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Counting linear extension majority cycles in partially ordered sets on up to 13 elements

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ABSTRACT

It is well known that the linear extension majority relation of a partially ordered set $(P, \leq_P)$ can contain cycles when at least 9 elements are present in P. Computer experiments have uncovered all posets with 9 elements containing such cycles and limited frequency estimates for linear extension majority cycles (or LEM cycles) in posets on up to 12 elements are available. In this contribution, we present an efficient approach which allows us to count and store all posets containing LEM cycles on up to 13 elements.

1. Preliminaries

A binary relation $\leq_P$ on a set P is called an *order relation* if it is reflexive ($x \leq_P x$), antisymmetric ($x \leq_P y$ and $y \leq_P x$ imply $x =_P y$) and transitive ($x \leq_P y$ and $y \leq_P z$ imply $x \leq_P z$). A *linear order relation* $\leq_P$ is an order relation in which every two elements are comparable ($x \leq_P y$ or $y \leq_P x$). If $x \leq_P y$ and $x \neq y$, we write $x <_P y$. If neither $x \leq_P y$ nor $x \geq_P y$, we say that x and y are *incomparable* and write $x \parallel_P y$. A couple $(P, \leq_P)$, where P is a set of objects and $\leq_P$ is an order relation on P, is called a partially ordered set or *poset* for short. The *size* of the poset $(P, \leq_P)$ is defined as the cardinality of P. In this paper, we will denote the size of $(P, \leq_P)$ as n and call a poset on n elements an n-element poset for short. A *chain* of a poset $(P, \leq_P)$ is a subset of P in which every two elements are comparable. Dually, an *antichain* of a poset $(P, \leq_P)$ is a subset of P in which every two elements are incomparable. The *width* of a poset $(P, \leq_P)$ is the size of the largest antichain of $(P, \leq_P)$. Dually, the *height* of a poset $(P, \leq_P)$ is the size of the largest chain of $(P, \leq_P)$. A poset $(P, \leq_P^\top)$ for which $y \leq_P^\top x$ if and only if $x \leq_P y$ for all $x, y \in P$ is called the *dual poset* of $(P, \leq_P)$. We say that a poset $(P, \leq_P)$ is a linear sum if there exist disjoint non-empty subsets P_1 and P_2 of P such that $P = P_1 \cup P_2$, and $x <_P y$ for any $x \in P_1$ and any $y \in P_2$.

The binary relation $\prec_P$, for which it holds that $(x, y) \in \prec_P$ if and only if $x <_P y$ and there exists no $z \in P$ such that $x <_P z <_P y$, is called the *covering relation* of $(P, \leq_P)$. A poset $(P, \leq_P)$ can be conveniently represented by a *covering graph* or so-called *Hasse diagram*, displaying the covering relation $\prec_P$. Note that $x <_P y$ if and only if there is a sequence of connected lines upwards from x to y.

A binary relation on a set P is called a *weak order relation* on P if it is reflexive, transitive and complete, *i.e.* every two elements are comparable. Moreover, if this relation is also antisymmetric, it is a *linear order relation*; in other words, a linear order relation is a weak order relation in which every two elements are comparable. Let Q be a set and R and S two binary

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relations on Q. If $R \subset S$, then (Q, S) is called an extension of (Q, R). A linear extension of a poset $(P, \leq_P)$ is an extension $(P, \leq_L)$ for which $\leq_L$ is a linear order relation. Let us denote the set of linear extensions of a poset $(P, \leq_P)$ as $\mathcal{E}(P)$ and its cardinality $|\mathcal{E}(P)|$ as $e(P)$.

