Existence of solutions for second-order perturbed nonconvex sweeping process

Dalila Azzam-Laouir*, Sabrina Izza
Laboratory of Pure and Applied Mathematics, University of Jijel, Algeria

A R T I C L E   I N F O

Article history:
Received 13 May 2010
Received in revised form 26 May 2011
Accepted 9 June 2011

Keywords:
Differential inclusion
Fixed point
Sweeping process

A B S T R A C T

We prove a theorem on the existence solutions for a second-order differential inclusion governed by a class of nonconvex sweeping process. Our result improves and generalizes many results from the literature of sweeping process.

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1. Introduction

In the present paper, we study, in a finite-dimensional space $E$, the existence of solutions for the second-order differential inclusion governed by a nonconvex sweeping process of the form

$$(P_{T,H}) \begin{cases} \dddot{u}(t) + N_{K(u(t))}(\dot{u}(t)) + F(t, u(t), \dot{u}(t)) + H(t, u(t), \dot{u}(t)), & \text{a.e. } t \in [0, T]; \\ \ddot{u}(t) \in K(u(t)), & \text{a.e. } t \in [0, T]; \\ u(0) = u_0; & \dot{u}(0) = v_0, \end{cases}$$

where $N_{K(u(t))}(\cdot)$ denotes the normal cone to $K(u(t))$, the sets $K(x)$ are uniformly $\rho$-prox-regular ($\rho \in (0, +\infty)$), $F : [0, T] \times E \times E \mapsto E$ is a closed convex valued set-valued mapping measurable on $[0, T]$ and upper semicontinuous on $E \times E$ and $H : [0, T] \times E \times E \mapsto E$ is a measurable set-valued mapping and mixed semicontinuous, that is, for a.e. $t \in [0, T]$, at each $(x, y) \in E \times E$ such that $H(t, x, y)$ is convex, the set-valued mapping $H(t, \cdot, \cdot)$ is upper semicontinuous on $E \times E$ and whenever $H(t, x, y)$ is not convex, the set-valued mapping $H(t, \cdot, \cdot)$ is lower semicontinuous on some neighborhood of $(x, y)$.

The existence of solutions for the second-order sweeping process has been thoroughly studied in the literature; see for example [1–7]. By using an important result of the coincidence between the solutions sets of a constrained and an unconstrained first-order differential inclusion proved in [8,9], we give a new proof of the perturbed second-order sweeping process which improves the ones given in [1–3,7].

2. Definition and preliminaries

Let $H$ be a real Hilbert space and let $S$ be a nonempty closed subset of $H$. We denote by $d(\cdot, S)$ the usual distance function associated with $S$, i.e., $d(u, S) := \inf_{y \in S} \|u - y\|$. For any $x \in H$ and $r \geq 0$, the closed ball centered at $x$ with radius $r$ will be denoted by $B_H(x, r)$. For $x = 0$ and $r = 1$, we will put $B_H$ in place of $B_H(0, 1)$.
By $L^1_{\mathcal{H}}(I)$, we denote the Banach space of all Lebesgue–Bochner integrable $\mathcal{H}$-valued mappings defined on the interval $I$, and by $w(L^1_{\mathcal{H}}, L^\infty_{\mathcal{H}})$, we denote the weak topology on $L^1_{\mathcal{H}}$.

For $A \subset \mathcal{H}$, $\text{co}(A)$ denotes the convex hull of $A$, and $\overline{\text{co}}(A)$ his closed convex hull.

Finally, $\mathcal{AC}_{\mathcal{H}}(I)$ denotes the Banach space of absolutely continuous $\mathcal{H}$-valued mappings defined on the interval $I$, endowed by the topology of uniform convergence.

The theorem below is a result characterizing the closed convex hull of a subset of a Banach space.

**Theorem 2.1.** Let $K$ be a nonempty subset of a Banach space $E$. Then

$$\overline{\text{co}}(K) = \{ x \in E : \forall x' \in E', \langle x', x \rangle \leq \delta^*(x', K) \},$$

where,

$$\delta^*(x', K) = \sup_{y \in K} \langle x', y \rangle$$

stands for the support function of $K$ at $x'$ and $E'$ is the topological dual of $E$.

We have also the following results which are needed in the proof of our main theorem.

**Lemma 2.1.** Let $E$ be a Banach space and $C$ a closed convex subset of $E$, then

$$d(x, C) = \sup_{x' \in \overline{\text{co}}(K)} [(x', x) - \delta^*(x', C)].$$

**Theorem 2.2** (See [10]). Let $E$ be a Banach space and $C$ a convex subset of $E$; then $C$ is weakly closed for $w(E, E')$ topology if and only if it is strongly closed.

