\xi(1)/\pi |H : I| = \alpha(1)$ and since $|H : I| = |G : T|$, we have $\xi(1)/\pi |G : T| = \alpha(1)$, and therefore $\chi(1)/\pi = \alpha(1)$ because $\xi(1)/\pi |G : T| = \chi(1)$ and $|G : T|$ is a $\pi$-number. Consequently $\alpha$ is a Fong character for $\chi$.

To prove that $\alpha$ is the only irreducible constituent of $\chi_H$ over $\delta$ is equivalent to showing that $[\chi_{N \cap H}, \delta] = [\alpha_{N \cap H}, \delta]$, since $[\chi_H, \alpha] = 1$ by Theorem 1.1. This follows easily since $[\chi_{N \cap H}, \delta] = [\chi_N, \varphi] [\varphi_{N \cap H}, \delta] = e$.  

Let $\xi$ be the Clifford correspondent of $\chi$ with respect to $\varphi$: thus $\xi \in Irr(T|\nu)$ and $\xi^G = \chi$. Let $[\chi_N, \varphi] = e$. By Corollary 7.6 in [5], the character $\xi \in B_\nu(T)$, and so applying our inductive hypothesis to $T$, we can conclude that there exists a unique Fong character $\gamma$ of $\xi$ such that $\gamma$ lies over $\delta$. Since $\delta$ is invariant in $I$ by hypothesis, we have $\gamma_{N \cap H} = [\gamma_{N \cap H}, \delta]$.

Diagram 1

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\begin{center}
\begin{tikzpicture}
  \node (G) at (0,0) {$G$};
  \node (H) at (0,2) {$H$};
  \node (T) at (-1,1) {$T$};
  \node (N) at (-2,0) {$N$};
  \node (I) at (-1,0) {$I$};
  \node (N\cap H) at (-2,0) {$N \cap H$};
  \node (\alpha) at (0,1) {$\alpha$};
  \node (\gamma) at (-1,1) {$\gamma$};
  \draw (G) -- (H);
  \draw (G) -- (T);
  \draw (H) -- (T);
  \draw (T) -- (N);
  \draw (T) -- (I);
  \draw (N) -- (I);
  \draw (N) -- (N\cap H);
  \draw (I) -- (N\cap H);
  \draw (\gamma) -- (\alpha);
\end{tikzpicture}
\end{center}

Diagram 1
Furthermore $[\alpha_{N \cap H}, \delta] = [\gamma_{N \cap H}, \delta] = \epsilon$, where the first equality is due to Clifford's Theorem and the second follows by the claim above. Therefore $[\chi_{N \cap H}, \delta] = [\alpha_{N \cap H}, \delta] = \epsilon$. So $\alpha$ is the only irreducible constituent of $\chi_H$ lying over $\delta$ and the lemma is proved. 

3.6. Lemma. Let $G$ be $\pi$-separable, $N \triangleleft G$, $H \in \text{Hall}_\pi(G)$, $\chi \in B_\pi(G)$, and $\alpha \in \text{Irr}(H)$ such that $\alpha | \chi_H$. If some irreducible constituent of $\alpha_{N \cap H}$ is Fong for $N$ and an irreducible constituent of $\chi_N$, then every irreducible constituent of $\alpha_{N \cap H}$ is Fong for $N$.

Proof. Let $\delta | \alpha_{N \cap H}$ such that $\delta$ is Fong for $N$ and $\phi | \chi_N$ such that $\delta | \phi_{N \cap H}$. Then since $\phi \in B_\pi(N)$ by 2.6(i), this $\delta$ must be Fong for $\phi$, hence $\phi(1)_\pi = \delta(1)$. Since $N \cap H \triangleleft H$, it follows by Clifford's Theorem, that all irreducible constituents of $\alpha_{N \cap H}$ have the same degree, that is to say $\delta^h(1) = \delta(1)$ for all $h \in H$. Hence, whenever $\beta | \alpha_{N \cap H}$ and $\psi | \chi_N$ such that $\beta | \psi_{N \cap H}$, we have $\beta(1) = \delta(1) = \phi(1)_\pi = \psi(1)_\pi$. The last equality follows from the fact that $\phi$ and $\psi$ are conjugate characters and as such they have the same degree. So $\beta(1) = \psi(1)_\pi$, and since $\psi \in B_\pi(N)$ by 2.6(i), the lemma follows.

3.7. Lemma. Let $G$ be a $\pi$-separable group, $M \triangleleft G$ such that $G/M$ is a $\pi'$-group, $H \in \text{Hall}_\pi(G)$, and $\alpha \in \text{Irr}(H)$. Then $\alpha$ is Fong for $G$ if and only if $\alpha$ is Fong for $M$.

Proof. Let $\alpha$ be Fong for $G$; then there exists $\chi \in B_\pi(G)$ such that $\alpha | \chi_H$ and $\alpha(1) = \chi(1)_\pi$. Since $M$ has $\pi'$-index in $G$ it contains $H$, and so there exists $\phi | \chi_M$ such that $\alpha | \phi_H$. By Clifford's Theorem we have that $\chi(1) = e s \phi(1)$, where $s$ is the length of the $G$-orbit of $\phi$ and $e$ is the ramification number. Since both $e$ and $s$ are $\pi'$-numbers we have that $\chi(1)_\pi = \phi(1)_\pi$, and so it follows that $\alpha$ is Fong for $\phi$. Hence $\alpha$ is Fong for $M$.

Conversely, if $\alpha$ is Fong for $M$, there exists a $\phi \in B_\pi(M)$ such that $\alpha | \phi_H$ and $\alpha(1) = \phi(1)_\pi$. By Theorem 2.6(iii) there exists a unique $\chi \in B_\pi(G)$ such that $\phi | \chi_M$, and arguing as above, $\chi(1)_\pi = \phi(1)_\pi$. Therefore $\alpha$ is Fong for $\chi$ and hence Fong for $G$.

3.8. Lemma. Let $G$ be $\pi$-separable, $H \in \text{Hall}_\pi(G)$, $N \triangleleft G$, such that $G/N$ is a $\pi'$-group. Let $\chi \in B_\pi(G)$ and $\alpha \in \text{Irr}(H)$ such that $\alpha$ is Fong for $\chi$. Assume further that $\alpha$ satisfies the following two conditions:

(i) every irreducible constituent of $\alpha_{N \cap H}$ is Fong for $N$;
(ii) if $\delta | \alpha_{N \cap H}$, then $\alpha$ is the only irreducible constituent of $\chi_H$ lying over $\delta$.

Then $I_H(\phi) = I_H(\delta)$, where $\phi$ is the irreducible constituent of $\chi_N$ such that $\delta$ is Fong for $\phi$. 1
Proof. By Lemma 3.4 we have $I_H(\phi) \geq I_H(\delta)$, and so to prove the lemma it is enough to show that $|H : I_H(\phi)| = |H : I_H(\delta)|$. Let $|H : I_H(\phi)| = r$ and $|H : I_H(\delta)| = s$. Clearly $I_H(\phi)$ is a Hall $\pi$-subgroup of $I_G(\phi)$ and so we have that $|G : I_G(\phi)| = |H : I_H(\phi)| = r$. It follows now by Clifford's Theorem that

$$\chi(1) = e\varphi(1)r \quad (3.8a)$$

and

$$\alpha(1) = \hat{e}\delta(1)s, \quad (3.8b)$$

where $e = [\chi_N, \varphi]$ and $\hat{e} = [\alpha_{N \cap H}, \delta]$.