The *mutual rank probability* $\mathrm{p}(x > y)$ of two elements x and y of a poset $(P, \leq_P)$ is defined as the probability that $x >_L y$ in a linear extension $(P, \leq_L)$ of $(P, \leq_P)$ that has been sampled uniformly at random from $\mathcal{E}(P)$. Stated differently, it is the number of linear extensions of $(P, \leq_P)$ in which $x >_L y$, divided by the number $e(P)$ of linear extensions of $(P, \leq_P)$. The mutual rank probability relation M_P is the $[0, 1]$-valued binary relation on P defined by $M_P[x, y] = \mathrm{p}(x > y)$. Note that M_P reciprocal, *i.e.* $\mathrm{p}(x > y) + \mathrm{p}(y > x) = 1$.

The *linear extension majority (LEM) relation* of a poset P is the binary relation $\succ_{\mathrm{LEM}}$ on P such that $x \succ_{\mathrm{LEM}} y$ if $\mathrm{p}(x > y) > \mathrm{p}(y > x)$. Since the mutual rank probability relation is reciprocal, it is equivalent to define $x \succ_{\mathrm{LEM}} y$ if $\mathrm{p}(x > y) > \frac{1}{2}$.

A downset or *ideal* of a poset $(P, \leq_P)$ is a subset $D \subseteq P$ such that $x \in D,\ y \in P$ and $y \leq_P x$ imply $y \in D$. Dually, an upset or *filter* of a poset $(P, \leq_P)$ is a subset $U \subseteq P$ such that $x \in U,\ y \in P$ and $x \leq_P y$ imply $y \in U$. Let us denote the set of all ideals of a poset $(P, \leq_P)$ as $\mathcal{I}(P)$. If we equip this set of ideals with the set inclusion $\subseteq$, a new poset $(\mathcal{I}(P), \subseteq)$ is obtained. Moreover, it is more than a poset: it is a distributive lattice [1]. The distributive lattice $(\mathcal{I}(P), \subseteq)$ is called the *lattice of ideals* of $(P, \leq_P)$ [2].

A *directed weighted graph* $G = (V, A, w)$ is defined as a triplet comprising a set V of vertices, a set A of arcs and a weight function $w : A \to \mathbb{R}$. Each arc $a \in A$ has a weight $w(a)$ attached to it. A *directed walk* in a graph $G = (V, A, w)$ from a vertex $v \in V$ to a vertex $w \in V$ is an alternating sequence of vertices and arcs $(v_0, a_1, v_1, a_2, \ldots, v_{l-1}, a_l, v_l)$ such that each a_i is the arc from v_{i-1} to v_i. The number of arcs l in the walk is called the *length* of the walk. A directed walk where $v_0 = v_l$ and each arc and each vertex aside from v_0 and v_l are unique is called a *directed cycle* of length l. In the remainder of this paper, the term walk will be used instead of directed walk, and the term cycle instead of directed cycle. If a walk starts and ends in the same vertex, we call it a *closed walk*.

2. Linear extension majority cycles

The linear extension majority relation $\succ_{\mathrm{LEM}}$ first appeared in 1968 in the work of Kislitsyn [3], and it was conjectured that $\succ_{\mathrm{LEM}}$ is transitive, and thus cannot contain cycles, *i.e.* subsets $\{x_1, x_2, \ldots, x_m\}$ of elements of P such that $x_1 \succ_{\mathrm{LEM}} x_2 \succ_{\mathrm{LEM}} \cdots \succ_{\mathrm{LEM}} x_m \succ_{\mathrm{LEM}} x_1$. However, in 1974 Fishburn [4] has shown that $\succ_{\mathrm{LEM}}$ can contain cycles, and thus is not transitive. These cycles are referred to as *LEM cycles* on m elements, or *m-cycles* for short. Since then, quite some attention has been given to LEM cycles in the literature. Examples of posets with LEM cycles in different contexts are given in [5–11], frequency estimates for LEM cycles have been reported in [12,13], and the occurrence of LEM cycles in certain subclasses of posets has been studied in [14,15].