**Theorem 2.3** (Banach–Mazur’s Lemma, See [11]). If $X$ is a Banach space and $(x_n)$ is a sequence of elements of $X$ converging weakly to $x$, then some sequence of convex combinations of the elements $x_n$ converges to $x$ in the norm topology of $X$.

**Theorem 2.4** (Theorem 4 in [12]). Let us consider a sequence of absolutely continuous functions $x_k(\cdot)$ from an interval $I$ of $\mathbb{R}$ to a finite-dimensional space $X$ satisfying

(a) $\forall t \in I$: $(x_k(t))_k$ is a relatively compact subset of $X$;

(b) there exists a positive function $c(\cdot) \in L^1_X(I)$ such that, for almost all $t \in I$, $\|\dot{x}_k(t)\| \leq c(t)$.

Then there exists a subsequence (again denoted by) $(x_k(\cdot))_k$ converging to an absolutely continuous function $x(\cdot)$ from $I$ to $X$ in the sense that

(i) $(x_k(\cdot))_k$ converges uniformly to $x(\cdot)$ over compact subsets of $I$;

(ii) $(x_k(\cdot))_k$ converges weakly to $x(\cdot)$ in $L^1_X(I)$.

We first need to recall some notation and definitions that will be used throughout the paper. Let $U$ be an open subset of $\mathcal{H}$ and $f: U \rightarrow (\infty, +\infty]$ a lower semicontinuous function.

The proximal subdifferential of $f(x)$, of $f$ at $x$ (see [13]) is defined by $\xi \in \partial^p f(x)$ iff there exist positive numbers $\sigma$ and $\gamma$ such that the following inequality is satisfied

$$f(y) - f(x) + \sigma \|y - x\|^2 \geq \langle \xi, y - x \rangle \quad \forall y \in x + \gamma \mathcal{B}_H.$$

Let $x$ be a point in $S \subset H$. We recall (see [13]) that the proximal normal cone of $S$ at $x$ is defined by $N^p_S(x) := \partial^p \psi_S(x)$, where $\psi_S$ denotes the indicator function of $S$, i.e., $\psi_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N^p_S(x) = \{ \xi \in H : \exists \alpha > 0 \text{ s.t. } x \in \text{Proj}_S(x + \alpha \xi) \},$$

where,

$$\text{Proj}_S(u) := \{ y \in S : d(u, S) := \| u - y \| \}.$$ 

If $X$ is a Banach space and $f$ is defined on a subset of $X$, the Clarke subdifferential $\partial^C f(x)$, of $f$ at $x$ (see [14]) is the subset of $X'$ given by

$$\partial^C f(x) = \{ \xi \in X' : f^C(x; v) \geq \langle \xi, v \rangle, \forall v \in X \},$$

where,

$$f^C(x; v) = \limsup_{y \rightarrow x, t \rightarrow 0} \frac{f(y + tv) - f(y)}{t}.$$
Let $S$ be an nonempty closed subset of $H$ and $x \in S$. The following assertions hold.

1. $\partial^2 d(x, S) = N^C_2(x) \cap \overline{B}_H$.
2. Let $\rho \in (0, +\infty]$. If $S$ is uniformly $\rho$-prox-regular, then one has
   (i) for all $x \in H$ with $d(x, S) < \rho$, $\text{Proj}_S(x) \neq \emptyset$;
   (ii) the proximal subdifferential of $d(\cdot, S)$ coincides with its Clarke subdifferential at all points $x \in H$ satisfying $d(x, S) < \rho$.

As a consequence of (ii), we obtain that for uniformly $\rho$-prox-regular sets, the proximal normal cone to $S$ coincides with all the normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N^C_2(x) = N_2^C(x)$. In such a case, we put $N_2(S) := N^C_2(x) = N_2^C(x)$.

Now, we recall some preliminaries concerning set-valued mappings. Given $T > 0$, let $C : [0, T] \rightrightarrows H$ and $K : H \rightrightarrows H$ be two set-valued mappings. We say that $C$ is absolutely continuous provided that there exists an absolutely continuous nonnegative function $a : [0, T] \to \mathbb{R}_+$ with $a(0) = 0$ such that

$$|d(C(t)) - d(y, C(s))| \leq \|x - y\| + |a(t) - a(s)|$$

for all $x, y \in H$ and all $t, s \in [0, T]$.

