Uniqueness of Best $L_1$-Approximations
from Periodic Spline Spaces

G. MEINARDUS

Fakultät für Mathematik und Informatik,
Universität Mannheim,
6800 Mannheim, West Germany

AND

G. NÜRNBERGER

Mathematisches Institut, Universität Erlangen-Nürnberg,
8520 Erlangen, West Germany

Communicated by Oved Shisha

Received December 8, 1987

It is shown that every periodic continuous function has a unique best $L_1$-approximation from a given periodic spline space, although these spaces are not weak Chebyshev in general. © 1989 Academic Press, Inc.

INTRODUCTION

Standard spaces for approximating periodic continuous functions $f: [a, b] \to \mathbb{R}$ (i.e., $f(a) = f(b)$) are spaces of periodic splines. We denote by $P_m(K_n)$ the $n$-dimensional space of periodic splines of order $m \geq 2$ with the set of knots $K_n = \{x_0, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

The space $P_m(K_n)$ is weak Chebyshev for odd $n$. We show that any periodic weak Chebyshev space $G$ (i.e., $g(a) = g(b)$ for all $g \in G$) with some additional property is necessarily of odd dimension. In particular, the space $P_m(K_n)$ is not weak Chebyshev for even $n$.

Our object is to prove a uniqueness result on best $L_1$-approximation by periodic splines. The standard spaces for which uniqueness of best $L_1$-approximations is known are all weak Chebyshev and have even a stronger property (A) (cf. Sommer [4] and Strauss [5]). We show that every periodic continuous function has a unique best $L_1$-approximation from $P_m(K_n)$, although $P_m(K_n)$ is not weak Chebyshev in general.
BEST $L_1$-APPROXIMATIONS

MAIN RESULTS

Let $C'[a, b]$ be the space of all $r$-times continuously differentiable real functions on the interval $[a, b]$. The space of polynomials of order at most $m$ is denoted by $\Pi_m$. Let a set of knots $K_n = \{x_0, ..., x_n\}$ with $n \geq 1$ and $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ be given. For $m \geq 2$ we call

$$P_m(K_n) = \{s \in C^{m-2}[a, b] : s\mid_{(x_{i-1}, x_i)} \in \Pi_m, i = 1, ..., n, s^{(j)}(a) = s^{(j)}(b), j = 0, 1, ..., m-2\}$$

the space of periodic splines of order $m$ with the set of knots $K_n$.

An $n$-dimensional subspace $G$ of $C[a, b]$ is called weak Chebyshev, if every function $g \in G$ has at most $n - 1$ sign changes; i.e., there do not exist points $a < t_1 < \cdots < t_{n+1} < b$ such that $g(t_i) g(t_{i+1}) < 0$, $i = 1, ..., n$.

We note that by induction on $m$ using Rolle's theorem it is not difficult to verify that every spline in $P_m(K_n)$ has at most $n - 1$ (respectively $n$) sign changes, if $n$ is odd (respectively even). In particular, the $n$-dimensional space $P_m(K_n)$ is weak Chebyshev for odd $n$ (compare also Schumaker [3]).

Our first result on weak Chebyshev spaces of periodic functions implies that this is not true for even $n$.

A subspace $G$ of $C[a, b]$ is called periodic, if $g(a) = g(b)$ for all $g \in G$. This definition differs from that given in Zielke [6, p. 20].

We next show that certain periodic weak Chebyshev spaces must have odd dimension. A similar result, which can be easily derived from Theorem 1, was proved in Zielke [6, p. 20].

THEOREM 1. Let $G$ be a periodic weak Chebyshev subspace of $C[a, b]$. If there exists a function $g_0 \in G$ with $g_0(a) \neq 0$, then the dimension of $G$ is odd.

Proof. Let $g_1, ..., g_n$ form a basis of the $n$-dimensional periodic weak Chebyshev subspace $G$ of $C[a, b]$. Since the functions $g_1, ..., g_n$ are linearly independent, there exist points $a \leq t_1 < \cdots < t_n \leq b$ such that the determinant $\det(g_j(t_i))_{i,j=1}^n$ is nonzero. Thus there exists a function $g \in G$ such that

$$g(t_i) = (-1)^i, \quad i = 1, ..., n.$$

We first consider the case $g(a) \neq 0$. Then we have $\sgn g(a) = -1$, since otherwise by considering the points $a, t_1, ..., t_n$ we see that $g$ has $n$ sign changes, contradicting the assumption that $G$ is weak Chebyshev. Since $g$ is a periodic function, we have $\sgn g(b) = -1$. For even $n$ we get $\sgn g(t_n) = 1$. By considering the points $t_1, ..., t_n, b$ we see that $g$ has again $n$ sign changes, contradicting our assumption. We now consider the case $g(a) = 0$. Let $g_0 \in G$ be the function with $g_0(a) \neq 0$. We may assume that
\( \text{sgn } g_0(a) = 1. \) For all \( \varepsilon > 0 \) we define the function \( g_\varepsilon \in G \) by \( g_\varepsilon = g + \varepsilon g_0. \) Then

\[ \text{sgn } g_\varepsilon(a) = 1 \]

and for sufficiently small \( \varepsilon > 0 \) we still have

\[ \text{sgn } g_\varepsilon(t_i) = (-1)^i, \quad i = 1, ..., n. \]