Aside from the fact that the existence of LEM cycles is an intriguing phenomenon in its own right, a better understanding of LEM cycles might help in the ongoing quest to characterize the transitivity of mutual rank probabilities in posets [16,7,17,18]. Furthermore, Gehrlein and Fishburn [13] discuss an interesting application of LEM cycles in which incomplete information about a linear order $\leq_L$ on a set P is given in the form of a partial order $\leq_P$. Assuming the partial information is correct, they consider the problem of attempting to reconstruct the linear order $\leq_L$. The choice of such a linear order amounts to the selection of a single extension from the set of all linear extensions of the poset. This is a problem frequently encountered in real world situations, e.g. when a decision maker insists on obtaining a linear order on all objects instead of a partial order obtained by comparing the attribute vectors of the objects [19–26]. The approach Gehrlein and Fishburn suggest assesses a conditional probability p_i that the corresponding linear extension $(P, \leq_{L_i})$ of the poset represents $\leq_L$ given the partial information contained in $\leq_P$. Once these probabilities p_i are obtained, it is possible to compute the probability that $x <_L y$ as the sum of all probabilities p_i corresponding to a linear extension for which it holds that $x <_{L_i} y$.

One model of interest describing the manner in which the partial order $\leq_P$ is obtained from $\leq_L$ implies that all of these probabilities p_i equal $1/e(P)$. In this case, the probability that $x <_L y$ is identical to the mutual rank probability $\mathrm{p}(y > x)$ according to the partial order $\leq_P$. Moreover, the maximum likelihood estimator $\prec^*$ of $\leq_L$ defined by $x \prec^* y$ if $\mathrm{p}(y > x) > \mathrm{p}(x > y)$, is nothing else but the LEM relation, in the sense that $x \prec^* y$ if and only if $y \succ_{\mathrm{LEM}} x$. Since posets exist with LEM cycles, the maximum likelihood estimator $\prec^*$ can be intransitive. This notion of a maximum likelihood estimator of $\leq_L$ being quite appealing, it would be interesting to obtain some measure of the propensity of this technique to produce intransitive maximum likelihood estimators.

Gehrlein and Fishburn [11] conducted a computer search to find all non-isomorphic posets with LEM cycles for poset cardinalities $n \leq 9$ and showed that no cycles exist for $n \leq 8$. Moreover, exactly 5 non-isomorphic posets for $n = 9$ are found. In a later paper, Gehrlein [12] estimates the likelihood of LEM cycles up to $n = 12$ by generating random connected posets.

In this paper, we count and store all posets that contain LEM cycles for $n \leq 13$, using an algorithm developed by the present authors [22,27]. Furthermore, the so-called worst balanced posets are found for $n \leq 13$, and the smallest poset of height one is shown to have 11 elements.

In Section 3 some algorithmic details are outlined, and in Section 4 the results are presented.

Table 1
Number of posets for $n = 9, 10, \ldots, 16$.

n	Number of posets
9	183 231
10	2 567 284
11	46 749 427
12	1 104 891 746
13	33 823 827 452
14	1 338 193 159 771
15	68 275 077 901 156
16	4 483 130 665 195 087

3. Algorithm

Unless some direct method is invented to avoid explicit enumeration, counting all posets with LEM cycles requires at least all posets to be enumerated and their mutual rank probability relations to be computed.

Brinkmann and McKay [28] developed a very efficient method to construct pairwise non-isomorphic posets, which allows them to enumerate posets on up to 16 elements. As an illustration of the size of the problem, the number of non-isomorphic posets of sizes 9 to 16 are shown in Table 1.

In order to compute the mutual rank probability relation for a given poset, a naïve approach would consist of enumerating all linear extensions. This would e.g. imply that only for the antichain of 13 elements, already more than 62 billion linear extensions need to be enumerated, which is clearly undesirable. This computationally extremely intensive process combined with the number of posets that grows quickly for increasing n, would make the counting procedure out of reach for $n = 12$, let alone for $n = 13$.