We will say that $K$ is Hausdorff-continuous (resp. Lipschitz with coefficient $\lambda > 0$), if for any $x \in H$, one has

$$\lim_{x' \to x} \mathcal{H}(K(x), K(x')) = 0$$

(resp. if for any $x, x' \in H$, one has

$$\mathcal{H}(K(x), K(x')) \leq \lambda \|x - x'\|,$$

where, $\mathcal{H}$ denotes the Hausdorff distance defined by

$$\mathcal{H}(A, B) = \sup_{a \in A} \sup_{b \in B} \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)).$$

The following result is the important theorem on the existence of measurable selection for measurable set-valued mappings (see Theorem III.6 in [16]).

**Theorem 2.5.** Let $X$ be a separable metric space, $(T, \Sigma)$ a measurable space and $\Gamma$ a multifunction from $T$ to complete nonempty subsets of $X$. If for each open set $U$ in $X$, $\Gamma^{-1}(U) = \{ t \in T : \Gamma(t) \cap U \neq \emptyset \}$ belongs to $\Sigma$, then $\Gamma$ admits a measurable selection.

We close this section with the following theorem in [17], which provides important closedness property of the subdifferential of the distance function associated with a set-valued mapping.

**Theorem 2.6.** Let $\rho \in (0, +\infty]$, $\Omega$ be an open subset in $H$, and $K : \Omega \rightrightarrows H$ be a Hausdorff-continuous set-valued mapping. Assume that $K(z)$ is uniformly $\rho$-prox-regular for all $z \in \Omega$. Then for a given $0 < \delta < \rho$, the following holds:

"for any $z \in \Omega$, $x_0 \in K(z)$ and $x_0 \in K(z) + (\rho - \delta)\overline{B}_H$, $x_n \to x$, $z_n \to z$ with $z_n \in K(x_n)$ not necessarily in $K(z_n)$ and $\xi_n \in \partial d(x_n, K(z_n))$ with $\xi_n \to \xi$ one has $\xi \in \partial d(x, K(z))$".

Here $\to^w$ means the weak convergence in $H$.

**Remark 2.1.** As a direct consequence of this theorem, we have for every $\rho \in (0, +\infty]$, for a given $0 < \delta < \rho$, and for every set-valued mapping $K : \Omega \rightrightarrows H$ with uniformly $\rho$-prox-regular values, the set-valued mapping $(z, x) \mapsto \partial d(x, K(z))$ is upper semicontinuous from $[z, x] \in \Omega \times H : x \in K(z) + (\rho - \delta)$ to $H$ endowed with the weak topology, which is equivalent to the upper semicontinuity of the function $(z, x) \mapsto \delta^*(\partial d(x, K(z)), p)$ on $[z, x] \in \Omega \times H : x \in K(z) + (\rho - \delta)$, for any $p \in H$. 


3. Main result

Our existence result is stated in a finite-dimensional space $E$ under the following assumptions.

(H1) For each $x \in E$, $K(x)$ is a nonempty closed subset in $E$ and uniformly $\rho$-prox-regular for some fixed $\rho \in (0, +\infty]$;

(H2) $K$ is Lipschitz with coefficient $\lambda > 0$;

(H3) $I = \sup_{x \in E} |K(x)| < +\infty$.

The proof of our main theorem uses a result of the coincidence between the solution sets of a constrained and an unconstrained first-order differential inclusion proved in [8,9] (see also Proposition 1.1 in [18]), the selection theorem proved in [19] and the Kakutani fixed point theorem for set-valued mappings. We begin by recalling them.

Theorem 3.1. Let $T_0, T$ be positive real numbers with $0 \leq T_0 < T$. For each $t \in [T_0, T]$, let $C(t)$ be a nonempty closed subset of a separable Hilbert space $H$. We will assume that

(i) the sets $C(t)$ are uniformly $\rho$-prox-regular for some fixed $\rho \in (0, +\infty]$;

(ii) $C(t)$ varies in an absolutely continuous way, that is, there exists an absolutely continuous nonnegative function $\zeta : [0, T] \to \mathbb{R}$ such that

$$|d(x, C(t)) - d(y, C(s))| \leq \|x - y\| + |\zeta(t) - \zeta(s)|$$

for all $x, y \in H$ and $s, t \in [T_0, T]$.

Then an absolutely continuous mapping $u(\cdot)$ is a solution of the constrained differential inclusion

$$\begin{cases}
\dot{u}(t) = -N_{C(t)}(u(t)) & \text{a.e. } t \in [T_0, T]; \\
u(T_0) = v_0 \in C(T_0)
\end{cases}$$

(implicitly subject to the constraints $u(t) \in C(t)$ for all $t \in [T_0, T]$) if and only if it is a solution of the unconstrained differential inclusion

$$\begin{cases}
\dot{u}(t) = -|\zeta(t)|\partial_d(u(t), C(t)), & \text{a.e. } t \in [T_0, T]; \\
u(0) = v_0 \in C(T_0).
\end{cases}$$

For the proof of our theorem we will also need the following theorem which is a direct consequence of Theorem 2.1 in [19].