Hence \( g_\varepsilon \) has at least \( n \) sign changes, contradicting our assumption. This proves Theorem 1.

We note that Theorem 1 is no longer true, if we drop the assumption that there exists a function \( g_0 \in G \) with \( g_0(a) \neq 0. \) This can be seen by following the example.

Let points \( a = x_1 < x_2 < \cdots < x_{n+m} = b \) be given and \( G = \text{span} \{ B_1^m, ..., B_n^m \}, \) where for each \( i \in \{ 1, ..., n \} \) the function \( B_i^m \) is the \( B \)-spline of order \( m \) with support \( (x_i, x_{i+m}) \). Then it is well known that \( G \) is an \( n \)-dimensional periodic weak Chebyshev subspace of \( C[a, b] \) such that \( g(a) = 0 \) for all \( g \in G \) (see Schumaker [3]).

Following the proof of Theorem 1 we see that the next result holds.

**Corollary 2.** Let \( G \) be a periodic weak Chebyshev subspace of \( C[a, b] \) of dimension \( n \). If there exists a function \( g_0 \in G \) with \( g_0(a) \neq 0 \), then there is no function \( g \in G \) with \( n - 1 \) sign changes on \( [a, b] \) satisfying \( g(a) = 0 \).

We now investigate the uniqueness of best \( L_1 \)-approximations from \( P_m(K_n) \) for periodic functions in \( C[a, b] \).

For all functions \( h \in C[a, b] \) the \( L_1 \)-norm is defined by

\[ \| h \|_1 = \int_a^b |h(t)| \, dt. \quad (1) \]

Let a subspace \( G \) of \( C[a, b] \) and a function \( f \in C[a, b] \) be given. A function \( g_f \in G \) is called a **best \( L_1 \)-approximation** of \( f \) from \( G \), if

\[ \| f - g_f \|_1 = \inf_{g \in G} \| f - g \|_1. \quad (2) \]

In the following we prove a global unicity result for best \( L_1 \)-approximations from \( P_m(K_n) \). For doing this we need some notations and results.

Given a function \( f \in C[a, b] \) we set \( Z(f) = \{ t \in [a, b] : f(t) = 0 \} \). Moreover, if \( A \) is a subset of \( [a, b] \), then we denote by \( |A| \) the number of points in \( A \).

The first result on zeros of periodic splines can be found in Schumaker [3].
Lemma 3. Let a spline \( s \in P_m(K_n) \) be given such that \( |Z(s)| < \infty \). If \( n \) is even (respectively odd), then \( |Z(s) \cap [a, b]| \leq n \) (respectively \( |Z(s) \cap [a, b]| \leq n - 1 \)). Moreover, if \( |Z(s) \cap [a, b]| = n \), then \( s \) changes sign at the zeros in \((a, b)\).

The next result on weak Chebyshev spaces is well known (see, e.g., Deutsch, et al. [1]).

Lemma 4. Let an \( n \)-dimensional weak Chebyshev subspace of \( C[a, b] \) and points \( a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b \) be given, where \( 0 \leq r \leq n - 1 \). Then there exists a nontrivial function \( g \in \mathcal{G} \) such that

\[
(-1)^i g(t) > 0, \quad t \in [t_{i-1}, t_i], \ i = 1, \ldots, r + 1.
\]

The following characterization of best \( L_1 \)-approximations can be found in Rice [2].

Theorem 5. Let \( G \) be a subspace of \( C[a, b] \) and \( f \in C[a, b] \). The following statements hold:

(i) A function \( g_f \in G \) is a best \( L_1 \)-approximation of \( f \) if and only if for all \( g \in G \),

\[
\int_a^b g(t) \operatorname{sgn}(f(t) - g_f(t)) \, dt \leq \int_{Z(f - g_f)} |g(t)| \, dt.
\]

(ii) If \( g_1, g_2 \in G \) are best \( L_1 \)-approximations of \( f \), then

\[
(f(t) - g_1(t))(f(t) - g_2(t)) \geq 0, \quad t \in [a, b].
\]

We are now in position to prove the announced unicity result.

