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Faces of polytopes and Koszul algebras

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ABSTRACT

Let $\mathfrak{g}$ be a semisimple Lie algebra and V a $\mathfrak{g}$-semisimple module. In this paper, we study the category $\mathscr{G}$ of $\mathbb{Z}$-graded finite-dimensional representations of $\mathfrak{g} \ltimes V$. We show that the simple objects in this category are indexed by an interval-finite poset and produce a large class of truncated subcategories which are directed and highest weight. In the case when V is a finite-dimensional $\mathfrak{g}$-module, we construct a family of Koszul algebras which are indexed by certain subsets of the set of weights $\mathrm{wt}(V)$ of V. We use these Koszul algebras to construct an infinite-dimensional graded subalgebra $\mathbf{A}_\Psi^{\mathfrak{g}}$ of the locally finite part of the algebra of invariants $(\mathrm{End}_{\mathbb{C}}(\mathbf{V}) \otimes \mathrm{Sym}\, V)^{\mathfrak{g}}$, where $\mathbf{V}$ is the direct sum of all simple finite-dimensional $\mathfrak{g}$-modules. We prove that $\mathbf{A}_\Psi^{\mathfrak{g}}$ is Koszul of finite global dimension.

0. Introduction

In this paper, we study the category of finite-dimensional representations of the semi-direct product Lie algebras $\mathfrak{g} \ltimes V$, where $\mathfrak{g}$ is a complex semisimple Lie algebra and V is a $\mathfrak{g}$-semisimple representation. There are several well-known classical families of such Lie algebras, for instance, the co-minuscule parabolic subalgebras of a simple Lie algebra. However, our primary motivation comes from two sources: the first is the truncated current algebras $\mathfrak{g} \otimes \mathbb{C}[t]/t^r\mathbb{C}[t]$, where $\mathbb{C}[t]$ is the polynomial ring in an indeterminate t, and their multi-variable generalizations, and the second motivation is our interest in the undeformed infinitesimal Hecke algebras (see [8,14,11,13]). The representation theory of the truncated current algebras has interesting combinatorial properties and is connected with important families of representations of quantum affine algebras. It appears likely that the more general setup we consider will also have such connections [3].

In this paper, we are primarily interested in understanding the homological properties of the category of finite-dimensional representations of the semi-direct product $\mathfrak{g} \ltimes V$. To be more precise, we shall regard the Lie algebra as being graded by the non-negative integers $\mathbb{Z}_+$. We assume that $\mathfrak{g}$ lives in grade zero and that V is finite-dimensional and lives in grade one. The universal enveloping algebra of $\mathfrak{g} \ltimes V$ is also $\mathbb{Z}_+$-graded and in fact has elements of grade s for all $s \in \mathbb{Z}_+$. We work with the category of $\mathbb{Z}$-graded modules for $\mathfrak{g} \ltimes V$, where the morphisms are just the degree zero maps. In the special case when V is the adjoint representation of a simple Lie algebra, this category was previously studied in [4,5]. The authors of those papers made certain choices which were not completely understood or explained. In the current paper, we recover as a special case the results of [4,5] and, using the results of [12], provide a more conceptual explanation for the choices.

We now explain the overall organization of the paper. The main result, which is the construction of a family of Koszul algebras, is given in Section 1 and can be stated independently of the representation theory of $\mathfrak{g} \ltimes V$. Thus, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathrm{wt}\,(V)$ be the set of weights of V. We consider the convex polytope defined by $\mathrm{wt}\,(V)$. For each subset Ψ of $\mathrm{wt}\,(V)$ which lies on a face of this polytope, we define an $\mathbb{Z}$-graded $\mathfrak{g}$-module algebra $\mathbf{A}_\Psi$. We prove that the

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(infinite-dimensional) subalgebra $\mathbf{A}_\Psi^\mathfrak{g}$ of $\mathfrak{g}$-invariants is Koszul of global dimension at most equal to the sum of the dimension of eigenspaces of V corresponding to the elements of Ψ. The strategy to prove this result is the following. We first observe that the set Ψ defines in a natural way a partial ordering $\leq_\Psi$ on $\mathfrak{h}^*$. Associated to each element ν of wt (V), we can define a subalgebra $\mathbf{A}_\Psi(\leq_\Psi \ \nu)^\mathfrak{g}$ of $\mathbf{A}_\Psi^\mathfrak{g}$. We relate this algebra to the endomorphism algebra of the projective generator of a full subcategory (with finitely many simple objects) of $\mathbb{Z}$-graded finite-dimensional representations of $\mathfrak{g} \ltimes V$. This is done in Sections two through four where we analyze the homological properties of the category. To do this, we need to work in a bigger category which has enough projectives. The Koszul complex associated to the symmetric algebra of V provides a projective resolution of the simple objects and we can compute arbitrary extensions between the simple modules. This allows us, in Section 5, to use results of [2] to prove that $\mathbf{A}_\Psi(\leq_\Psi \ \nu)^\mathfrak{g}$ is Koszul. The final step is to prove that this implies that $\mathbf{A}_\Psi^\mathfrak{g}$ is Koszul. A natural question that arises from our work is the realization of Koszul duals of these algebras as module categories arising from Lie theory. The authors hope to pursue this question in the future.