Theorem 3.2 (See Theorem 2.1 in [19]). Let $E$ be a finite-dimensional space and let $M : [0, T] \times E \times E \to E$ be a closed valued set-valued mapping satisfying the following hypotheses.

(i) $M$ is $\mathcal{L}([0, T]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable;

(ii) for almost every $t \in [0, T]$, at each $(x, y) \in E \times E$ such that $M(t, x, y)$ is convex, $M(t, \ldots)$ is upper semicontinuous, and whenever $M(t, x, y)$ is not convex, $M(t, \ldots)$ is lower semicontinuous on some neighborhood of $(x, y)$;

(iii) there exists a Caratheodory function $\zeta : [0, T] \times E \times E \to \mathbb{R}_+$, which is integrably bounded and such that $M(t, x, y) \cap \overline{B}_E(0, \zeta(t, x, y)) \neq \emptyset$ for all $(t, x, y) \in [0, T] \times E \times E$.

Then for any $\varepsilon > 0$ and any compact set $K \subset AC_E([0, T])$, there is a nonempty closed convex valued set-valued mapping $\Phi : K \rightrightarrows L^1_E([0, T])$, which has a strongly weakly sequentially closed graph such that for any $u \in K$ and $\phi \in \Phi(u)$, one has

$$\phi(t) \in M(t, u(t), \dot{u}(t));$$

$$\|\phi(t)\| \leq \zeta(t, u(t), \dot{u}(t)) + \varepsilon,$$

for almost every $t \in [0, T]$.

Now we are able to prove our main result.

Theorem 3.3. Let $E$ be a finite-dimensional space, $K : E \rightrightarrows E$ be a set-valued mapping satisfying assumptions (H1), (H2) and (H3). Let $T > 0$, and let $F : [0, T] \times E \times E \rightrightarrows E$ be a convex closed valued set-valued mapping, Lebesgue measurable on $[0, T]$ and upper semicontinuous on $E \times E$ and such that

$$F(t, x, y) \subset (m_1(t) + p_1(t)\|x\| + q_1(t)\|y\|)\overline{B}_E$$

for all $(t, x, y) \in [0, T] \times E \times E$, for some nonnegative functions $m_1, p_1, q_1 \in L^1_{\mathbb{R}}([0, T])$. Let $H : [0, T] \times E \times E \rightrightarrows E$ be an other set-valued mapping satisfying the following assumptions.

(i) $H$ is $\mathcal{L}([0, T]) \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$-measurable;

(ii) for almost every $t \in [0, T]$, at each $(x, y) \in E \times E$ such that $H(t, x, y)$ is convex, $H(t, \ldots)$ is upper semicontinuous, and whenever $H(t, x, y)$ is not convex, $H(t, \ldots)$ is lower semicontinuous on some neighborhood of $(x, y)$;
(iii) there are nonnegative functions $m_2, p_2, q_2 \in L_1^∞([0, T])$ such that

$$H(t, x, y) \subset (m_2(t) + p_2(t)\|x\| + q_2(t)\|y\|)\mathbb{B}_E$$

for all $(t, x, y) \in [0, T] \times E \times E$.

Let $u_0 \in E$ and $v_0 = K(u_0)$ and suppose that for each $t \in [0, T]$

$$\|v_0\| + (\lambda l + 2M(t))T \leq l$$

(3.1)

where

$$M(t) := \frac{1}{2} + (m_1(t) + m_2(t)) + (p_1(t) + p_2(t))(\|u_0\| + \lambda) + (q_1(t) + q_2(t))l.$$

Then, there exist two Lipschitz mappings $u, v : [0, T] \to E$ such that

$$\begin{align*}
\mu(t) &= u_0 + \int_0^t v(s)ds, \quad \forall t \in [0, T]; \\
-\dot{v}(t) &\in N_K(u(t))(v(t)) + F(t, u(t), v(t)) + H(t, u(t), v(t)), \quad \text{a.e. } t \in [0, T]; \\
v(t) &\in K(u(t)), \quad \text{a.e. } t \in [0, T]; \\
u(0) &= u_0; \quad v(0) = v_0.
\end{align*}$$

In other words, there is a Lipschitz solution $u : [0, T] \to E$ to the Cauchy problem ($\mathcal{P}_{\mathcal{F}, H}$).