Since $\varphi$ is the only irreducible constituent of $\chi_N$ lying over $\delta$ by 1.1(iii), we have

$$[\chi_{N \cap H}, \delta] = [\chi_N, \varphi][\varphi_{N \cap H}, \delta] = e[\varphi_{N \cap H}, \delta] = e. \quad (3.8c)$$

By part (ii) of the hypothesis, $\alpha$ is the only irreducible constituent of $\chi_H$ lying over $\delta$; hence

$$[\chi_{N \cap H}, \delta] = [\chi_N, \alpha][\alpha_{N \cap H}, \delta] = 1[\alpha_{N \cap H}, \delta] = \hat{e}. \quad (3.8d)$$

Combining (3.8c) and (3.8d) we get

$$e = \hat{e}. \quad (3.8e)$$

Since both $e$ and $r$ divide $|G : I_G(\phi)|$ they are both $\pi$-numbers, and so considering $\pi$-parts of (3.8a) we obtain

$$\chi(1)_{\pi} = e\varphi(1)_{\pi}r. \quad (3.8f)$$

By Theorem 1.1(iii) we have $\chi(1)_{\pi} = \alpha(1)$ and $\varphi(1)_{\pi} = \delta(1)$, and so (3.8f) becomes

$$\alpha(1) = e\delta(1)r.$$

By (3.8b) and (3.8e) it follows that $r = s$. }

The next result shows that if Theorem B holds for $O^*(G)$, then it holds for any intermediate normal subgroup $N$. ($O^*(G) \leq N \leq G.$)

3.9. Lemma. Let $G$ be $\pi$-separable, $N \triangleleft G$ such that $G/N$ is a $\pi$-group, and $H \in \text{Hall}_\pi(G)$. Let $\chi \in B_\pi(G)$, and $\alpha \in \text{Irr}(H)$ be a Fong character for $\chi$ such that $\alpha$ satisfies:

(i) every irreducible constituent of $\alpha_{O^*(G) \cap H}$ is Fong for $O^*(G)$ and
(ii) if $\delta \mid \chi_{O^*(G) \cap H}$, then $I_H(\delta) = I_H(\psi)$ for the unique $\psi \mid \chi_{O^*(G)}$ such that $\delta$ is Fong for $\psi$.

Then every irreducible constituent of $\chi_{N \cap H}$ is Fong for $N$ and if $\beta \mid \chi_{N \cap H}$ such that $\beta$ is Fong for some $\theta \mid \chi_N$, then $I_H(\theta) = I_H(\beta)$.

**Proof.** Let $\theta \mid \chi_N$ and $\varphi \mid \chi_{O^*(G)}$. Since $G = NH$, we have

$$[\varphi \mid \chi_{O^*(G) \cap H}, \chi_{O^*(G) \cap H}] = [(\varphi \mid \chi_{O^*(G) \cap H})^H, \chi_{O^*(G) \cap H}] = [\varphi, \chi_{O^*(G) \cap H}] \geq [\chi_{O^*(G) \cap H}, \chi] > 0.$$  

In particular, there exists $\delta \mid \chi_{O^*(G) \cap H}$ such that $\delta$ is Fong for $\varphi$. Since there exists a unique such $\varphi \mid \chi_{O^*(G)}$, it follows that condition (ii) is satisfied by $\psi = \varphi$. Since $N \cap H < H$ we have

$$I_{N \cap H}(\varphi) = I_H(\varphi) \cap (N \cap H) = I_H(\varphi) \cap N$$
$$I_{N \cap H}(\delta) = I_H(\delta) \cap (N \cap H) = I_H(\delta) \cap N$$

and since $I_H(\varphi) = I_H(\delta)$ by hypothesis, it follows that

$$I_{N \cap H}(\varphi) = I_{N \cap H}(\delta).$$

Thus we have the following diagram:

![Diagram 2](image-url)

By Lemma 3.5 there exists a unique Fong character $\beta$ of $\theta$ such that $\beta$ lies over $\delta$ and $\beta$ is the only irreducible constituent of $\theta_{N \cap H}$ lying over $\delta$. Since $\alpha$ is the only irreducible constituent of $\chi_H$ lying over $\delta$ by Lemma 3.5, the character $\beta$ must be a constituent of $\alpha_{N \cap H}$ and by Lemma 3.6 every irreducible constituent of $\alpha_{N \cap H}$ is Fong for $N$. 
It remains to show that \( I_H(\beta) = I_H(\theta) \). By Lemma 3.8 it is sufficient to show that \( \alpha \) is the only irreducible constituent of \( \chi_H \) lying over \( \beta \); but this follows easily from the fact that \( \alpha \) is the only irreducible constituent of \( \chi_H \) lying over \( \delta \) by Lemma 3.5.

We are now ready to prove Theorem B:

3.10. Theorem. Let \( G \) be \( \pi \)-separable, \( H \in \text{Hall}_\pi(G) \), and \( \chi \in B_\pi(G) \). Then there exists a Fong character \( \alpha \in \text{Irr}(H) \) associated with \( \chi \) such that

(i) every irreducible constituent of \( \alpha_{O^\pi(G) \cap H} \) is Fong for \( O^\pi(G) \), and also

(ii) if \( \delta | \alpha_{O^\pi(G) \cap H} \), then \( I_H(\phi) = I_H(\delta) \), where \( \phi | \chi_{O^\pi(G)} \) such that \( \delta \) is Fong for \( \phi \).

Proof. We use induction on \(|G|\). We may assume that \( O^\pi(G) < G \) for if \( O^\pi(G) = G \), then Theorem 3.10 is trivially satisfied by all the Fong characters of \( \chi \). Let \( N = O^\pi(G) \) and \( \phi | \chi_N \). Let also \( T = I_G(\phi) \) and \( I = I_H(\phi) \).

Once again since \( T = NI \) we have that \( I \in \text{Hall}_\pi(T) \).

Assume first that \( T < G \) and hence that \( I < H \). Let \( \zeta \in \text{Irr}(T | \phi) \) be the Clifford correspondent of \( \chi \) with respect to \( \phi \), that is to say \( \zeta^G = \chi \). By Corollary 7.6 in [5], we have that \( \zeta \in B_\pi(T) \), and since \( O^\pi(T) = O^\pi(G) \) we can apply the inductive hypothesis to \( T \) to conclude that there exists a Fong character \( \gamma \) of \( \zeta \) such that every irreducible constituent of \( \gamma_{N \cap H} \) is Fong for \( N \) and if \( \delta \) is one of them, then \( I_I(\delta) = I_I(\phi) = I \), in other words \( \delta \) is invariant in \( I \). Thus we have

\[
\begin{array}{c}
G \\
\chi \\
H \\
\alpha \\
T \\
\zeta \\
I \\
\gamma \\
N \\
\phi \\
N \cap H \\
\delta \\
\end{array}
\]
Appealing to 3.4, we have \( I = I_T(\delta) \leq I_H(\delta) \leq I_H(\varphi) = I \), and so
\[
I_H(\delta) = I_H(\varphi).
\] (3.10a)

Since \( \gamma \in \text{Irr}(I|\delta) \), it follows by Clifford's Theorem that \( \gamma^H \) is irreducible. Since \( T H = G \), by Mackey's Theorem we obtain \( (\xi^G)_H = (\xi_{T \cap H})^H \); in particular, \( \gamma^H \) is an irreducible constituent of \( (\xi^G)_H = \chi_H \). Let \( \gamma^H = \alpha \in \text{Irr}(H) \). Then \( \alpha \) is Fong for \( \chi \), because
\[
\alpha(1) = \gamma(1) |H : I| = \xi(1)|G : T| = (\xi(1)|G : T|)_{\alpha} = \chi(1)_{\alpha}.
\]

Hence it follows by Lemma 3.6 and (3.10a) that \( \alpha \) satisfies the conclusions of Theorem 3.10 and the result is true in this case.