1. The main results

We shall denote by $\mathbb{Z}$ (resp. $\mathbb{Z}_+$, $\mathbb{C}$) the set of integers (resp. non-negative integers, complex numbers).

1.1

Throughout this paper, we fix a complex semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We let R be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and fix a set of simple roots $\Delta = \{\alpha_i | i \in I\}$ of R. If $\{\omega_i | i \in I\}$ is a set of fundamental weights, we denote by P^+ the $\mathbb{Z}_+$-span of the fundamental weights, and by Q^+ the $\mathbb{Z}_+$-span of Δ.

1.2

Given $\mu \in P^+$, let $V(\mu)$ be the finite-dimensional simple $\mathfrak{g}$-module with highest weight μ. Define

$$\mathbf{V} := \bigoplus_{\mu \in P^+} V(\mu) \quad \text{and} \quad \mathbf{V}^\circledast := \bigoplus_{\mu \in P^+} V(\mu)^*.$$

The natural embedding $\mathbf{V}^\circledast \otimes \mathbf{V} \to \operatorname{End} \mathbf{V}$ (respectively $\mathbf{V}^\circledast \otimes \mathbf{V} \to \operatorname{End} \mathbf{V}^\circledast$) of $\mathfrak{g}$-modules is an anti-homomorphism (resp., a homomorphism). For each $\mu \in P^+$, this anti-homomorphism (resp., homomorphism) restricts to an isomorphism $V(\mu)^* \otimes V(\mu) \to (\operatorname{End} V(\mu))^{\mathrm{op}}$ (resp., $V(\mu)^* \otimes V(\mu) \to \operatorname{End} V(\mu)^*$). Under this isomorphism, the preimage of the identity element in $(\operatorname{End} V(\mu))^{\mathrm{op}}$ is the canonical $\mathfrak{g}$-invariant element $1_\mu \in V(\mu)^* \otimes V(\mu)$.

1.3

Suppose that A is an $\mathbb{Z}$-graded $\mathfrak{g}$-module algebra; i.e., A is an associative $\mathbb{Z}$-graded algebra which admits a compatible action of $\mathfrak{g}$:

$$A = \bigoplus_{k \in \mathbb{Z}} A[k], \qquad \mathfrak{g}.A[k] \subset A[k],\ k \in \mathbb{Z}.$$

The space

$$\mathbf{A} = A \otimes \mathbf{V}^\circledast \otimes \mathbf{V},$$

has a natural $\mathbb{Z}$-grading given by

$$\mathbf{A}[k] = A[k] \otimes \mathbf{V}^\circledast \otimes \mathbf{V}.$$

Moreover, it acquires the structure of an $\mathbb{Z}$-graded $\mathfrak{g}$-module algebra as follows: the multiplication is given by linearly extending the assignment

$$(a \otimes f \otimes v)(b \otimes g \otimes w) = g(v) ab \otimes f \otimes w,$$

while the $\mathfrak{g}$-module structure is just given by the usual action on tensor products of $\mathfrak{g}$-modules.

Abusing notation, we set

$$1_\mu = 1_A \otimes 1_\mu.$$

The next Lemma is immediate.

Lemma. *For $\mu, \nu \in P^+$, we have,*

$$1_\mu \mathbf{A} 1_\nu = A \otimes V(\mu)^* \otimes V(\nu). \quad \square$$

1.4

From now on, we fix a finite-dimensional $\mathfrak{g}$-module V and write

$$V = \oplus_{\mu\in\mathfrak{h}^*} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v \ \forall h \in \mathfrak{h}\}.$$

Set $\mathrm{wt}\,(V) = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$ and assume that $\mathrm{wt}\,(V) \neq \{0\}$. We shall also set

$$V^{\mathfrak{g}} = \{v \in V : xv = 0,\ \forall\, x \in \mathfrak{g}\}.$$

Suppose that $\Psi \subset \mathrm{wt}\,(V)$ is nonempty. Define a reflexive, transitive relation $\leq_\Psi$ on $\mathfrak{h}^*$ via

$$\mu \leq_\Psi \nu \iff \nu - \mu \in \mathbb{Z}_+\Psi,$$

and set

$$d_\Psi(\mu, \nu) := \min \left\{ \sum_{\beta\in\Psi} m_\beta : \nu - \mu = \sum_{\beta\in\Psi} m_\beta \beta,\ m_\beta \in \mathbb{Z}_+ \forall \beta \in \Psi \right\}.$$

1.5

For $\mu \leq_\Psi \nu \in P^+$, define

$$\mathbf{A}_\Psi(\nu, \mu) := 1_\nu \mathbf{A}[d_\Psi(\mu, \nu)] 1_\mu,$$

and given $F \subset P^+$, define

$$\mathbf{A}_\Psi(F) := \bigoplus_{\mu,\nu\in F, \mu\leq_\Psi \nu} \mathbf{A}_\Psi(\nu, \mu).$$

Note that $\mathbf{A}_\Psi(F)$ is a $\mathfrak{g}$-module.