**Proof.** Step 1. Put $l = [0, T]$.

$$M_i(\cdot) = m_i(\cdot) + p_i(\cdot)(\|u_0\| + \lambda) + q_i(\cdot)l + \frac{1}{4} (i = 1, 2),$$

and observe that $M(\cdot) = M_1(\cdot) + M_2(\cdot)$. Let us consider the sets

$$\begin{align*}
\mathcal{X} &= \left\{ u \in AC_E(l) : u(t) = u_0 + \int_0^t \dot{u}(s)ds, \quad \forall t \in I \text{ and } \|\dot{u}(t)\| \leq l, \text{ a.e. on } I \right\}, \\
\mathcal{U} &= \left\{ v \in AC_E(l) : v(t) = v_0 + \int_0^t \dot{v}(s)ds, \quad \forall t \in I \text{ and } \|\dot{v}(t)\| \leq \lambda l + 2M(t), \text{ a.e. on } I \right\}, \\
\mathcal{K} &= \{ h \in L_1^∞(l) : \|h(t)\| \leq M(t), \text{ a.e. on } I \}.
\end{align*}$$

It is clear that $\mathcal{X}$ is a convex $w(L_1^∞(l), L_1^∞(l))$ compact subset of $L_1^∞(l)$, and that $\mathcal{X}$ and $\mathcal{U}$ are convex compact sets in $AC_E(l)$. Indeed, let $(u_n)$ be a sequence in $\mathcal{X}$. Then

$$u_n(t) = u_0 + \int_0^t \dot{u}_n(s)ds, \quad \forall t \in I \text{ and } \|\dot{u}_n(s)\| \leq l, \text{ a.e. on } I.$$

Therefore, for all $t \in I$

$$\|u_n(t)\| \leq \|u_0\| + \lambda,$$

this shows that $(u_n(t))$ is a bounded sequence in the finite-dimensional space $E$, then it is relatively compact in $E$, and since $\|\dot{u}_n(t)\| \leq l$ a.e. on $I$, we conclude, by Theorem 2.4, that there exists a subsequence (again denoted by $(u_n)$ converging to an absolutely continuous mapping $u$ and $\dot{u}_n$ converges uniformly to $\dot{u}$ and $\dot{u}_n$ converges weakly to $\dot{u}$ in $L_1^∞(l)$). Using Lebesgue’s Theorem, we obtain

$$u(t) = \lim_{n \to \infty} u_n(t) = u_0 + \lim_{n \to \infty} \int_0^t \dot{u}_n(s)ds = u_0 + \int_0^t \dot{u}(s)ds, \quad \forall t \in I,$$

and, by Theorem 2.2, $\|\dot{u}(t)\| \leq l$ a.e. on $I$ since the set $\{ y \in L_1^∞(l) : \|y(t)\| \leq l \}$ is convex and strongly closed in $L_1^∞(l)$ and hence it is weakly closed in $L_1^∞(l)$. Consequently $u \in \mathcal{X}$, that is, $\mathcal{X}$ is compact in $AC_E(l)$.

Using the same arguments above, we obtain that $\mathcal{U}$ is compact in $AC_E(l)$.

By Theorem 3.2, there are nonempty closed convex valued set-valued mappings $\Phi_i : \mathcal{X} \to L_1^∞(l) (i = 1, 2)$, which have strongly weakly sequentially closed graphs, such that for any $u \in \mathcal{X}$ and $\varphi \in \Phi_1(u)$ for a.e. $t \in I$, we have

$$\varphi(t) \in F(t, u(t), \dot{u}(t)) \quad \text{and} \quad \|\varphi(t)\| \leq m_1(t) + p_1(t)\|u(t)\| + q_1(t)\|\dot{u}(t)\| + \frac{1}{4}$$

and for any $u \in \mathcal{X}$ and $\psi \in \Phi_2(u)$ for a.e. $t \in I$, we have

$$\psi(t) \in H(t, u(t), \dot{u}(t)) \quad \text{and} \quad \|\psi(t)\| \leq m_2(t) + p_2(t)\|u(t)\| + q_2(t)\|\dot{u}(t)\| + \frac{1}{4}.$$
Since \( u \in X \), we have \( \| \dot{u}(t) \| \leq l \) and

\[
\| u(t) \| = \| u_0 + \int_0^t \dot{u}(s)ds \| \leq \| u_0 \| + IT,
\]

hence

\[
\| \varphi(t) \| \leq m_1(t) + p_1(t)(\| u_0 \| + IT) + q_1(t)l + \frac{1}{4} = M_1(t)
\]

and

\[
\| \psi(t) \| \leq m_2(t) + p_2(t)(\| u_0 \| + IT) + q_2(t)l + \frac{1}{4} = M_2(t).
\]