We may from now on assume that \( T = G \). Let \( K = O_{\pi'}(G) \) and \( L = O_{\pi''}(G) \). We may also assume that \( K < N \), for if \( K = N \), then \( N = 1 \) by \( \pi \)-separability, and this implies that \( G \) is a \( \pi \)-group, whence \( G = H \) and the result is trivial in this case:

\[
\begin{array}{c}
\text{Diagram 4} \\
G & \xrightarrow{\chi} & KH \\
\downarrow & & \downarrow \\
N & \xrightarrow{\varphi} & H \\
\downarrow & & \downarrow \\
K & \xrightarrow{\theta} & K \cap H \\
\downarrow & & \downarrow \\
L & \xrightarrow{\psi} & L \cap H \\
\downarrow & & \downarrow \\
\zeta & & \zeta
\end{array}
\]

Since \( \varphi \) is invariant in \( G \), it follows by Clifford's Theorem that
\[
\chi_N = e \varphi,
\] (3.10b)

where \( e \) is a \( \pi \)-number. Since \( |N : K| \) is a \( \pi' \)-number, it follows by Lemma 6.5 in [5] that
\[
\varphi_K = \theta_1 + \cdots + \theta_s,
\] (3.10c)

where all the \( \theta_i \)'s are distinct. Moreover, \( s \) is a \( \pi' \)-number because \( s | |N : K| \).
Combining (3.10b) and (3.10c) we obtain
\[ \chi_K = e(\theta_1 + \cdots + \theta_s), \]  
and hence by Clifford's Theorem we have \(|G : I_G(\theta_i)| = s\) for all \(i \in \{1, \ldots, s\}\).

Therefore, since \(s\) is a \(\pi\)'-number, there exists a \(i \in \{1, \ldots, s\}\) such that \(I_G(\theta_i)\) contains \(H\). Let \(\theta = \theta_i\) and note that \(\theta\) is invariant in \(KH\). Let \(\rho | \chi_{KH}\) such that \(\rho | \chi_{KH}\). Since \(\theta \in B_\pi(K)\), it follows by Theorem 2.6(ii) that \(\rho \in B_\pi(KH)\).

It is easy to see that \(O^\pi(KH) = L\), and so by the inductive hypothesis applied to \(KH\), there exists a Fong character \(\alpha\) for \(\rho\) such that every irreducible constituent \(\zeta\) of \(\alpha_{L \cap H}\) is Fong for \(L\) and is such that \(I_H(\zeta) = I_H(\psi)\), for the \(\psi | \rho_L\) such that \(\zeta\) is Fong for \(\psi\).

Since \(\theta\) is invariant in \(KH\), it follows by Lemma 4.1 in [S] that \(\chi(1) \chi(1) = \varphi(1) \varphi(1)\) which implies that \([\chi_N, \varphi] = e = [\rho_K, \theta]\), that is, \(\rho_K = e\theta\).

Applying Lemmas 3.9 and 3.5 to \(KH\) gives \(\alpha_{K \cap H} = e\delta\), where \(\delta\) is Fong for \(\theta\). By Lemma 3.7 we have that \(\delta\) is also Fong for \(\varphi\) and in particular, since both \(\varphi\) and \(\delta\) are invariant in \(H\), we have that
\[ I_H(\varphi) = I_H(\delta) = H. \]  
Finally, since \(e\) is a \(\pi\)-number, we have
\[ \alpha(1) = e\delta(1) = e\varphi(1) = (e\varphi(1)) = \chi(1), \]  
and so \(\alpha\) is a Fong character for \(\chi\). Equation (3.10f) now ensures that \(\alpha\) satisfies the conclusions of Theorem 3.10 and the proof is complete.

3.11. COROLLARY. Let \(G\) be \(\pi\)-separable, \(H \in \text{Hall}_\pi(G)\), \(N \triangleleft G\) such that \(G/N\) is a \(\pi\)-group, and \(\chi \in B_\pi(G)\). Then there exists a Fong character \(\alpha\) of \(\chi\) such that every irreducible constituent of \(\alpha_{N \cap H}\) is Fong for \(N\) and if \(\beta | \alpha_{N \cap H}\), then \(I_H(\theta) = I_H(\beta)\), where \(\theta | \chi_N\) such that \(\beta\) is Fong for \(\theta\).

Proof. Since \(O^\pi(G) \trianglelefteq N\), by Theorem 3.10 there exists a Fong character \(\alpha\) of \(\chi\) such that every irreducible constituent of \(\alpha_{O^\pi(G) \cap H}\) is Fong for \(O^\pi(G)\) and if \(\delta | \alpha_{O^\pi(G) \cap H}\) is associated with some \(\varphi | \chi_{O^\pi(G)}\), then \(I_H(\varphi) = I_H(\delta)\). The result now follows by Lemma 3.9.

3.12. COROLLARY. Let \(G\) be \(\pi\)-separable, \(N \triangleleft G\) such that \(G/N\) is a \(\pi\)-group, \(H \in \text{Hall}_\pi(G)\), and \(\chi \in B_\pi(G)\). Also let \(\alpha | \chi_H\) and \(\varphi | \chi_N\) be such that \(\alpha_{N \cap H}\) and \(\varphi_{N \cap H}\) have a common irreducible constituent \(\delta\) such that \(\delta\) is
Fong for \( \varphi \). Then \( \alpha \) is Fong for \( \chi \) and \( \alpha \) is the only irreducible constituent of \( \chi_H \) lying over \( \delta \) if and only if \( I_H(\varphi) = I_H(\delta) \).

**Proof.** This is true by Lemma 3.8 and Lemma 3.5.

Let \( \text{Schar}(G) \) denote the set \( \{ \varphi \in \text{Irr}(N) \mid N \leq G \} \). Whenever \( \varphi \) and \( \theta \) are elements of \( \text{Schar}(G) \) such that \( \text{Dom}(\varphi) \leq \text{Dom}(\theta) \), we shall use the notation \( \varphi < \theta \) to denote the fact that \( \varphi \) is an irreducible constituent of \( \theta_{\text{Dom}(\varphi)} \).

3.13. **Definition.** Given the lower \( \pi\pi' \)-series of \( G \),

\[
1 = N_m \lhd K_m \lhd \cdots \lhd N_0 \lhd K_0 = G,
\]

(3.13a)

and \( \chi \in B_\pi(G) \), a chain of irreducible characters:

\[
1 = \varphi^{(m)} < \theta^{(m)} < \cdots < \varphi^{(0)} < \theta^{(0)} = \chi,
\]

(3.13b)

where \( \theta^{(i)} \in B_\pi(K_i) \) and \( \varphi^{(i)} \in B_\pi(N_i) \) satisfying

\[
I_{K_i \cap H}(\varphi^{(i)}) = I_{K_i \cap H}(\theta^{(i+1)})
\]

(3.13c)

for all \( i \in \{0, 1, \ldots, m\} \) is called a **good \( B_\pi \)-chain** of \( \chi \).

Next we show that given \( \chi \in B_\pi(G) \), where \( G \) is \( \pi \)-separable, a good \( B_\pi \)-chain of \( \chi \) always exists. We prove first the following lemma:

3.14. **Lemma.** Let \( G \) be \( \pi \)-separable, \( H \in \text{Hall}_\pi(G) \), and \( K \triangleleft N \leq G \) both normal in \( G \) such that \( G/N \) is a \( \pi \)-group and \( N/K \) a \( \pi' \)-group. Let also \( \varphi \in B_\pi(N) \). Then there exists a \( \theta \mid \varphi_K \) such that

\[
I_H(\theta) = I_H(\varphi).
\]

**Proof.** Since \( N \) has \( \pi \)-index in \( G \), we must have that \( G = NH \). Let \( T = I_G(\varphi) \) and \( I = I_H(\varphi) \). Notice that \( T = NI \) and thus \( I \) is in fact a Hall \( \pi \)-subgroup of \( T \). Since \( (|I|, |G : N|) = 1 \), it follows by Glauberman's lemma, 13.8 in [3], that \( \varphi_K \) has an \( I \)-invariant irreducible constituent, say \( \theta \). Thus \( I_H(\theta) \supseteq I_H(\varphi) \). The reverse inclusion is obtained by noticing that \( \theta \in B_\pi(K) \) by Theorem 2.6(i), and by part (iii) of the same theorem it uniquely determines \( \varphi \). Hence \( I_H(\theta) \subseteq I_H(\varphi) \) and the result follows.