Lemma. *Suppose that $\Psi \subset \mathrm{wt}\,(V)$ is the set of weights of V which lie on some proper face of the weight polytope of V, and let A be as above.*

(i) *$\mathbf{A}_\Psi(F)$ is a graded subalgebra of $\mathbf{A}_\Psi(G)$ for all $F \subset G \subset P^+$.*
(ii) *If $F \subset G \subset P^+$, then $\mathbf{A}_\Psi(F)^{\mathfrak{g}}$ is a graded subalgebra of $\mathbf{A}_\Psi(G)^{\mathfrak{g}}$.*

This result is clear once we show that d_Ψ satisfies the following in Proposition 5.2:

$$d_\Psi(\eta, \mu) + d_\Psi(\mu, \nu) = d_\Psi(\eta, \nu) \quad \forall\, \eta \leq_\Psi \mu \leq_\Psi \nu \in \mathfrak{h}^*. \tag{1.1}$$

1.6

Given $\mu, \nu \in P^+$, define $[\mu, \nu]_\Psi := (\leq_\Psi \nu) \cap (\mu \leq_\Psi)$, where

$$\leq_\Psi \nu := \{\eta \in P^+ : \eta \leq_\Psi \nu\}, \quad \text{and} \quad \mu \leq_\Psi := \{\eta \in P^+ : \mu \leq_\Psi \eta\}.$$

The main theorem of this paper is the following.

Theorem. *Suppose that Ψ is a subset of $\mathrm{wt}\,(V)$ which lies on some proper face of the weight polytope of V, and let A be the symmetric algebra of V. Given $\mu \leq_\Psi \nu \in P^+$, the algebras $\mathbf{A}_\Psi^{\mathfrak{g}}$, $\mathbf{A}_\Psi(\leq_\Psi \nu)^{\mathfrak{g}}$, $\mathbf{A}_\Psi(\mu \leq_\Psi)^{\mathfrak{g}}$, and $\mathbf{A}_\Psi([\mu, \nu]_\Psi)^{\mathfrak{g}}$ are Koszul with global dimension at most $N_\Psi := \sum_{\xi\in\Psi} \dim V_\xi$. Moreover, there exist $\mu \leq_\Psi \nu \in P^+$ such that the global dimension of all these algebras is exactly N_Ψ.*

2. The categories $\mathcal{G}$ and $\widehat{\mathcal{G}}$

In this section, we define and study the elementary properties of the category of $\mathbb{Z}$-graded finite-dimensional representations of $\mathfrak{g} \ltimes V$. We classify the simple objects in this category and describe their projective covers. We denote by $\mathbf{U}(\mathfrak{b})$ the universal enveloping algebra of a Lie algebra $\mathfrak{b}$. We use freely the notation established in Section 1.

2.1

We begin by working in the following general situation. Thus, we assume that $\mathfrak{a}$ is an $\mathbb{Z}_+$-graded complex Lie algebra,

$$\mathfrak{a} = \bigoplus_{n\in\mathbb{Z}_+} \mathfrak{a}_n,$$

with the additional assumptions that $\mathfrak{a}_0 = \mathfrak{g}$ and $\dim \mathfrak{a}_n < \infty$ for all $n \in \mathbb{Z}_+$. Set $\mathfrak{a}_+ = \bigoplus_{n>0} \mathfrak{a}_n$ and note that it is an $\mathbb{Z}_+$-graded ideal in $\mathfrak{a}$.

Set $R^+ = R \cap Q^+$ and fix a Chevalley basis $\{x_\alpha^\pm, h_i : \alpha \in R^+,\ i \in I\}$ of $\mathfrak{g}$, and

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}, \quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

2.2

Let $\mathcal{F}(\mathfrak{g})$ be the semisimple tensor category whose objects are finite-dimensional $\mathfrak{g}$-modules and morphisms are maps of $\mathfrak{g}$-modules. The simple objects of $\mathcal{F}(\mathfrak{g})$ are just the modules $V(\lambda)$, $\lambda \in P^+$. We shall need the fact that $V(\lambda)$ is generated by an element v_λ with relations:

$$\mathfrak{n}^+ v_\lambda = 0, \qquad h v_\lambda = \lambda(h) v_\lambda, \qquad (x_{\alpha_i}^-)^{\lambda(h_i)+1} v_\lambda = 0, \tag{2.1}$$

for all $h \in \mathfrak{h}$ and $i \in I$. Any object V of $\mathcal{F}(\mathfrak{g})$ is a weight module, i.e.,

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, \quad V_\mu = \{v \in V : hv = \mu(h)v,\ h \in \mathfrak{h}\},$$

and we set $\mathrm{wt}\,(V) = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$.

Let $\widehat{\mathcal{G}}$ be the category whose objects are $\mathbb{Z}$-graded $\mathfrak{a}$-modules V with finite-dimensional graded components $V[r]$; i.e.,

$$V = \bigoplus_{r \in \mathbb{Z}} V[r], \quad V[r] \in \mathrm{Ob}\, \mathcal{F}(\mathfrak{g})\ \forall r \in \mathbb{Z}.$$

The morphisms in $\widehat{\mathcal{G}}$ are $\mathfrak{a}$-module maps $f : V \to W$ such that $f(V[r]) \subset W[r]$ for all $r \in \mathbb{Z}$. For $V \in \widehat{\mathcal{G}}$, we have

$$V_\mu = \bigoplus_{r \in \mathbb{Z}} V_\mu[r], \quad V_\mu[r] = V_\mu \cap V[r].$$

Observe that the adjoint representation of $\mathfrak{a}$ is an object of $\widehat{\mathcal{G}}$.