These last inequalities show that \( \Phi_i (i = 1, 2) \) has \( w(L^1_{\leq}, L^\infty_{\leq}) \) compact values in \( L^1_{\leq}(I) \). Let us consider the set-valued mapping \( \Phi : X \rightrightarrows L^1_{\leq}(I) \) where \( \Phi (\cdot) = \Phi_1 (\cdot) + \Phi_2 (\cdot) \). It is clear that \( \Phi \) is strongly measurable. By extracting subsequences (that we do not relabel), we can conclude that \( (\xi_n) \) converges to some mapping \( \xi \) in \( M_1(I)\mathcal{B}_{\leq} \). Consequently, \( \Phi \) is strongly measurable. In view of the existence theorem of measurable selection, there is a measurable selection \( \xi \) of \( \Phi \). This shows that \( \Phi \) is strongly measurable.

Step 2. Let us define the set-valued mapping \( \Psi : X \rightrightarrows C(I) \) by

\[
\Psi(f) = \left\{ u \in AC_I : \exists v \in U, u(t) = u_0 + \int_0^t v(s)ds, \forall t \in I \right\}.
\]

Observe that, for all \( f \in X \), the set-valued mapping \( K \circ f \) is Lipschitz with coefficient \( l\lambda \). Indeed, for all \( t, t' \in I \)

\[
\mathcal{H}((K \circ f)(t), (K \circ f)(t')) = \mathcal{H}(f, f(t')) \leq (\lambda l + M(t))d(f(t), f(t')) - h(t) \text{ a.e.},
\]

\[
\leq \lambda \mathcal{M}(f(t) - f(t')) \leq \lambda \mathcal{M}(f(t)) \leq \lambda \mathcal{M}(f(t)) - h(t).
\]

Hence, for all \( t \in I \), \( \mathcal{M}(f(t)) \) is a nonempty closed convex subset of \( E \). Furthermore, for any \( f \in X \) and \( h \in \Phi(f) \), the mapping \( t \rightarrow (\lambda l + M(t))d(f(t), f(t')) + h(t) \) is measurable, in view of the measurability of \( \mathcal{M}(f(t)) \).

\textbf{Theorem 2.5.} There is a measurable mapping \( \gamma : I \rightarrow E \) such that \( \gamma(t) = -(\lambda l + M(t))d(f(t), f(t')) - h(t) \) for all \( t \in I \).

Let \( v : I \rightarrow E \) be the mapping defined by \( v(t) = v_0 + \int_0^t \gamma(s)ds \); then, \( \dot{v}(t) = -(\lambda l + M(t))d(f(t), f(t')) - h(t) \) a.e., and

\[
\| \dot{v}(t) \| \leq \| \dot{v}(t) \| \leq \| v_0 \| + \int_0^t \| \dot{v}(s) \|ds \leq \| v_0 \| + \int_0^t (\lambda l + M(s))ds \leq \| v_0 \| + \int_0^t l - \| v_0 \| \leq T, \quad T = l,
\]

we have \( \| v_0 \| \leq l \) since \( v_0 \in K(u_0) \). Then, \( u \in X \). We conclude that \( \Psi \) maps \( X \) into itself. On the other hand, it is clear that for any \( f \in X \), \( \Psi(f) \) is a convex subset of \( X \) since \( d(f(t), f(t')) \) and \( \Psi(f) \) are convex.

Let us prove now that, for any \( f \in X \), \( \Psi(f) \) is a compact subset of \( X \). Since \( X \) is compact, it is sufficient to prove that \( \Psi(f) \) is closed. Let \( (u_n) \) be a sequence in \( \Psi(f) \) converging uniformly to \( u \in X \), that is, there is a sequence \( (v_n) \subset U \) such that for each \( n \in N \), \( u_n(t) = u_0 + \int_0^t v_n(s)ds \), for all \( t \in I \),

\[
\dot{v}_n(t) = -(\lambda l + M(t))d(f(t), f(t')) - h_n(t) \quad \text{a.e.},
\]