However, as the present authors have shown in [27], a more direct way to compute the mutual rank probability relation using the lattice of ideals representation of a poset could save considerable computing time. This approach no longer necessitates the enumeration of all linear extensions, though requires additional memory for storing the lattice of ideals. Since we are precisely interested in generating small posets, the lattice of ideals of such posets nicely fits into memory of current computer architectures. The algorithm consists of two main parts. After the lattice of ideals of $(P, \leq_P)$ has been constructed, the first part traverses the lattice in a breadth-first as well as a depth-first manner as to attach counting information to each ideal. The second part subsequently derives all mutual rank probabilities from this counting information in one pass over the lattice. The time complexity of this algorithm contains as a factor the number of ideals of the poset, which can still be exponential in n. However, in posets of limited width, the number of ideals is much smaller than the number of linear extensions allowing the computation of the mutual rank probability relation in a fraction of the time that would be needed to enumerate all linear extensions and subsequently inferring the mutual rank probability relation.

We combined the poset generation algorithm of Brinkmann and McKay [28] and our algorithm [27] to compute the mutual rank probability relation for each poset enumerated. This approach enabled us to obtain exact counts for posets on up to 13 elements.

For each poset $(P, \leq_P)$ generated by the algorithm of Brinkmann and McKay, Algorithm 1 is executed. This algorithm checks whether a given poset $(P, \leq_P)$ contains a LEM cycle of length l where $l = 3, 4, \ldots, k$ with $k \leq n$. The result of the check is returned as an array of booleans in which the element at index l is **true** if $(P, \leq_P)$ contains a LEM cycle of length l and **false** in the negative case.

On the first line of Algorithm 1, the lattice of ideals $(\mathcal{I}(P), \subseteq)$ is constructed. The most efficient algorithm currently known for constructing $(\mathcal{I}(P), \subseteq)$ is the algorithm of Habib et al. [29], which has an optimal complexity up to a constant factor. In line number 2 the algorithm suggested by the present authors [27] for computing the mutual rank probability relation M_P for which $M_P[x, y] = \mathrm{p}(x > y)$ for each $x, y \in P$ is invoked.

Let us construct the rank probability graph $G = (V, E, w)$ in which the vertices are the elements of our poset $(P, \leq_P)$ and the arcs the couples $(x, y) \in P^2$ for which $M_P[x, y] > 1/2$. Furthermore, let us attribute a weight $M_P[x, y]$ to each arc (x, y). Clearly, a cycle in G of length l is a LEM cycle of length l.

Once the relation M_P is known, in order to know whether there is a closed walk in the rank probability graph from a vertex $x \in V$ to the same vertex x with l arcs, we calculate M_P^l, in which the matrix multiplication $M_P^{l-1} \times M_P$ is defined as the usual matrix multiplication where min is substituted for $\cdot$ and max for $+$. Remark that the first multiplication $M_P \times M_P$ assigned to M_P^2 takes place before the **for**-loop in line 3, since it is impossible for cycles of length 2 to occur due to the reciprocity of the mutual rank probability relation M_P. Each subsequent multiplication is executed inside the **for**-loop in line 6. If it holds that $M_P^l[x, x] > 1/2$ (line 9), there is a closed walk with l arcs. It is clear that, if there is no closed walk with l arcs, an l-cycle is impossible, so no further check is needed. If there is a closed walk of length l and $l < 6$, there is an l-cycle (line 10). Indeed, due to the fact that if an arc (x, y) is present in the rank probability graph, no arc (y, x) can be present, the smallest cycles that can occur have length 3. For $l = 6$, a closed walk of length 6 could arise from two cycles of length 3 sharing one common vertex. Therefore it is clear that for $l \geq 6$ a situation can occur in which a closed walk of length l is the composition of two or more cycles.