Let $\mathcal{G}$ be the full subcategory of $\widehat{\mathcal{G}}$ given by

$$V \in \mathrm{Ob}\, \mathcal{G} \iff V \in \mathrm{Ob}\, \widehat{\mathcal{G}}, \quad \dim V < \infty.$$

Given $V \in \mathrm{Ob}\, \mathcal{F}(\mathfrak{g})$, let $\mathrm{ev}\,(V) \in \mathrm{Ob}\, \mathcal{G}$ be given by

$$\mathrm{ev}\,(V)[0] = V, \qquad \mathrm{ev}\,(V)[k] = 0 \quad \forall k > 0,$$

with the $\mathfrak{a}$-module structure defined by setting $\mathfrak{a}_+ \mathrm{ev}\,(V) = 0$ and leaving the $\mathfrak{g}$-action unchanged. Clearly, any $\mathfrak{g}$-module morphism extends to a morphism of graded $\mathfrak{a}$-modules, and we have a covariant functor $\mathrm{ev} : \mathcal{F}(\mathfrak{g}) \to \mathcal{G}$.

2.3

For $r \in \mathbb{Z}$, define a grading shift operator $\tau_r : \widehat{\mathcal{G}} \to \widehat{\mathcal{G}}$ via

$$\tau_r(V)[k] = V[k - r].$$

For $\lambda \in P^+$ and $r \in \mathbb{Z}$, set

$$V(\lambda, r) = \tau_r \mathrm{ev}\, V(\lambda), \qquad v_{\lambda,r} = \tau_r v_\lambda.$$

Proposition. *For $(\lambda, r) \in P^+ \times \mathbb{Z}$ we have that $V(\lambda, r)$ is a simple object in $\widehat{\mathcal{G}}$. Moreover if $(\mu, s) \in P^+ \times \mathbb{Z}$, then,*

$$V(\lambda, r) \cong_{\widehat{\mathcal{G}}} V(\mu, s) \iff \lambda = \mu, \quad \text{and} \quad r = s.$$

Conversely, if $M \in \widehat{\mathcal{G}}$ is simple then

$$M \cong V(\lambda, r), \quad \text{for some } (\lambda, r) \in P^+ \times \mathbb{Z}.$$

Proof. The first two statements are trivial. Suppose now that M is a simple object of $\widehat{\mathcal{G}}$ and that $r, s \in \mathbb{Z}$ are such that $M[r] \neq 0$ and $M[s] \neq 0$, and assume that $r > s$. Then the subspace $\oplus_{k \geq r} M[k] \neq 0$ and is a proper submodule of M, contradicting the fact that M is simple. Hence there must exist a unique $r \in \mathbb{Z}$ such that $M[r] \neq 0$. In particular, we have

$$M \cong_{\widehat{\mathcal{G}}} \tau_r \mathrm{ev}\, M[r],$$

and also that $M[r] \cong V(\lambda)$ for some $\lambda \in P^+$. This completes the proof of the Proposition. □

From now on, we set $\Lambda = P^+ \times \mathbb{Z}$ and observe that this set parametrizes the set of simple objects in $\widehat{\mathcal{G}}$ and $\mathcal{G}$. Given $V \in \mathrm{Ob}\, \widehat{\mathcal{G}}$, we set

$$[V : V(\lambda, r)] = \dim \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), V[r]).$$

2.4

We now turn our attention to constructing projective resolutions of the simple objects of $\widehat{\mathcal{G}}$. The algebra $\mathbf{U}(\mathfrak{a})$ has a $\mathbb{Z}_+$-grading inherited from the grading on $\mathfrak{a}$: namely, the grade of a monomial $a_1 \cdots a_k$, where $a_i \in \mathfrak{a}_{s_i}$, is $s_1 + \cdots + s_k$. The ideal $\mathbf{U}(\mathfrak{a}_+)$ is a graded ideal with finite-dimensional graded pieces. By the Poincare–Birkhoff–Witt Theorem, we have an isomorphism of $\mathbb{Z}_+$-graded vector spaces

$$\mathbf{U}(\mathfrak{a}) \cong \mathbf{U}(\mathfrak{a}_+) \otimes \mathbf{U}(\mathfrak{g}).$$

If we regard $\mathfrak{a}_+$ as a $\mathfrak{g}$-module via the adjoint action, then we have the following result, again by using the Poincare–Birkhoff–Witt Theorem.

Proposition. *As $\mathfrak{g}$-modules,*

$$\mathbf{U}(\mathfrak{a}_+)[k] \cong \bigoplus_{(r_1,\dots,r_k)\in\mathbb{Z}_+^k : \sum_{j=1}^k jr_j = k} \operatorname{Sym}^{r_1}(\mathfrak{a}_1) \otimes \cdots \otimes \operatorname{Sym}^{r_k}(\mathfrak{a}_k). \quad \square$$

Given $M \in \operatorname{Ob}\widehat{\mathcal{G}}$, we can regard $\mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{g})} M$ as a module for $\mathfrak{a}$ by left multiplication. Moreover, if we set

$$(\mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{g})} M)[k] = (\mathbf{U}(\mathfrak{a}_+) \otimes M)[k] = \bigoplus_{i\in\mathbb{Z}_+} (\mathbf{U}(\mathfrak{a}_+)[i] \otimes M[k-i])$$

then we have the following Corollary of Proposition 2.4.