(3.2)
and \((h_n) \subset \Phi(f)\). As \(v_n(t) = v_0 + \int_0^t \dot{v}_n(s) \, ds\) for all \(t \in I\) and \(\|\dot{v}_n(t)\| \leq \lambda I + 2M(t)\) a.e., we have by (3.1)
\[
\|v_n(t)\| \leq \|v_0\| + \int_0^t \|\dot{v}_n(s)\| \, ds \leq \|v_0\| + (\lambda I + 2M(t))T \leq I,
\]
that is, the sequence \((v_n(t))\) is relatively compact in the finite-dimensional space \(E\). By Theorem 2.4, we conclude that there exists a subsequence \((v_{n_k}(t))\) (again denoted by \((v_n(t))\)) converging uniformly to an absolutely continuous mapping \(v(\cdot)\) and that \((\dot{v}_n(\cdot))\) converges weakly to \(\dot{v}(\cdot)\) in \(L^1_E(I)\). As \(U\) is compact, it is clear that \(v(\cdot) \in U\). In particular, we have for all \(t \in I\)
\[
u(t) = \lim_{n \to +\infty} u_n(t) = u_0 + \lim_{n \to +\infty} \int_0^t v_n(s) \, ds = u_0 + \int_0^t v(s) \, ds.
\]
As \((h_n) \subset \Phi(f)\) and as \(\Phi(f) = w(L^1_E(I), L^\infty_E(I))\) compact, by extracting a subsequence, we may conclude that \((h_n)\) weakly converges in \(L^1_E(I)\) to some mapping \(h \in \Phi(f)\). Consequently, \((\dot{v}_n + h_n)w(L^1_E(I), L^\infty_E(I))\) converges to \(\dot{v} + h \in L^1_E(I)\). Banach–Mazur’s Lemma (Theorem 2.3) ensures that for a.e. \(t \in I\)
\[
\dot{v}(t) + h(t) \in \bigcap_n \overline{co}[\dot{v}_k(t) + h_k(t) : k \geq n].
\]
Fix such \(t \in I\) and any \(\mu \in H\), then relation (3.2) and Theorem 2.1 give
\[
\langle \dot{v}(t) + h(t), \mu \rangle \leq \delta^*[-(\lambda I + M(t))\partial d(\dot{f}(t), K(f(t))), \mu],
\]
and since \(\partial d(\dot{f}(t), K(f(t)))\) is a closed convex set, we obtain by Lemma 2.1
\[
d(\dot{v}(t) + h(t), -(\lambda I + M(t))\partial d(\dot{f}(t), K(f(t)))) \leq 0.
\]
Then we obtain
\[
\dot{v}(t) + h(t) \in -(\lambda I + M(t))\partial d(\dot{f}(t), K(f(t))) \quad \text{a.e.}
\]
This shows that \(\Psi(f)\) is a compact subset of \(X\).

Finally, let us prove that the set-valued mapping \(\Psi\) is upper semicontinuous for \(X\) equipped with the topology of uniform convergence, or equivalently the graph of \(\Psi\), \(gph(\Psi) = \{(x, y) \in X \times X : y \in \Psi(x)\}\) is closed. Let \((f_n, u_n)\) be a sequence in \(gph(\Psi)\) converging to \((f, u) \in X \times X\). For each \(n \in \mathbb{N}\), there is \(v_n \in U\) such that \(u_n(t) = u_0 + \int_0^t v_n(s) \, ds\) for all \(t \in I\) and \(v_n(t) \in (-\lambda I + M(t))\partial d(\dot{f_n}(t), K(f_n(t))) + h_n(t) \quad \text{a.e.},\)
\[
\dot{v}_n(t) + h_n(t) \in -(\lambda I + M(t))\partial d(\dot{f}_n(t), K(f_n(t))) \quad \text{a.e.},
\]
and \((\dot{v}_n + h_n)w(L^1_E(I), L^\infty_E(I))\) converges to \(\dot{v} + h \in L^1_E(I)\). Banach–Mazur’s Lemma (see Theorem 2.3) ensures that for a.e. \(t \in I\)
\[
\dot{v}(t) + h(t) \in \bigcap_n \overline{co}[\dot{v}_k(t) + h_k(t) : k \geq n].
\]
Fix such \(t \in I\) and any \(\mu \in E\), then the last relation and Theorem 2.1 give
\[
\langle \dot{v}(t) + h(t), \mu \rangle \leq \lim_n \sup \delta^*[-(\lambda I + M(t))\partial d(\dot{f}_n(t), K(f_n(t))), \mu] \leq \delta^*[-(\lambda I + M(t))\partial d(\dot{f}(t), K(f(t))), \mu],
\]
where the second inequality follows from Remark 2.1 by using Lemma 2.1. As the set \(\partial d(\dot{f}(t), K(f(t)))\) is closed and convex, we obtain
\[
\dot{v}(t) + h(t) \in -(\lambda I + M(t))\partial d(\dot{f}(t), K(f(t))).
\]
This says that $\Psi$ is upper semicontinuous. An application of the Kakutani theorem gives a fixed point of $\Psi$ that is, there is $f \in \mathcal{X}$ such that $f \in \Psi(f)$, which means that there is $v \in \mathcal{U}$ and $h \in \Phi(f)$ such that

$$f(t) = u_0 + \int_0^t v(s)ds \quad \text{for all } t \in I$$

and

$$-\dot{v}(t) \in (\lambda I + M(t))\partial d(v(t), K(f(t))) + h(t) \quad \text{a.e.,}$$

with $v(0) = v_0$. Let for all $t \in I$

$$z(t) = \int_0^t h(s)ds, \quad D(t) = K(f(t)) + z(t) \quad \text{and} \quad w(t) = v(t) + z(t).$$