By applying Lemma 3.14 repeatedly to pairs of terms in the lower \( \pi\pi' \)-series of \( G \), we see that, given any \( \chi \in B_\pi(G) \), a good \( B_\pi \)-chain of \( \chi \) can always be constructed. Thus we have the following corollary:

3.15. **Corollary.** Let \( G \) be \( \pi \)-separable, \( H \in \text{Hall}_\pi(G) \), and \( \chi \in B_\pi(G) \). Then a good \( B_\pi \)-chain of \( \chi \) always exists.
It is clear from the definition that conjugating a good $B_\pi$-chain of $\chi$ with any element of $H$ gives us another good $B_\pi$-chain of $\chi$.

3.16. **Definition.** Let $G$ be $\pi$-separable and $H \in \text{Hall}_\pi(G)$. Let $\chi \in B_\pi(G)$ and let $x \in \text{Irr}(H)$ be a Fong character for $\chi$. Then we say that $x$ is strongly $\pi$-Fong if

(i) $x$ is $\pi$-Fong, and

(ii) whenever

$$1 = \delta^{(m)} < \delta^{(m-1)} < \ldots < \delta^{(0)} < x$$

is a chain of irreducible characters such that $\delta^{(i)} \in \text{Irr}(H \cap N_i)$ is Fong for $\phi^{(i)} \in B_\pi(N_i)$ in a good $B_\pi$-chain of $\chi$, then

$$I_{K_i \cap H}(\phi^{(i)}) = I_{K_i \cap H}(\delta^{(i)})$$

for all $i \in \{0, 1, \ldots, m\}$.

Notice that if $x$ is strongly $\pi$-Fong for $\chi$, then $\delta^{(i)}$ is strongly $\pi$-Fong for $\phi^{(i)}$ for all $i \in \{0, 1, \ldots, m\}$. Since $N_i \cap H = K_{i+1} \cap H$, it follows by Lemma 3.7 that $\delta^{(i)}$ is Fong for $\theta^{(i+1)}$ and in fact strongly $\pi$-Fong. Notice also that by (3.13c) and (3.16b) it follows that $I_{K_i \cap H}(\theta^{(i+1)}) = I_{K_i \cap H}(\delta^{(i)})$.

3.17. **Lemma.** Let $G$ be $\pi$-separable, $H \in \text{Hall}_\pi(G)$, and $M \triangleleft G$ such that $G/M$ is a $\pi$-group. Let $\chi \in B_\pi(G)$ and $x$ a strongly $\pi$-Fong character for $\chi$. Then every irreducible constituent of $\alpha_M \cap H$ is strongly $\pi$-Fong for $M$.

**Proof.** We first make the obvious remark that $O^\pi(M) = O^\pi(G)$. Let

$$1 = N_m \triangleleft K_m \triangleleft \ldots \triangleleft N_0 \triangleleft K_0 = G$$

be the lower $\pi\pi'$-series of $G$. Let

$$1 = \phi^{(m)} < \theta^{(m)} < \ldots < \phi^{(0)} < \theta^{(0)} = \chi$$

be a good $B_\pi$-chain of $\chi$ and

$$1 = \delta^{(m)} < \delta^{(m-1)} < \ldots < \delta^{(0)} < x$$

the corresponding chain of Fong characters. Since $M$ has $\pi$-index in $G$, we have $N_0 \triangleleft M \triangleleft K_0 = G$ and, in particular $N_0 \cap H \triangleleft M \cap H$. Let $\beta | M \cap H$. Then by Lemma 3.9, we have that $\beta$ is Fong for $M$ associated with some $\psi | \chi_M$. Since $M \cap H \triangleleft H$, all irreducible constituents of $\alpha_M \cap H$ are conjugate, and so we may assume without any loss of generality that $\beta$ lies over $\delta^{(0)}$. 

To show that $\beta$ is strongly $\pi$-Fong for $\psi$ it is enough to show that
\[ 1 = \phi^{(m)} < \theta^{(m)} < \cdots < \phi^{(0)} < \psi \] (3.17d)
is a good $B_{\pi}$-chain of $\psi$ and also that the corresponding chain of Fong characters
\[ 1 = \delta^{(m)} < \delta^{(m-1)} < \cdots < \delta^{(0)} < \beta \] (3.17e)
satisfies the conditions
\[ I_{K_i \cap H}(\phi^{(i)}) = I_{K_i \cap H}(\delta^{(i)}) \quad \text{for all } i \in \{1, \ldots, m\} \] (3.17f)
and
\[ I_{M \cap H}(\phi^{(0)}) = I_{M \cap H}(\delta^{(0)}). \] (3.17g)

By (3.17b) we have that $I_{K_i \cap H}(\phi^{(i)}) = I_{K_i \cap H}(\theta^{(i+1)})$ and by (3.17f) we see that $I_{K_i \cap H}(\phi^{(i)}) = I_{K_i \cap H}(\delta^{(i)})$ for all $i \geq 1$. To complete the proof we only need to show that
\[ I_{M \cap H}(\phi^{(0)}) = I_{M \cap H}(\theta^{(1)}) = I_{M \cap H}(\delta^{(0)}). \] (3.17h)

But by (3.17a) and (3.17b) we have
\[ I_{H}(\phi^{(0)}) = I_{H}(\theta^{(1)}) = I_{H}(\delta^{(0)}), \]
and intersecting these equalities with $M$ gives (3.17h) and this completes the proof.
3.18. Theorem. Let $G$ be a $\pi$-separable group, $H \in \text{Hall}_{\pi}(G)$, and $\chi \in B_{\pi}(G)$. Then there exists a strongly $\pi$-Fong character $\alpha$ associated with $\chi$.

Proof. We use induction on $|G|$.

Let

$$1 = N_m < K_m < \cdots < N_0 < K_0 = G$$

be the lower $\pi\pi'$-series of $G$, and let by Corollary 3.15

$$1 = \phi^{(m)} < \theta^{(m)} < \cdots < \phi^{(0)} < \theta^{(0)} = \chi$$

be a good $B_{\pi}$-chain of $\chi$.

We may assume that $N_0 < G$, for if $N_0 = G$, then $K_1$ is the first proper subgroup in (3.18a), and by the inductive hypothesis applied to $K_1$, there exists a strongly $\pi$-Fong character $\delta^{(1)}$ for $\theta^{(1)}$. By Lemma 3.7, we have that $\delta^{(1)}$ is also Fong for $\chi$, since $G/K_1$ is a $\pi'$-group. Since $\delta^{(1)}$ satisfies conditions (3.16b) obviously for $i = 1$ and by the inductive hypothesis for $i > 1$, we may choose $\alpha = \delta^{(1)}$ and theorem is true in this case. Let $T^{(0)} = I_G(\phi^{(0)})$ and $\xi^{(0)}$ be the Clifford correspondent of $\chi$ with respect to $\phi^{(0)}$. Assume first that $T^{(0)} < G$. Then we have:

![Diagram 6]

By Lemma 7.6 in [5] we have that $\xi^{(0)} \in B_{\pi}(T^{(0)})$, and since $O^\pi(T^{(0)}) - O^\pi(G)$, by the inductive hypothesis applied to $T^{(0)}$ there exists a Fong character $\gamma$ of $\xi^{(0)}$ such that $\gamma$ is strongly $\pi$-Fong for $\xi^{(0)}$. Let us now consider the following good $B_{\pi}$-chain of $\xi^{(0)}$:

$$1 = \phi^{(m)} < \theta^{(m)} < \cdots < \phi^{(0)} < \xi^{(0)}.$$  \hspace{1cm} (3.18c)