Corollary. *For all $M \in \operatorname{Ob}\widehat{\mathcal{G}}$ with $M[s] = 0$ for $s \ll 0$, we have $\mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{g})} M \in \operatorname{Ob}\widehat{\mathcal{G}}$.* $\square$

2.5

For $(\lambda, r) \in \Lambda$, set

$$P(\lambda, r) = \mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda, r) \in \operatorname{Ob}\widehat{\mathcal{G}}, \qquad p_{\lambda,r} = 1 \otimes v_{\lambda,r}.$$

Proposition. (i) *For $(\lambda, r) \in \Lambda$, we have that $P(\lambda, r)$ is generated as an $\mathbb{Z}$-graded $\mathfrak{a}$-module by $p_{\lambda,r}$ with defining relations*

$$\mathfrak{n}^+ p_{\lambda,r} = 0, \qquad h p_{\lambda,r} = \lambda(h) p_{\lambda,r}, \qquad (x_{\alpha_i}^-)^{\lambda(h_i)+1} p_{\lambda,r} = 0, \tag{2.2}$$

for all $h \in \mathfrak{h}$ and $i \in I$. In particular, we have that if $V \in \operatorname{Ob}\widehat{\mathcal{G}}$, then

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) \cong \operatorname{Hom}_{\mathfrak{g}}(V(\lambda), V[r]).$$

(ii) *$P(\lambda, r)$ is the projective cover of its unique irreducible quotient $V(\lambda, r)$ in $\widehat{\mathcal{G}}$.*
(iii) *Let $K(\lambda, r)$ be the kernel of the morphism $P(\lambda, r) \to V(\lambda, r)$ which maps $p_{\lambda,r} \to v_{\lambda,r}$. Then*

$$K(\lambda, r) = \mathbf{U}(\mathfrak{a})(\mathfrak{a}_+ \otimes V(\lambda, r)),$$

and hence

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(K(\lambda, r), V(\mu, s)) \neq 0 \implies \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{a}_{s-r} \otimes V(\lambda), V(\mu)) \neq 0.$$

Proof. It is clear that the element $p_{\lambda,r}$ generates $P(\lambda, r)$ as a $\mathfrak{a}$-module. Moreover, since $v_{\lambda,r}$ satisfies relations (2.1), it follows that $p_{\lambda,r}$ satisfies (2.2). The fact that they are the defining relations is immediate from the Poincare–Birkhoff–Witt Theorem. It is now easily seen that the map

$$\varphi \to \varphi|_{1\otimes V(\lambda,r)}$$

gives an isomorphism $\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) \cong \operatorname{Hom}_{\mathfrak{g}}(V(\lambda), V[r])$. Since $\mathbf{U}(\mathfrak{a}_+)[0] = \mathbb{C}$, we see that $\dim P(\lambda, r)_\lambda[r] = 1$, and hence $P(\lambda, r)$ has a unique maximal graded submodule with corresponding quotient $V(\lambda, r)$. The fact that $P(\lambda, r)$ is projective is standard. Suppose that there exists a projective module $P \in \operatorname{Ob}\widehat{\mathcal{G}}$ and a surjective morphism $\psi : P \to V(\lambda, r)$; and choose $p \in P_\lambda[r]$ such that $\psi(p) = v_{\lambda,r}$. The induced morphism $\widetilde{\psi} : P \to P(\lambda, r)$ must satisfy $\widetilde{\psi}(p) = p_{\lambda,r}$ and, hence, is surjective, proving that $P(\lambda, r)$ is the projective cover. This also implies that $V(\lambda, r)$ is the quotient of $P(\lambda, r)$ by imposing the additional relation $\mathfrak{a}_+ p_{\lambda,r} = 0$. This proves that $K(\lambda, r)$ is generated as a $\mathfrak{a}_+$-module by $\mathfrak{a}_+ \otimes V(\lambda, r)$. Hence if $\varphi \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(K(\lambda, r), V(\mu, s)) \neq 0$, then $\varphi(\mathfrak{a}_+[s-r] \otimes V(\lambda, r)) \neq 0$, and the proof of the Proposition is complete. $\square$

The following is obvious.

Corollary. *Suppose that $M \in \widehat{\mathcal{G}}$ is such that $M[s] = 0$ for all $s \ll 0$. Then M is a quotient of a projective object $\mathbb{P}(M) \in \widehat{\mathcal{G}}$.* $\square$

2.6

Motivated by the preceding Proposition, we define a partial order on Λ as follows. Say that (μ, s) *covers* (λ, r) if $s > r$ and $\mu - \lambda \in \mathrm{wt}\,\mathfrak{a}_{s-r}$. Define $\preccurlyeq$ to be the transitive and reflexive closures of this relation. In particular, if $(\lambda, r) \preccurlyeq (\mu, r)$, then $\lambda = \mu$. It is easily checked that $\preccurlyeq$ is a partial order on Λ.