Algorithm 1 Checking whether a given poset $(P, \leq_P)$ contains a LEM cycle of length l where $l = 2, 3, \ldots, k$ with $k \leq n$

1: construct the lattice of ideals $(\mathcal{I}(P), \subseteq)$ of $(P, \leq_P)$
2: compute the mutual rank probability relation M_P using $(\mathcal{I}(P), \subseteq)$
3: $M_P^2 \leftarrow M_P \times M_P$
4: **for each** $l = 3, 4, \ldots, k$ **do**
5:     cycle[l] $\leftarrow$ **false**
6:     $M_P^l \leftarrow M_P^{l-1} \times M_P$
7:     $c \leftarrow 0$
8:     **for each** $j = 1, \ldots, n$ **do**
9:         **if** $M_P^l[j, j] > 1/2$ **then**
10:             **if** $l < 6$ **then**
11:                 cycle[l] $\leftarrow$ **true**
12:                 **break for**
13:             $c \leftarrow c + 1$
14:             elem[c] $\leftarrow j$
15:     **if** $c \geq l$ **then**
16:         **for each** $i = 1, 2, \ldots, c$ **do**
17:             **if** there is an l-cycle starting in elem[i] **then**
18:                 cycle[l] $\leftarrow$ **true**
19:                 **break for**
20: **return** cycle

Since we are searching for l-cycles, it is clear that, in order for such a cycle to occur, at least l elements should have a closed walk of length l (line 15). Finally, an explicit depth-first search should be done for each candidate element (line 17), *i.e.* each element x having $M_P^l[x, x] > 1/2$, to verify whether an l-cycle is present. For each such candidate, recursively all possible successor arcs which have not yet been visited are selected until exactly l arcs have been chosen. Subsequently, if the vertex at the last arc is the starting vertex, a cycle is detected.

The algorithm we implemented is a slight variant of the above one, in the sense that we search for all possible LEM cycles of length l instead of just returning whether P contains a LEM cycle of length l. Moreover, each poset in which a LEM cycle occurs is stored in a database for future reference.

4. Results

When generating posets on n elements, the generation algorithm of Brinkmann and McKay [28] will, for arbitrary $r, m \in \mathbb{N}, r < m$, generate all posets on $n - 4$ elements and number them in the order they occur, while only generating successors of those posets whose number equals r mod m. This option allowed us to split the generation process, and thus the counting procedure. For $n = 12$, we divided the generation process into 100 parts and for $n = 13$ into 1000 parts. Rescaled to a single 2.4 GHz processor the entire process for $n = 13$ would take around 4 computing years. Because of the fact that the number of posets for $n = 14$ is almost 40 times larger than the number of posets for $n = 13$, combined with the exponential behaviour in n of the number of ideals, it is not possible to obtain results for $n > 13$ in a feasible time frame with our approach, unless substantially more computing power is available. Moreover, due to the growing size of the lattice of ideals, the size of the memory also becomes a constraining factor for larger values of n.

In order to verify the correctness of the implementation of the algorithm, for posets on up to 9 elements all mutual rank probability relations were compared to the results of an independent implementation based on the Varol–Rotem algorithm [30] that generates all linear extensions of a given poset, and then deduces the mutual rank probability relation for each poset.

4.1. LEM cycles

In Table 2, the number of n-element posets with l-cycles is shown, while in Table 3 the relative number of n-element posets with l-cycles, multiplied by 10^4, is shown.

Clearly, the results in Table 3 provide additional support for the conjecture formulated by Gehrlein and Fishburn [11] that the likelihood of observing a random poset with a LEM cycle increases as n increases. The rate at which the likelihood increases, however, seems to decrease with increasing n. This is in accordance with the LEM cycle frequency estimates made by Gehrlein [12].

When using nonlinear optimization to fit a function $f(x) = a + b \cdot x + x \cdot \log(x)$ with two parameters $a, b \in \mathbb{R}$ to the total relative frequencies in Table 3, we found values $a = 5.2304$ and $b = -2.7572$ such that $f(x)$ explains 99.59% of the variance. Of course, using such a formula to estimate the incidence of n-elements with LEM cycles should be done with great caution, especially for larger n. We expect, however, that for n smaller than 50, a reasonable approximation can be obtained by using $f(x)$.

Table 2
Number of n-element posets with LEM cycles of length l, for $n = 9, 10, \ldots, 13$ and $l = 3, 4, \ldots, 8$.