(3.3)

It is clear that $D$ is uniformly $\rho$-prox-regular since $Kof$ is uniformly $\rho$-prox-regular. Furthermore, $D$ is absolutely continuous. Indeed, for all $x$, $y \in E$ and all $t$, $s \in I$

$$|d(x, D(t)) - d(y, D(s))| = |d(x, K(f(t)) + z(t)) - d(x, K(f(s)) + z(s))|$$

$$= |d(x - z(t), K(f(t))) - d(y - z(s), K(f(s)))|$$

$$\leq \|x - z(t) - y - z(s)\| + \lambda |t - s|$$

$$\leq \|x - y\| + \|z(t) - z(s)\| + \lambda |t - s|$$

$$= \|x - y\| + \int_s^t h(\tau)d\tau + \lambda |t - s|$$

$$\leq \|x - y\| + \int_s^t (\|h(\tau)\| + \lambda) d\tau$$

$$\leq \|x - y\| + \int_s^t (M(\tau) + \lambda) d\tau.$$ By setting, for all $t \in I$

$$a(t) = \lambda t \quad \text{and} \quad \zeta(t) = \int_0^t (M(s) + \lambda) ds = \int_0^t (M(s) + \dot{a}(s)) ds,$$

relation (3.3) can be rewritten as follows

$$-\dot{v}(t) + h(t) \in (\dot{a}(t) + M(t))\partial d(v(t), K(f(t))) \quad \text{a.e},$$

and by (3.4)

$$-\dot{w}(t) \in \dot{\zeta}(t)\partial d(w(t), D(t)) \quad \text{a.e},$$

with $w(0) = v_0$, that is, $w$ is a solution of the unconstrained differential inclusion

$$\begin{cases}
-\dot{w}(t) \in \dot{\zeta}(t)\partial d(w(t), D(t)) & \text{a.e. } t \in I; \\
w(0) = v_0 \in D(0).
\end{cases}$$

Consequently, by Theorem 3.1, we conclude that $w$ is a solution of the constrained differential inclusion

$$\begin{cases}
-\dot{w}(t) \in N_{D(t)}(w(t)), & \text{a.e. } t \in I; \\
w(t) \in D(t), & \text{a.e. } t \in I, \\
w(0) = v_0 \in D(0).
\end{cases}$$

in other words, $v$ is a solution of the differential inclusion

$$\begin{cases}
-\dot{v}(t) \in N_{K(f(t))}(v(t)) + h(t), & \text{a.e. } t \in I; \\
v(t) \in K(f(t)), & \text{a.e. } t \in I, \\
v(0) = v_0 \in K(u_0).
\end{cases}$$

Putting $u = f$ we obtain,

$$\begin{cases}
-\dot{u}(t) \in N_{K(u(t))}(u(t)) + h(t), & \text{a.e. } t \in I; \\
\dot{u}(t) \in K(u(t)), & \text{a.e. } t \in I, \\
u(0) = u_0, \quad \dot{u}(0) = v_0 \in K(u_0).
\end{cases}$$
As $h \in \Phi(u) = \Phi_1(u) + \Phi_2(u)$, there exists $h_i \in \Phi_i(u)$ ($i = 1, 2$) such that $h = h_1 + h_2$, $h_1(t) \in F(t, u(t), \dot{u}(t))$ and $h_2(t) \in H(t, u(t), \dot{u}(t))$, a.e. on $I$. Hence

\[
\begin{aligned}
-\ddot{u}(t) &\in N_{K(u(t))}(\dot{u}(t)) + F(t, u(t), \dot{u}(t)) + H(t, u(t), \dot{u}(t)), \; \text{a.e. } t \in I; \\
\dot{u}(t) &\in K(u(t)), \; \text{a.e. } t \in I; \\
u(0) = u_0, \quad \dot{u}(0) = v_0 \in K(u_0).
\end{aligned}
\]

with for almost all $t \in I$

\[
\|\ddot{u}(t)\| \leq \lambda_1 + 2M(t).
\]

This completes the proof of our theorem. □

**Remark 3.1.** Note that our main result above is new even when the sets $K(x)$ are convex.

**Acknowledgment**

The authors would like to thank Professor L. Thibault for his important comments and suggestions.

**References**