Lemma. *Let $(\lambda, r), (\mu, s) \in \Lambda$. Then,*

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(V(\lambda, r), V(\mu, s)) \neq 0 \implies (\mu, s) \text{ covers } (\lambda, r).$$

Proof. Using Proposition 2.5(ii), we see that

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(V(\lambda, r), V(\mu, s)) \cong \mathrm{Hom}_{\widehat{\mathcal{G}}}(K(\lambda, r), V(\mu, s)).$$

The Lemma follows from Proposition 2.5(iii). □

2.7

We can now produce a projective resolution of the simple objects of $\widehat{\mathcal{G}}$. The resolution is not minimal, and, in fact, it is unclear how to produce a minimal resolution. However, as we shall see, it is adequate to compute extensions between the simple objects.

For $(\lambda, r) \in \Lambda$ and $j \in \mathbb{Z}_+$, define

$$P_j(\lambda, r) := \mathbb{P}(\wedge^j(\mathfrak{a}_+) \otimes V(\lambda, r)) = \mathbf{U}(\mathfrak{a}_+) \otimes \wedge^j(\mathfrak{a}_+) \otimes V(\lambda, r) \in \widehat{\mathcal{G}}.$$

In particular, notice that

$$P_j(\lambda, r)[k] = 0, \quad k < r + j, \qquad P_0(\lambda, r) = V(\lambda, r).$$

For $j \geq 0$, define linear maps $d_j : P_j(\lambda, r) \to P_{j-1}(\lambda, r)$, (where we understand that $P_{-1}(\lambda, r) = V(\lambda, r)$) by

$$d_0 : P(\lambda, r) \to V(\lambda, r), \quad d_0(u \otimes v) = u.v,$$

for $u \in \mathbf{U}(\mathfrak{a}_+)$ and $v \in V(\lambda, r)$ and

$$d_j = D \otimes \mathrm{id}_{V(\lambda, r)}, \quad j > 0,$$

where D is the Koszul differential on the Chevalley–Eilenberg complex [6] for $\mathfrak{a}_+$.

Proposition. (i) *If $j > 0$ and $[P_j(\lambda, r) : V(\mu, s)] \neq 0$, then $(\lambda, r) \prec (\mu, s)$.*
(ii) *The following is a projective resolution of $V(\lambda, r)$ in $\widehat{\mathcal{G}}$:*

$$\cdots \xrightarrow{d_3} P_2(\lambda, r) \xrightarrow{d_2} P_1(\lambda, r) \xrightarrow{d_1} P(\lambda, r) \xrightarrow{d_0} V(\lambda, r) \longrightarrow 0.$$

Proof. If $j > 0$, then $P_j(\lambda, r)[r] = 0$ and so $[P_j(\lambda, r) : V(\lambda, r)] = 0$. The rest of the proof follows by an argument similar to the one in Proposition 2.5. Note that $P_j(\lambda, r)$ is projective in $\widehat{\mathcal{G}}$ for all $j \in \mathbb{Z}_+$ by Corollary 2.5. It is straightforward to check that d_j is an $\mathfrak{a}$-module map, and hence a morphism in $\widehat{\mathcal{G}}$ for all $j \in \mathbb{Z}_+$. Finally, since $d_j = D \otimes \mathrm{id}_{V(\lambda,r)}$ for all j, it follows that the sequence is exact in $\widehat{\mathcal{G}}$. □

2.8

For $s \in \mathbb{Z}$, let $\widehat{\mathcal{G}}_{\leq s}$ be the full subcategory of $\widehat{\mathcal{G}}$ satisfying

$$V \in \mathrm{Ob}\,\widehat{\mathcal{G}} \implies V[k] = 0, \quad k > s.$$

The subcategory $\mathcal{G}_{\leq s}$ is defined similarly. Notice that

$$V \in \mathrm{Ob}\,\widehat{\mathcal{G}}, \qquad V[k] = 0, \quad k \ll 0 \implies V_{\leq r} \in \mathcal{G}_{\leq r}, \; r \in \mathbb{Z}.$$

For $V \in \mathrm{Ob}\,\widehat{\mathcal{G}}$, define

$$V_{>s} = \bigoplus_{k>s} V[k], \qquad V_{\leq s} = V/V_{>s},$$

and note that $V_{\leq s} \in \mathrm{Ob}\,\widehat{\mathcal{G}}_{\leq s}$. If $f \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(V, W)$ and $s \in \mathbb{Z}$, there is a natural morphism

$$f_{\leq s} \in \mathrm{Hom}_{\widehat{\mathcal{G}}_{\leq s}}(V_{\leq s}, W_{\leq s}), \qquad v + V_{>s} \xmapsto{f_{\leq s}} f(v) + W_{>s}.$$

The assignment $V \mapsto V_{\leq s}$ and $f \mapsto f_{\leq s}$ defines a full, exact, and essentially surjective functor : $\widehat{\mathcal{G}} \to \widehat{\mathcal{G}}_{\leq s}$ for each $s \in \mathbb{Z}_+$. The following is immediate from Propositions 2.4 and 2.5.