$n\|l$	3	4	5	6	7	8	All
9	5	–	–	–	–	–	5
10	148	6	–	–	–	–	153
11	5 740	101	–	–	–	–	5 815
12	216 573	2 885	5	21	–	–	218 097
13	9 318 881	102 127	471	363	1	–	9 348 400

Table 3
Relative number of n-element posets with LEM cycles of length l, multiplied by 10^{4}, for $n = 9, 10, \ldots, 13$ and $l = 3, 4, \ldots, 8$.

$n\|l$	3	4	5	6	7	8	All
9	0.273	–	–	–	–	–	0.273
10	0.576	0.023	–	–	–	–	0.596
11	1.228	0.022	–	–	–	–	1.244
12	1.960	0.026	0.000	0.000	–	–	1.974
13	2.755	0.030	0.000	0.000	0.000	–	2.764

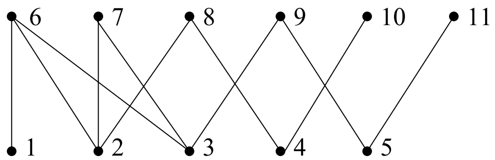

Fig. 1. The smallest poset with height one having a LEM cycle, where $p(9 > 7) = p(10 > 8) = 174660/349260$ and $p(8 > 9) = p(7 > 10) = 174790/349260$.

Table 4
Number of n-element posets of height 1 with LEM cycles of length l, for $n = 11, 12, 13$ and $l = 3, 4, \ldots, 7$.

$n\|l$	3	4	5	6	7	All
11	–	2	–	–	–	2
12	11	9	–	1	–	20
13	175	123	–	3	–	296

Table 5
Relative number of n-element posets with height 1 having LEM cycles of length l, multiplied by 10^4, for $n = 11, 12, 13$ and $l = 3, 4, \ldots, 7$.

$n\|l$	3	4	5	6	7	All
11	–	0.308	–	–	–	0.308
12	0.219	0.179	–	0.020	–	0.399
13	0.345	0.273	–	0.006	–	0.584

4.2. Other results

4.2.1. Posets of height 1

Ewacha et al. [15] have shown that posets with height 1 can have LEM cycles. Our results indicate that the smallest posets having this property have 11 elements. Actually, there are only two such 11-element posets: the poset depicted in Fig. 1 and its dual poset which have a cycle of length 4. The results of this counting operation for $n = 11, 12, 13$ are shown in Table 4, while the relative number of n-element posets with height 1 having LEM cycles are shown in Table 5. Remark that no 11-element poset with height 1 has cycles of length 3. An analogous observation can be made for 12 and 13-element posets with height 1: although cycles with length 6 occur, no poset has cycles of length 5. It is clear that the probability of encountering an n-element poset with a LEM cycle in a poset of height 1 is much lower than the probability in the whole space of n-element posets. Note, also that the relative number of posets with height 1 and with a LEM cycle seems to increase at a lower pace than the total number of n-element posets with LEM cycles.

4.2.2. Worst balanced posets

For a poset $(P, \leq_P)$, the balance constant $b(P)$ is defined as the maximum over all pairs $(x, y) \in P$ of $\min(p(x > y), p(y > x))$. Worst balanced n-element posets are n-element posets of which the balance constant is the smallest and which are not a linear sum of other posets. The importance of these worst balanced posets stems from a well-known conjecture made by Kislitsyn in 1968 [3], known as the 1/3-2/3-conjecture. It states that in any non-chain poset P one can always find a couple of elements $(x, y) \in P$ such that $1/3 \leq p(x > y) \leq 2/3$. Brightwell, Felsner and Trotter [31] proved that there