Theorem 6. Every periodic function in \( C[a, b] \) has a unique best \( L_1 \)-approximation from \( P_m(K_n) \).

Proof. Suppose that the claim is false. Then there exists a function \( f \in C[a, b] \) such that \( s_1 = 0 \) and \( s_0 \in P_m(K_n), s_0 \neq 0 \), are best \( L_1 \)-approximations of \( f \) from \( P_m(K_n) \). It follows from Theorem 5 that

\[
f(t)(f(t) - s_0(t)) \geq 0, \quad t \in [a, b].
\]

This implies that for all \( t \in [a, b] \),

\[
|f(t) - \frac{1}{2}s_0(t)| = \frac{1}{2}(f(t) - s_0(t)) + \frac{1}{2}f(t) = \frac{1}{2} |f(t) - s_0(t)| + \frac{1}{2} |f(t)|.
\]
Therefore, if \( f(t) - \frac{1}{2}s_0(t) = 0 \), then \( \frac{1}{2} | f(t) - s_0(t) | + \frac{1}{2} | f(t) | = 0 \) which implies that \( s_0(t) = 0 \). This shows that
\[
Z(f - \frac{1}{2}s_0) \subseteq Z(s_0). \tag{6}
\]

**Claim.** There exists a nontrivial function \( s \in P_m(K_n) \) such that
\[
(f(t) - \frac{1}{2}s_0(t)) s(t) \geq 0, \quad t \in [a, b],
\tag{7}
\]
and
\[
s(t) = 0, \quad t \in [c, d], \quad \text{if } f(t) - \frac{1}{2}s_0(t) = 0, t \in [c, d],
\tag{8}
\]
for all \( c < d \).

Suppose for the moment that the claim is true. Then it follows that
\[
\int_a^b s(t) \text{sgn}(f(t) - \frac{1}{2}s_0(t)) > 0 = \int_{Z(f - (1/2)s_0)} |s(t)| \, dt.
\]

Then by Theorem 5 the spline \( \frac{1}{2}s_0 \) is not a best \( L_1 \)-approximation of \( f \) from \( P_m(K_n) \) which is a contradiction, since \( s_1 = 0 \) and \( s_0 \in P_m(K_n) \) are best \( L_1 \)-approximations of \( f \). Therefore, it remains to prove the existence of the spline \( s \) as in the claim. It suffices to consider three cases.

**Case 1.** \( |Z(s_0)| < \infty \). We first consider the case when \( n \) is odd. It follows from Lemma 3 that \( |Z(s_0) \cap (a, b)| \leq n - 1 \). Then by (6) the function \( f - \frac{1}{2}s_0 \) has at most \( n - 1 \) sign changes. Thus there exists a sign \( \sigma \in \{-1, 1\} \) and points \( a = t_0 < t_1 < \cdots < t_r < t_{r+1} = b \), where \( 0 \leq r \leq n - 1 \), such that
\[
\sigma(-1)^i (f(t) - \frac{1}{2}s_0(t)) \geq 0, \quad t \in [t_{i-1}, t_i], \quad i = 1, \ldots, r. \tag{9}
\]

Since \( n \) is odd, \( P_m(K_n) \) is an \( n \)-dimensional weak Chebyshev space. Therefore, by Lemma 4 there exists a nontrivial function \( s \in P_m(K_n) \) such that
\[
\sigma(-1)^i s(t) \geq 0, \quad t \in [t_{i-1}, t_i], \quad i = 1, \ldots, r. \tag{10}
\]

Then it follows from (9) and (10) that the spline \( s \) has the desired property (7).

We now consider the case when \( n \) is even. We set \( K_{n-1} = \{y_0, \ldots, y_{n-1}\} \), where \( y_i = x_i \), \( i = 0, \ldots, n - 2 \), and \( y_{n-1} = b \). Since \( n - 1 \) is odd, \( P_m(K_{n-1}) \) is an \((n - 1)\)-dimensional weak Chebyshev space.