Lemma. *For $(\lambda, r) \in \Lambda$ and $s \in \mathbb{Z}$, we have $P(\lambda, r)_{\le s} = 0$ if $s < r$. If $s \ge r$, then $P(\lambda, r)_{\le s}$ is the projective cover in $\mathcal{G}_{\le s}$ of $V(\lambda, r)$ and $P(\lambda, r)[s] \neq 0$ if $\mathfrak{a}_+ \neq 0$.* □

2.9

We end this section with the following result, which shows that it is necessary to work in $\widehat{\mathcal{G}}$ rather than $\mathcal{G}$.

Lemma. *$\mathcal{G}$ has projective objects if and only if $\mathfrak{a}_+ = 0$.*

Proof. First suppose that $\mathfrak{a}_+ = 0$ and $\mathfrak{a} = \mathfrak{g}$ is semisimple. Then $\mathcal{G}$ is a semisimple category and all Ext^1-groups vanish. In particular, every object in $\mathcal{G}$ is projective.

Suppose that $\mathfrak{a}_+ \neq 0$ and let $P \in \mathcal{G}$ be a non-zero projective object in $\mathcal{G}$. Since P is finite-dimensional, we may assume without loss of generality that P is indecomposable and maps onto $V(\lambda, r)$ for some $(\lambda, r) \in \Lambda$. For any $s \in \mathbb{Z}$ such that $P \in \mathcal{G}_{\le s}$, we see from Lemma 2.8 that there exists a surjective map from $P \to P(\lambda, r)_{\le s}$. Suppose now that s is such that $P[s-1] \neq 0$ but $P[s] = 0$. Then we would have that $P(\lambda, r)_{\le s}[s] = 0$, which contradicts Lemma 2.8. □

3. Truncated categories

In this section, we study certain Serre subcategories of $\widehat{\mathcal{G}}$, and prove that they are directed categories with finitely many simple objects.

3.1

Given $\Gamma \subset \Lambda$, let $\widehat{\mathcal{G}}[\Gamma]$ be the full subcategory of $\widehat{\mathcal{G}}$ consisting of all M such that

$$M \in \operatorname{Ob}\widehat{\mathcal{G}}, \quad [M : V(\lambda, r)] > 0 \implies (\lambda, r) \in \Gamma.$$

The subcategories $\mathcal{G}[\Gamma]$ are defined in the obvious way. Observe that if $(\lambda, r) \in \Gamma$, then $V(\lambda, r) \in \widehat{\mathcal{G}}[\Gamma]$, and we have the following trivial result.

Lemma. *The isomorphism classes of simple objects of $\widehat{\mathcal{G}}[\Gamma]$ are indexed by Γ.* □

3.2

For $V \in \widehat{\mathcal{G}}$, set

$$V_\Gamma^+ := \{v \in V[r]_\lambda : (\lambda, r) \in \Gamma,\ \mathfrak{n}^+ v = 0\},$$
$$V_\Gamma := \mathbf{U}(\mathfrak{g})V_\Gamma^+ \qquad V^\Gamma := V/V_{\Lambda\setminus\Gamma}.$$

It is clear that V_Γ and V^Γ are $\mathbb{Z}$-graded $\mathfrak{g}$-modules, and that they are finite-dimensional if Γ is a finite set. If $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(V, W)$ then $f(V_\Gamma^+) \subset W_\Gamma^+$, and hence the restriction f_Γ of f to V_Γ is an element of $\operatorname{Hom}_{\mathfrak{g}}(V_\Gamma, W_\Gamma)$. Moreover, since $f(V_{\Lambda\setminus\Gamma}) \subset W_{\Lambda\setminus\Gamma}$, we also have a natural induced map of $\mathfrak{g}$-modules $f^\Gamma : V^\Gamma \to W^\Gamma$. It is not true in general that V_Γ and V^Γ are in $\widehat{\mathcal{G}}[\Gamma]$. However, in the case when V_Γ and W_Γ (resp. V^Γ and W^Γ) are $\mathfrak{a}$-submodules, f_Γ (resp. f^Γ) is a morphism in $\widehat{\mathcal{G}}[\Gamma]$.

The following is the first step in determining a sufficient condition for this to be true. Set

$$\Lambda(V) = \{(\lambda, r) \in \Lambda : V_\lambda[r] \neq 0\}.$$

Proposition. *Suppose $V \in \widehat{\mathcal{G}}$ and $\Gamma \subset \Lambda$. If V_Γ is not an $\mathfrak{a}$-submodule of V, then there exist $(\nu, s) \in \Lambda(V) \setminus \Gamma$ and $(\lambda, r) \in \Gamma \cap \Lambda(V)$ such that (ν, s) covers (λ, r).*