(Notice that (3.18c) is indeed a good $B_{\pi}$-chain of $\xi^{(0)}$ since $I_{T^{(0)} \cap H}(\phi^{(0)}) = T^{(0)} \cap H = I_H(\phi^{(0)}) = I_H(\theta^{(1)})$, that is to say both $\phi^{(0)}$ and $\theta^{(1)}$ are invariant in $T^{(0)} \cap H$ and hence $I_{T^{(0)} \cap H}(\phi^{(0)}) = I_{T^{(0)} \cap H}(\theta^{(1)}) = T^{(0)} \cap H$; furthermore $I_{K_i \cap H}(\phi^{(i)}) = I_{K_i \cap H}(\theta^{(i+1)})$ for all $i \geq 1$ by (3.18b).)
Since $\gamma$ is strongly $\pi$-Fong for $\xi^{(0)}$ there exists a chain of Fong characters corresponding to (3.18c), namely

$$1 = \delta^{(m)} < \delta^{(m-1)} < \ldots < \delta^{(0)} < \gamma$$ \hfill (3.18d)

such that

$$I_{K_i \cap H}(\delta^{(i)}) = I_{K_i \cap H}(\phi^{(i)}) \quad \text{for all } i \geq 1 \quad (3.18e)$$

and

$$I_{T^{(0)} \cap H}(\delta^{(0)}) = I_{T^{(0)} \cap H}(\phi^{(0)}) = T^{(0)} \cap H. \quad (3.18f)$$

Hence, by Lemma 3.4 we have $I_H(\delta^{(0)}) \leq I_H(\phi^{(0)}) = T^{(0)} \cap H = I_{T^{(0)} \cap H}(\delta^{(0)}) \leq I_H(\delta^{(0)})$ and therefore

$$I_H(\delta^{(0)}) = I_H(\phi^{(0)}). \quad (3.18g)$$

In view of (3.18g) we can deduce from Lemma 3.5 that there exists a Fong character $\alpha$ of $\chi$ such that $\alpha$ lies over $\delta^{(0)}$ and the argument of the proof shows that in fact $\alpha = \gamma^H$. Hence

$$1 = \delta^{(m)} < \delta^{(m-1)} < \ldots < \delta^{(0)} < \chi$$

is a chain of Fong characters corresponding to the good $B_{\pi}$-chain in (3.18b), and by (3.18e) and (3.18g), it follows that $\alpha$ is strongly $\pi$-Fong. (Condition (i) of Definition 3.16 easily follows from Lemma 3.6.) Therefore, from now on we may assume that $\phi^{(0)}$ is invariant in $G$, and hence in $H$. Since (3.18b) is a good $B_{\pi}$-chain of $\chi$, it follows that $\theta^{(1)}$ is also invariant in $H$. Let $\rho^{(1)}|_{\chi_{K_1 H}}$ such that $\rho^{(1)}|_{\chi_{K_1 H}}$. Since $K_1 H/K_1$ is a $\pi$-group, it follows by Theorem 2.6(b) that $\rho^{(1)} \in B_{\pi}(K_1 H)$, and by the inductive hypothesis applied to $K_1 H$, there exists a Fong character $\alpha$ of $\rho^{(1)}$ such that $\alpha$ is strongly $\pi$-Fong for $\rho^{(1)}$:  

![Diagram](image-url)
By Lemma 3.9 it follows that
\[ \alpha_{K_1 \cap H} = e^{(1)} \delta^{(0)}, \] (3.18h)
where \( e^{(1)} = [\alpha_{K_1 \cap H}, \delta^{(0)}] = [\rho^{(1)}(\alpha), \theta^{(1)}]. \) By Lemma 3.7 we have that \( \delta^{(0)} \) is also Fong for \( \varphi^{(0)} \) and it follows easily from Lemma 4.1 in [5] that
\[ e^{(1)} = [\chi_{\mathcal{N}_0}, \varphi^{(0)}]. \] (3.18i)

Using (3.18i) and Clifford's Theorem, we have that \( \chi(1) = e^{(1)} \varphi^{(0)}(1), \) and taking \( \pi \)-parts of this equality we obtain
\[ (\chi(1))_{\pi} = e^{(1)}(\varphi^{(0)}(1))_{\pi} = e^{(1)}(\theta^{(1)}(1))_{\pi} = e^{(1)} \delta^{(0)}(1) = \alpha(1). \]

Therefore \( \alpha \) is a Fong character for \( \chi. \) By Lemma 3.17, we have that \( \delta^{(0)} \) is strongly \( \pi \)-Fong for \( K_1, \) and by assumption \( I_H(\delta^{(0)}) = H = I_H(\varphi^{(0)}) = I_H(\theta^{(1)}). \) This completes the proof that \( \alpha \) is strongly \( \pi \)-Fong character, since the chain of characters
\[ 1 = \delta^{(m)} < \delta^{(m-1)} < \cdots < \delta^{(0)} < \alpha \]
is the corresponding chain of Fong characters to the good \( B_\pi \)-chain (3.18b) and \( I_{K_1 \cap H}(\varphi^{(i)}) = I_{K_1 \cap H}(\delta^{(i)}) \) for all \( i \in \{0, 1, \ldots, m\}. \]

As a corollary to Theorem 3.17 we get Theorem C.

3.19. Corollary. Let \( G \) be \( \pi \)-separable, \( H \in \text{Hall}_\pi(G), \) and \( \chi \in B_\pi(G). \) Then some Fong character \( \alpha \in \text{Irr}(H) \) associated with \( \chi \) is \( \pi \)-Fong.

Our aim will to be show that strongly \( \pi \)-Fong characters are in fact subnormally Fong, for then Theorem 3.18 will guarantee their existence and this will answer Isaacs' question. We need one more proposition.

3.20. Proposition. Let \( G \) be \( \pi \)-separable and \( M \triangleleft G \) such that \( G/M \) is a \( \pi' \)-group. Let \( \chi \in B_\pi(G), \) and \( \alpha \) a strongly \( \pi \)-Fong character for \( \chi. \) Then \( \alpha \) is strongly \( \pi \)-Fong for \( M. \)

To prove this proposition we need the following lemma.

3.21. Lemma. Let \( G \) be \( \pi \)-separable, \( M \) and \( N \) normal subgroups of \( G \) such that \( G = MN \) with \( G/M \) being a \( \pi \)-group and \( G/N \) a \( \pi' \)-group. Let \( \rho \in B_\pi(M) \) and \( \theta \in B_\pi(N) \) such that \( \rho_{M \cap H} \) and \( \theta_{M \cap H} \) have common irreducible constituent \( \varphi. \) Then
\[ I_N(\rho) = I_N(\varphi). \]
Proof. By Theorem 2.6(ii), \( \rho \) is the only irreducible constituent of \( \varphi^M \) that lies in \( B_\pi(M) \). Thus \( \varphi \) uniquely determines \( \rho \) and \( I_N(\rho) \supseteq I_N(\varphi) \). By Glauberman's lemma, 13.9 in [3], every irreducible constituent of \( \rho_{M \cap H} \) is invariant in \( I_N(\rho) \) and in particular \( I_N(\varphi) \supseteq I_N(\rho) \) and the result follows.

3.22. Corollary. Let \( G, M, N \) be as in Lemma 3.21. Let \( \chi \in B_\pi(G) \), let \( \theta | \chi_N \), and let \( \rho | \chi_M \). Then there exists a unique \( \psi \in \text{Irr}(M \cap N) \) such that \([\theta_{M \cap H}, \psi] \neq 0 \neq [\rho_{M \cap H}, \psi] \).