**Case 1.1.** \( f(a) - \frac{1}{2}s_0(a) = 0 \). It follows from (6) that \( s_0(a) = 0 \). Then by Lemma 3 we have \( |Z(s_0) \cap (a, b)| \leq n - 1 \). Therefore, by (6) the function \( f - \frac{1}{2}s_0 \) has at most \( n - 1 \) sign changes. If \( f - \frac{1}{2}s_0 \) has at most \( n - 2 \) sign changes, then analogously as in the case when \( n \) is even, there exists a
spline $s \in P_m(K_{n-1}) \subset P_m(K_n)$ satisfying (7). If $f - \frac{1}{2}s_0$ changes sign at $n - 1$ points $t_1 < \cdots < t_{n-1}$ in $(a, b)$, then by (6) we have $t_1, \ldots, t_{n-1} \in Z(s_0)$. Since $s_0(a) = 0$, it follows from Lemma 3 that $Z(s_0) \cap (a, b) = \{t_1, \ldots, t_{n-1}\}$ and $s_0$ changes sign at the points $t_1, \ldots, t_{n-1}$. Therefore, the spline $s = s_0$ or $s = -s_0$ satisfies (7).

**Case 1.2.** $f(a) - \frac{1}{2}s_0(a) \neq 0$. It follows from Lemma 3 that $|Z(s_0) \cap (a, b)| \leq n$. Then by (6) we have $|Z(f - \frac{1}{2}s_0) \cap (a, b)| \leq n$. Moreover, since $f(a) - \frac{1}{2}s_0(a) = f(b) - \frac{1}{2}s_0(b) \neq 0$, the function $f - \frac{1}{2}s_0$ has an even number of sign changes. If $f - \frac{1}{2}s_0$ has at most $n - 2$ sign changes, then analogously as in Case 1.1 there exists a spline $s \in P_m(K_{n-1}) \subset P_m(K_n)$ satisfying (7). If $f - \frac{1}{2}s_0$ changes sign at $n$ points $t_1 < \cdots < t_n$ in $(a, b)$, then by (6) we have $t_1, \ldots, t_n \in Z(s_0)$. Moreover, it follows from Lemma 3 that $Z(s_0) \cap (a, b) = \{t_1, \ldots, t_n\}$ and $s_0$ changes sign at the points $t_1, \ldots, t_n$. Therefore, the spline $s = s_0$ or $s = -s_0$ satisfies (7).

**Case 2.** $s_0(t) = 0$, $t \in [x_k, x_i] \cup [x_p, x_q]$, where $k < l < p < q$, and $|Z(s_0) \cap (x_l, x_p)| < \infty$. It is well known that

$$G = \{s_{[x_k, x_q]}: s \in P_m(K_n) \text{ and } s(t) = 0, t \in [x_k, x_l] \cup [x_p, x_q]\}$$

is a $(p - l - m + 1)$-dimensional weak Chebyshev space. Since $s_0_{[x_k, x_q]} \in G$ and $|Z(s_0) \cap (x_l, x_p)| < \infty$, we have $|Z(s_0) \cap (x_l, x_p)| \leq p - l - m$ (see Schumaker [3]). Then by (6) the function $f - \frac{1}{2}s_0$ has at most $p - 1 - m$ sign changes in $(x_l, x_p)$. Therefore, analogously as above there exists a spline $s \in G$ such that

$$(f(t) - \frac{1}{2}s_0(t)) s(t) \geq 0, \quad t \in [x_l, x_p].$$

We now extend $s$ to $[a, b]$ by defining

$$s(t) = 0, \quad t \in [a, x_k] \cup [x_q, b],$$

which implies that $s \in P_m(K_n)$ has the desired properties (7) and (8).

**Case 3.** $s_0(t) = 0$, $t \in [x_p, x_q]$, where $p < q$, and $|Z(s_0) \cap ([a, b] \setminus [x_p, x_q])| < \infty$. By identifying $b$ with $a$ we may consider the interval $[a, b)$ as a circle $T$ with circumference $b - a$. We set

$$y_i = x_{i+q}, \quad i = 0, \ldots, n - q,$$

and

$$y_i = x_{i-n+q}, \quad i = n-q+1, \ldots, n-q+p.$$
may be identified with the space

\[ H = \{ s \in C^{m-2}(T) : s|_{[y_{i-1}, y_i]} \in \Pi_m, i = 1, ..., n - q + p, \]

and \( s(t) = 0, t \in [y_{n-q+p}, y_0] \} \).

The space \( H \) may be considered as a usual spline space and it is well known that \( H \) is a \((n + p - q - m + 1)\)-dimensional weak Chebyshev space. Since \( s_0 \in H \) and \( |Z(s_0) \cap ([a, b] \setminus [x_p, x_q])| < \infty \), we have \( |Z(s_0) \cap ([a, b] \setminus [x_p, x_q])| \leq n + p - q - m \) (see Schumaker [3]). Then by (6) the function \( f - \frac{1}{2}s_0 \) has at most \( n + p - q - m \) sign changes in \([a, b] \setminus [x_p, x_q]\).

Therefore, analogously as above there exists a spline \( s \in H \) satisfying (7) and (8). This proves Theorem 6.

REFERENCES