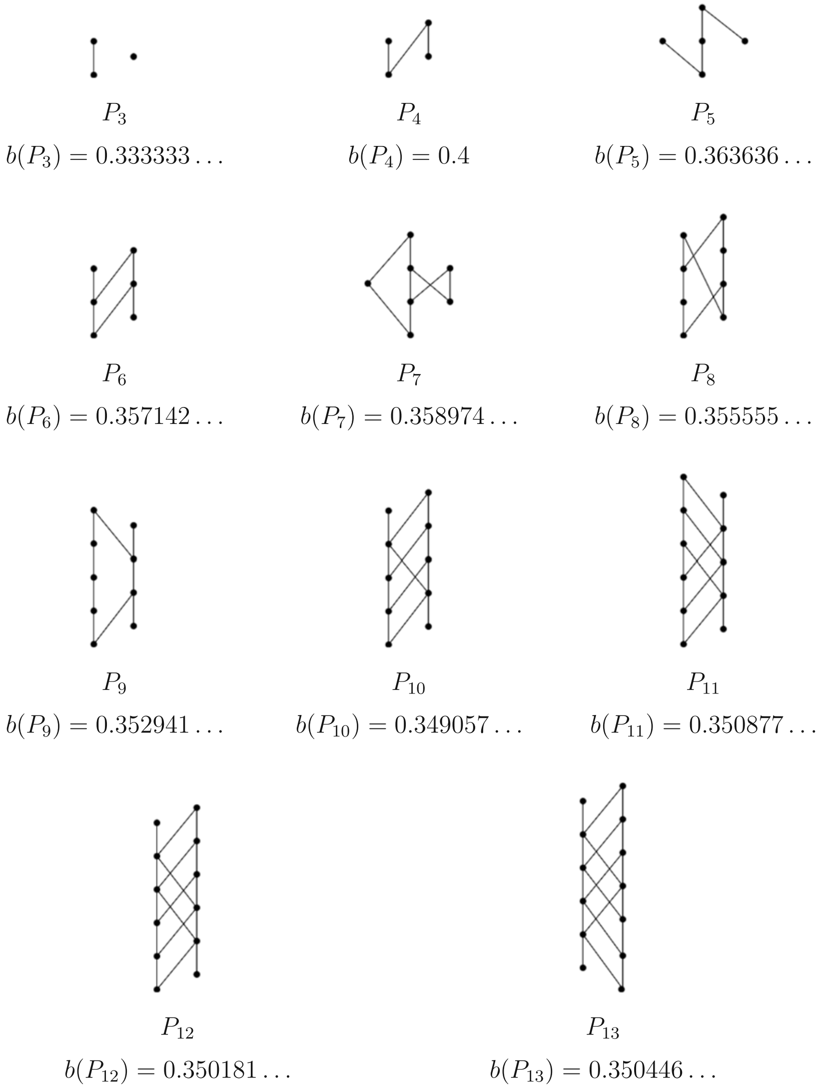

Fig. 2. All worst balanced posets for $n = 3, 4, \ldots, 13$.

exists a couple of elements $(x, y) \in P$ such that $(5-\sqrt{5})/10 \leq \mathrm{p}(x > y) \leq (5+\sqrt{5})/10$ and showed that for a class of countably infinite posets for which the notion of mutual rank probabilities makes sense, it is the best possible bound. A finite non-chain poset for which $b(P) < 1/3$ would be a counterexample to this conjecture. In this context it would be interesting to know the structure of posets of which the balance constant approaches $1/3$ as close as possible.

Brightwell [32] presented all worst balanced posets for n up to 8, and Peczarski [33] found the worst balanced posets for $n = 9, 10, 11$. Due to the regularity one can observe, Peczarski introduced a new class of badly balanced posets, which he called *ladders with broken rungs*. We obtained the worst balanced posets for $n = 12, 13$, and, as can be seen in Fig. 2, they indeed fall into Peczarski's class of ladders with broken rungs.

5. Conclusion

In this paper, an approach using the lattice of ideals representation of a poset enabled us to enumerate all posets on up to 13 elements containing LEM cycles. The complete list of all posets with LEM cycles can be obtained from the authors and can be helpful in the search for counterexamples to conjectures concerning LEM cycles. Furthermore, all posets of height 1 on up to 13 elements as well as all worst balanced posets on up to 13 elements are found.

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