Proof. Let \( \varphi \) and \( \zeta \) be irreducible constituents of both \( \theta_{M \cap H} \) and \( \rho_{M \cap H} \). Then by Clifford's theorem we have that \( \varphi^n = \zeta \) for some \( n \in N \). We now have that \( \varphi \) is an irreducible constituent of both \( \rho \) and \( \rho^n \). Since \( M/N \) is a \( \pi' \)-group, Theorem 2.6(ii) forces \( \rho = \rho^n \). Since \( n \) fixes \( \rho \) it must also fix \( \varphi \), by Lemma 3.21 above. It therefore follows that \( \varphi = \varphi^n = \zeta \).

We are now ready to prove Proposition 3.20.

Proof (of Proposition 3.20). We use induction on \(|G|\).

Case 1. \( O^n(G) < G \). Let

\[
1 = N_m < K_m < \cdots < N_0 < K_0 = G \tag{3.20a}
\]

be the lower \( \pi\pi' \)-series of \( G \). Since \( \alpha \) is a strongly \( \pi \)-Fong character for \( \chi \) we can find a good \( B_\pi \)-chain

\[
1 = \varphi^{(m)} < \theta^{(m)} < \cdots < \varphi^{(0)} < \theta^{(0)} = \chi \tag{3.20b}
\]

of \( \chi \), and a corresponding chain

\[
1 = \delta^{(m)} < \delta^{(m-1)} < \cdots < \delta^{(0)} < \alpha \tag{3.20c}
\]

of Fong characters, such that

\[
I_{K_i \cap H}(\varphi^{(i)}) = I_{K_i \cap H}(\delta^{(i)}) \tag{3.20d}
\]

for all \( i \in \{0, 1, \ldots, m\} \). Let \( M \) be any normal subgroup of \( G \) with \( \pi' \)-index in \( G \). Then \( M \cap O^n(G) < O^n(G) \), and since \( M \cap O^n(G) \) has \( \pi' \)-index in \( O^n(G) \), we have

\[
M \cap O^n(G) \supseteq O^{n\pi'}(G) \tag{3.20e}
\]

Similarly \( M \cap O^n(G) \) is normal and has \( \pi \)-index in \( M \), and hence

\[
M \cap O^n(G) \supseteq O^n(M) \tag{3.20f}
\]
Let 

\[ 1 = \tilde{N}_n \triangleleft \tilde{K}_n \triangleleft \cdots \triangleleft \tilde{N}_0 \triangleleft \tilde{K}_0 = M \]  

be the lower \( \pi \pi' \)-series of \( M \). It follows by (3.20e) that \( M \cap N_0 \geq K_1 \), and by (3.20f) that \( M \cap N_0 \geq \tilde{N}_0 \). Let \( \psi^{(0)} | (\theta^{(1)})^{M \cap N_0} \) such that \( \psi^{(0)} \in B_\pi(M \cap N_0) \). By 2.6(iii) there exists a unique such \( \psi^{(0)} \) lying over \( \theta^{(1)} \). Clearly \( \psi^{(0)} \) also lies below \( \phi^{(0)} \). So we have the following configuration:

![Diagram 8](image_url)

It is easy to see that \( I_H(\theta^{(1)}) \leq I_H(\psi^{(0)}) \leq I_H(\phi^{(0)}) = I_H(\theta^{(1)}) \), where the last equality follows from the fact that (3.20b) is a good \( R_\pi \)-chain for \( \chi \). Hence

\[ I_H(\psi^{(0)}) = I_H(\phi^{(0)}) = I_H(\theta^{(1)}). \]  

(3.20h)

By the remarks following Definition 3.16 we know that \( \delta^{(0)} \) is strongly \( \pi \)-Fong for \( \phi^{(0)} \), and since \( M \cap N_0 \) has \( \pi' \)-index in \( N_0 \) (= \( O^\pi(G) < G \)), we can apply our inductive hypothesis to \( N_0 \) for the normal subgroup \( M \cap N_0 \) to conclude that \( \delta^{(0)} \) is strongly \( \pi \)-Fong for \( \psi^{(0)} \).

Let

\[ 1 = \tilde{\phi}^{(n)} < \tilde{\theta}^{(n)} < \cdots < \tilde{\theta}^{(1)} < \tilde{\phi}^{(0)} < \psi^{(0)} \]  

(3.20i)

be a good \( B_\pi \)-chain for \( \psi^{(0)} \) where \( \tilde{\phi}^{(i)} \in B_\pi(\tilde{N}_i) \) for \( 0 \leq i \leq n \), and \( \tilde{\theta}^{(i)} \in B_\pi(\tilde{K}_i) \) for \( 1 \leq i \leq n \). (As remarked earlier in the proof, \( M \cap N_0 \) has \( \pi \)-index in \( M \) and so the lower \( \pi \pi' \)-series of \( M \cap N_0 \) is (3.20g), apart from the term \( \tilde{K}_0 \) which is replaced by \( M \cap N_0 \).)
Also let
\[
1 = \delta^{(n)} < \delta^{(n-1)} < \cdots < \delta^{(0)} < \delta^{(0)}
\]  
be the corresponding chain of Fong characters such that
\[
I_{K_i \cap H}(\tilde{\phi}^{(i)}) = I_{K_i \cap H}(\tilde{\delta}^{(i)}) \quad \text{for all } 1 \leq i \leq n
\]
and
\[
I_{(M \cap N_0) \cap H}(\tilde{\phi}^{(0)}) = I_{(M \cap N_0) \cap H}(\tilde{\delta}^{(0)}).
\]

Let \( \rho \mid (\psi^{(0)})^M \) such that \( \rho \) is a constituent of \( \chi_M \). Then by Theorem 2.6(ii) it follows that \( \rho \in B_\pi(M) \). Now by (3.20d) and (3.20h) we have \( I_H(\psi^{(0)}) = I_H(\delta^{(0)}) \), and so by Lemma 3.5, there exists a Fong character \( \alpha' \) for \( \rho \) such that \( \alpha' \) lies over \( \delta^{(0)} \); furthermore \( \alpha' \) is the only irreducible constituent of \( \rho_H \) that lies over \( \delta^{(0)} \). Since \( I_H(\psi^{(0)}) = I_H(\delta^{(0)}) \), a further application of Lemma 3.5 tells us that \( \alpha = \alpha' \), since \( \alpha \) is the only constituent of \( \chi_H \) lying over \( \delta^{(0)} \).

It remains to show that \( I_H(\tilde{\psi}^{(0)}) = I_H(\tilde{\delta}^{(0)}) \). We will do that by showing that \( \alpha \) is in fact the only irreducible constituent of \( \rho_H \) that lies over \( \delta^{(0)} \), and thus the result will then follow by Corollary 3.12. Observe first that \( \delta^{(0)} \) is in fact strongly \( \pi \)-Fong for \( \tilde{\phi}^{(0)} \). We also have that \( I_{M \cap N_0 \cap H}(\tilde{\phi}^{(0)}) = I_{M \cap N_0 \cap H}(\tilde{\delta}^{(0)}) \), and thus it follows by Lemma 3.5 that \( \text{Irr}(M \cap N_0 \cap H \mid \delta^{(0)}) \) consists entirely of irreducible characters that are Fong (in fact strongly \( \pi \)-Fong) for \( M \cap N_0 \). Therefore it follows that there exists a bijection from \( \text{Irr}(M \cap N_0 \mid \tilde{\phi}^{(0)}) \) into \( \text{Irr}(M \cap N_0 \cap H \mid \delta^{(0)}) \).

Clearly \( \psi^{(0)} \in \text{Irr}(M \cap N_0 \mid \tilde{\phi}^{(0)}) \), and under the above bijection, it corresponds to \( \delta^{(0)} \in \text{Irr}(M \cap N_0 \cap H \mid \delta^{(0)}) \). Thus if \( \psi' \mid \rho_{M \cap N_0} \) is such that \( \psi' \in \text{Irr}(M \cap N_0 \mid \tilde{\phi}^{(0)}) \), then there exists an \( h_0 \in H \), such that \( \psi' = (\psi^{(0)})^{h_0} \). Hence it follows that \( (\delta^{(0)})^{h_0} \in \text{Irr}(M \cap N_0 \cap H \mid \delta^{(0)}) \), and clearly \( (\delta^{(0)})^{h_0} \) is the unique irreducible constituent of \( \psi' \) that lies over \( \delta^{(0)} \). But as remarked earlier \( \alpha \) is the only irreducible constituent of \( \rho_H \), and hence of \( \rho_H \), that lies over the \( H \)-orbit of \( \delta^{(0)} \). It follows now that \( \alpha \) is the only irreducible constituent of \( \rho_H \) that lies over \( \delta^{(0)} \), and Corollary 3.12 yields the required equality, that is to say,
\[
I_H(\tilde{\psi}^{(0)}) = I_H(\tilde{\delta}^{(0)}).
\]

We next claim that
\[
1 = \tilde{\phi}^{(n)} < \tilde{\delta}^{(n)} < \cdots < \tilde{\phi}^{(0)} < \tilde{\delta}^{(0)} = \rho
\]
is a good \( B_\pi \)-chain of \( \rho \), where \( \tilde{\phi}^{(i)} \) and \( \tilde{\delta}^{(i)} \) are as in (3.20i) for \( 0 \leq i \leq n \). To prove this claim we need only show that \( I_H(\tilde{\phi}^{(i)}) = I_H(\tilde{\delta}^{(i)}) \), because
by (3.20i) the remaining conditions are satisfied. But this is clear from the fact that \( I_H(\tilde{\theta}^{(1)}) \leq I_H(\tilde{\varphi}^{(0)}) = I_H(\tilde{\delta}^{(0)}) \leq I_H(\tilde{\theta}^{(1)}) \), for then we have equality throughout and the claim is proved.

The chain of Fong characters corresponding to (3.20m) is clearly
\[ 1 = \tilde{\delta}^{(n)} < \tilde{\delta}^{(n-1)} < \ldots < \tilde{\delta}^{(0)} < \alpha, \]
and so by (3.20k) and (3.20m) it follows that \( \alpha \) is strongly \( \pi \)-Fong for \( \rho \). Hence \( \alpha \) is strongly \( \pi \)-Fong for \( M \), and the proposition follows in this case.

**Case 2.** \( O_\pi(G) = G \). In this case \( \pi \)-separability implies that \( K_1 = O_\pi(G) < G \). In the notation of (3.20a) and (3.20b), we have that \( \varphi^{(0)} = \theta^{(0)} = \chi \) and \( \delta^{(0)} = \alpha \). Let \( \theta^{(1)} \) be the unique irreducible constituent of \( \chi_{K_1} \) such that \( \alpha \) is Fong for \( \theta^{(1)} \). By the remarks following Definition 3.16, we see that \( \alpha \) is strongly \( \pi \)-Fong for \( \theta^{(1)} \). Let \( \rho \) be the unique irreducible constituent of \( \chi_{M} \) lying over \( \alpha \). Then by Lemma 3.7 we know that \( \alpha \) is Fong for \( \rho \). Let \( \tilde{\varphi}^{(0)} \) be an irreducible constituent of \( \rho_{N_0} \). (We use the same notation for the lower \( \pi \pi' \)-series of \( M \) as in (3.20g).) Clearly \( \theta^{(1)} \) is also an irreducible constituent of \( \rho_{K_1} \), and so by Corollary 3.22, it follows that, upon restriction to \( \tilde{N}_0 \cap K_1 \), the characters \( \tilde{\varphi}^{(0)} \) and \( \theta^{(1)} \) have unique common irreducible constituent \( \psi^{(1)} \). (We may, without any loss of generality, assume that \( \psi^{(1)} \) lies over \( \varphi^{(1)} \), since \( \varphi^{(1)} \) is just any one of the irreducible constituents of \( (\theta^{(1)})_{N_1} \).) Thus we have

\[ \begin{array}{c}
G \\
\chi \\
\rho \\
\theta^{(1)} \\
\psi^{(1)} \\
\varphi^{(1)} \\
\delta^{(0)} \\
\delta^{(1)} \\
\delta^{(1)} \cap H \\
\delta^{(0)} \cap H \\
N_1 \cap H \\
N_0 \cap K_1 \\
K_1 \\
K_2 \\
H \\
M \\
\end{array} \]

**Diagram 9**

Let \( \tilde{\delta}^{(0)} \) be an irreducible constituent of \( \alpha_{N_0 \cap K_1} \) such that \( \tilde{\delta}^{(0)} \) is Fong for \( \psi^{(1)} \). By Lemma 3.17, we have that \( \tilde{\delta}^{(0)} \) is strongly \( \pi \)-Fong for \( \psi^{(1)} \) since
\(\alpha\) is strongly \(\pi\)-Fong for \(\theta^{(1)}\). Let \(\tilde{\theta}^{(1)}|\,(\psi^{(1)})_{K_1}\) such that \(\tilde{\theta}^{(0)}\) is Fong for \(\tilde{\theta}^{(1)}\). Then by the inductive hypothesis applied to \(\tilde{N}_0 \cap K_1\) (with \(\tilde{K}_1\) in the place of \(M\)), we can conclude that \(\tilde{\theta}^{(0)}\) is strongly \(\pi\)-Fong for \(\theta^{(1)}\) since \(\tilde{\theta}^{(0)}\) is strongly \(\pi\)-Fong for \(\psi^{(1)}\).

Let
\[
1 = \tilde{\phi}^{(n)} < \tilde{\theta}^{(n)} < \cdots < \tilde{\phi}^{(1)} < \tilde{\theta}^{(1)}
\]
be a good \(B_{\pi}\)-chain for \(\tilde{\theta}^{(1)}\), where \(\tilde{\phi}^{(i)} \in B_{\pi}(\tilde{N}_i)\), and \(\tilde{\theta}^{(i)} \in B_{\pi}(\tilde{K}_i)\) for all \(1 \leq i \leq n\), using the notation of (3.20g). Since \(\tilde{\theta}^{(0)}\) is strongly \(\pi\)-Fong for \(\tilde{\theta}^{(1)}\), let
\[
1 = \tilde{\theta}^{(n)} < \tilde{\theta}^{(n-1)} < \cdots < \tilde{\theta}^{(1)} < \tilde{\theta}^{(0)}
\]
be the chain of Fong characters corresponding to (3.20o). Then
\[
I_{K_1 \cap H}(\tilde{\phi}^{(i)}) = I_{K_1 \cap H}(\tilde{\theta}^{(i)}) \quad \text{for all} \quad 1 \leq i \leq n.
\]

Next we claim that
\[
1 = \phi^{(n)} < \theta^{(n)} < \cdots < \theta^{(1)} < \phi^{(0)} < \rho
\]
is a good \(B_{\pi}\)-chain for \(\rho\), that
\[
1 = \tilde{\theta}^{(n)} < \tilde{\theta}^{(n-1)} < \cdots < \tilde{\theta}^{(1)} < \tilde{\theta}^{(0)} < \alpha
\]
is the corresponding chain of Fong characters, and that \(I_{K_1 \cap H}(\tilde{\phi}^{(i)}) = I_{K_1 \cap H}(\tilde{\theta}^{(i)})\) for all \(0 \leq i \leq n\). By (3.20o) and (3.20q), we need only show that \(I_H(\tilde{\phi}^{(0)}) = I_H(\tilde{\theta}^{(1)}) = I_H(\tilde{\theta}^{(0)})\).

By Lemmas 3.4 and 3.5 we have
\[
I_H(\tilde{\theta}^{(0)}) \subseteq I_H(\tilde{\theta}^{(1)}) \subseteq I_H(\tilde{\phi}^{(0)}).
\]
Clearly we have that \(I_H(\tilde{\phi}^{(0)}) \in \text{Hall}_n(I_M(\tilde{\phi}^{(0)}))\). By Lemma 3.21 it follows that \(I_{K_1}(\tilde{\phi}^{(0)}) = I_{K_1}(\psi^{(1)})\) and since \(I_{K_1}(\tilde{\phi}^{(0)})\) is normal and has \(\pi\)-index in \(I_M(\tilde{\phi}^{(0)})\), it follows that \(I_{K_1}(\tilde{\phi}^{(0)})\) contains all Hall \(\pi\)-subgroups of \(I_M(\tilde{\phi}^{(0)})\). Since \(I_H(\psi^{(1)}) \in \text{Hall}_n(I_{K_1}(\psi^{(1)}))\), we have that \(I_H(\psi^{(1)}) \in \text{Hall}_n(I_{K_1}(\tilde{\phi}^{(0)}))\) and by the above \(I_H(\psi^{(1)}) \in \text{Hall}_n(I_M(\tilde{\phi}^{(0)}))\). Therefore
\[
I_H(\psi^{(1)}) = I_H(\tilde{\phi}^{(0)}).
\]
By Lemma 3.9 we have \(I_H(\psi^{(1)}) = I_H(\tilde{\theta}^{(0)})\); hence by (3.20u) we have that \(I_H(\tilde{\phi}^{(0)}) = I_H(\tilde{\theta}^{(0)})\), and by (3.20t) it follows that
\[
I_H(\tilde{\theta}^{(0)}) = I_H(\tilde{\theta}^{(1)}) = I_H(\tilde{\phi}^{(0)}).
\]
The above equality ensures that (3.20r) is a good $B_\pi$-chain for $\rho$ and that (3.20s) is the corresponding chain of Fong characters such that

$$I_{K_i \cap H}(\tilde{\phi}^{(i)}) = I_{K_i \cap H}(\tilde{\delta}^{(i)})$$

for all $0 \leq i \leq n,$

thereby proving that $\alpha$ is strongly $\pi$-Fong for $M$. This completes the proof of Proposition 3.20.

3.23. THEOREM. Let $G$ be $\pi$-separable, $\chi \in B_\pi(G)$, and $\alpha$ a strongly $\pi$-Fong character of $\chi$, then $\alpha$ is subnormally Fong for $\chi$.

Proof. We argue by induction on $|G|$. Let $S$ be a proper subnormal subgroup of $G$. Then $S$ is a subnormal subgroup of some maximal normal subgroup $M$ of $G$ and since $G$ is $\pi$-separable, the quotient $G/M$ is either a $\pi$-group or a $\pi'$-group.

If $G/M$ is a $\pi'$-group, then $\alpha$ is strongly $\pi$-Fong for $M$ be Proposition 3.20, and by the inductive hypothesis applied to $M$, $\alpha$ is subnormally Fong for $M$. In particular if $\beta \mid \alpha_{H \cap S}$ then $\beta$ is Fong for $S$.

If $G/M$ is $\pi$-group, then by Lemma 3.17 every irreducible constituent of $\alpha_{M \cap H}$ is strongly $\pi$-Fong for $M$, and by the inductive hypothesis applied to $M$, every irreducible constituent of $\alpha_{M \cap H}$ is subnormally Fong for $M$. In particular, if $\beta \mid \alpha_{H \cap S}$, then $\beta$ is Fong for $S$.

Hence, in either case, $\alpha$ satisfies the requirements of a subnormally Fong character of $G$.

Clearly Theorem A is an easy corollary of Theorems 3.18 and 3.23 above.

We finish by looking at Isaacs Example 9.1 in [6]. Isaacs provides this example to demonstrate that not all Fong characters associated with a $\chi \in B_\pi(G)$ behave well with respect to normal subgroups. We will apply our foregoing theory to show that there exists a Fong character $\alpha$ associated with $\chi$ such that $\alpha$ behaves well with respect not only to normal subgroups but to subnormal subgroups as well.

EXAMPLE (9.1 in [6]). Let $D_8$, the dihedral group of order 8, act on $A_4$ with the cyclic $C$ of order 4 as the kernel of this action. Let $G$ denote the semidirect product of $A_4$ by $D_8$ with respect to this action. For simplicity of notation write $A = A_4$ and $D = D_8$. Take $\pi = \{2\}$ and notice that since $G$ is soluble it is clearly $\pi$-separable.

Let $\theta \in \text{Irr}(A)$ be the unique faithful character of $A_4$ (of degree 3) and $\lambda \in \text{Irr}(C)$ be one of the two faithful characters of $C$. It is not difficult to show that $(\theta \lambda)^G$ is irreducible, equal to $\chi$ say, and in fact $\chi \in B_\pi(G)$. 

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Let $V$ denote the Klein 4 group contained in $A$, then $H = VD$ is a Hall $\pi$-subgroup of $G$. The following three claims are justified in Isaacs [6].

(i) $\chi_H = \alpha_1 + \alpha_2 + \alpha_3$, where $\alpha_i$ is Fong for $\chi$ for $i = 1, 2, 3$.

Let $K$ be a normal subgroup of $D$ such that $K \cong V_4$ and consider the subgroup $N = AK$ of $G$. Clearly $N \lhd G$ since $|G : N| = 2$.

(ii) $\chi_N$ is the sum of two distinct irreducible characters, say $\psi_1$ and $\psi_2$.

(iii) Only one of the $\alpha_i$ for $i = 1, 2, 3$, reduces to the sum of two linear characters upon restriction to $H \cap N$.

Following claim (iii) above let $(\alpha_1)_{N \cap H} = \lambda_1 + \lambda_2$, where $\lambda_1$ is Fong for $\psi_1$ and $\lambda_2$ is Fong for $\psi_2$. It is not hard to show that $(\alpha_2)_{N \cap H} = (\alpha_3)_{N \cap H} = \beta$ which by claim (iii) is irreducible.

We show that $\chi = \alpha_1$ is in fact subnormally Fong. Consider the lower $\pi\pi'$-series of $G$,

$$1 \triangleleft V \triangleleft A \triangleleft G,$$

where $A = O^{\pi}(G)$ and $V = O^{\pi\pi}(G)$. We now have that $\chi_A = 2\theta$ and $\alpha_{A \cap H} = 2\tau$, where $\tau = (\lambda_1)_{A \cap H} = (\lambda_2)_{A \cap H}$ and $(\alpha_2)_{A \cap H} = (\alpha_3)_{A \cap H} = \rho_1 + \rho_2$. We clearly have that $I_H(\theta) = H$ and $I_H(\tau) = H$. It is easy to see that since $\tau$ is a linear character it is strongly $\pi$-Fong for $\theta$ and by the stabilizer condition above, the character $\chi$ is stronger $\pi$-Fong. All that remains to show is that $\chi$ is in fact subnormally Fong.

Clearly any subnormal subgroup is contained in a maximal normal subgroup of $G$ and the maximal normal subgroups of $G$ are $A \times C$ and two more of type $AK$ where $K$ is a Klein-4 subgroup of $D$. In either case, the character $\chi$ restricted to a Hall $\pi$-subgroup of a maximal normal subgroup is the sum of two linear characters. Therefore if $S \lhd G$ and $S \neq G$, then the constituents of $\alpha_{H \cap S}$ are linear and thus they are Fong for $S$.

In the above example and all the other examples we looked at it seems as though every Fong character is in fact $\pi$-Fong. So we naturally ask the following question:

3.24. Question. Let $G$ be $\pi$-separable and $\chi \in B_\pi(G)$. Is every Fong character associated with $\chi$ a $\pi$-Fong character?

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