Proof. Since V_Γ is $\mathbb{Z}$-graded and generated as a $\mathfrak{g}$-module by V_Γ^+, we may assume without loss of generality that there exists $a \in \mathfrak{a}_k$ and $v \in V_\Gamma^+ \cap V[r]_\lambda$ with $a.v \notin V_\Gamma$ for some $k \in \mathbb{Z}_+$, $r \in \mathbb{Z}$, $\lambda \in P^+$. Let U be a $\mathfrak{g}$-module complement of V_Γ in V. Then, the projection of av onto U is non-zero and so there exists $\nu \in P^+$ such that the composition of $\mathfrak{g}$-module maps,

$$\mathfrak{a}_k \otimes V(\lambda, r) \to V \twoheadrightarrow U \twoheadrightarrow U[r+k] \twoheadrightarrow V(\nu, r+k)$$

is non-zero. Call this nonzero composite map ξ. Now $\mathfrak{a}_k.V(\lambda, r) \neq 0$, so one can show that no nonzero maximal vector $v_\lambda \in V(\lambda, r)_\lambda$ (i.e., a weight vector killed by $\mathfrak{n}^+$) is killed by all of $\mathfrak{a}_k$. Since $\mathfrak{a}_k \otimes \mathbb{C}v_\lambda$ is a $\mathbf{U}(\mathfrak{b}^+)$-submodule of $\mathfrak{a}_k \otimes V(\lambda, r)$, $\xi(\mathfrak{a}_k \otimes \mathbb{C}v_\lambda)$ is a nonzero $\mathbf{U}(\mathfrak{b}^+)$-submodule of $V(\nu, r+k)$. Since $\xi(\mathfrak{a}_k \otimes \mathbb{C}v_\lambda)$ is finite-dimensional, it must contain a maximal weight vector in $V(\nu, r+k)$. In particular, $v_\nu \in \xi(\mathfrak{a}_k \otimes \mathbb{C}v_\lambda)$ where $\mathbb{C}v_\nu = V(\nu, r+k)_\nu$. Hence $v_\nu \in \mathfrak{a}_k.v_\lambda \subset V[r+k]$, so $\nu - \lambda \in \operatorname{wt}(\mathfrak{a}_k)$, and we conclude that $(\nu, r+k)$ covers (λ, r). □

3.3

A subset Γ of Λ is said to be *interval-closed* if

$$(\lambda, r) \preccurlyeq (\nu, p) \preccurlyeq (\mu, s), \quad (\lambda, r), (\mu, s) \in \Gamma \implies (\nu, p) \in \Gamma.$$

Proposition. *Suppose Γ is a finite interval-closed subset of Λ. Let $V \in \operatorname{Ob}\widehat{\mathcal{G}}$.*

(i) *Assume that for any $(\lambda, r) \in \Lambda(V) \setminus \Gamma$ there exists $(\mu, s) \in \Gamma$ with $(\lambda, r) \prec (\mu, s)$. Then $V_\Gamma \in \operatorname{Ob}\widehat{\mathcal{G}}[\Gamma]$. Furthermore, if U is a submodule of V, then*

$$U_\Gamma, (V/U)_\Gamma \in \operatorname{Ob}\widehat{\mathcal{G}}[\Gamma], \quad (V/U)_\Gamma \cong V_\Gamma/U_\Gamma.$$

(ii) *Assume that for any $(\lambda, r) \in \Lambda(V) \setminus \Gamma$ there exists $(\mu, s) \in \Gamma$ with $(\mu, s) \preccurlyeq (\lambda, r)$. Then $V^\Gamma \in \operatorname{Ob}\widehat{\mathcal{G}}[\Gamma]$. Furthermore, if U is a submodule of V, then*

$$U^\Gamma, (V/U)^\Gamma \in \operatorname{Ob}\widehat{\mathcal{G}}[\Gamma], \quad (V/U)^\Gamma \cong V^\Gamma/U^\Gamma.$$

Proof. Suppose that V_Γ is not an $\mathfrak{a}$-module. By Proposition 3.2 there exists $(\lambda, r) \in \Lambda(V) \cap \Gamma$ and $(\nu, s) \in \Lambda(V) \setminus \Gamma$ such that $(\lambda, r) \preccurlyeq (\nu, s)$. By hypothesis we can choose $(\mu, k) \in \Gamma$ with $(\lambda, r) \preccurlyeq (\nu, s) \preccurlyeq (\mu, k)$ which contradicts the fact that Γ is interval-closed. Suppose now that we have a short exact sequence

$$0 \to U \to V \to W \to 0$$

of objects of $\widehat{\mathcal{G}}$. Since $\Lambda(V) = \Lambda(U) \cup \Lambda(W)$, it is clear that U and W both satisfy the hypothesis of (*i*) and, hence, $U_\Gamma, W_\Gamma \in \widehat{\mathcal{G}}[\Gamma]$ and hence the inclusion of U in V induces a $\widehat{\mathcal{G}}[\Gamma]$-morphism $U_\Gamma \to V_\Gamma$ which is obviously injective since $U_\Gamma^+ \subset V_\Gamma^+$. Similarly W_Γ is a quotient of V_Γ as objects of $\widehat{\mathcal{G}}[\Gamma]$ and the exactness follows by noting that $V_\Gamma = U_\Gamma \oplus W_\Gamma$ as $\mathfrak{g}$-modules.

The proof of part (ii) is similar and hence omitted. □

3.4

We now construct projective objects and projective resolutions of simple objects in $\widehat{\mathcal{G}}[\Gamma]$ when Γ is finite and interval-closed.

Proposition. *Suppose $\Gamma \subset \Lambda$ is finite and interval-closed with respect to $\preccurlyeq$, and assume that $(\lambda, r), (\mu, s) \in \Gamma$.*

(i) *$P(\lambda, r)^\Gamma$ is the projective cover in $\mathcal{G}[\Gamma]$ of $V(\lambda, r)$.*

(ii) *We have*

$$[P(\lambda, r) : V(\mu, s)] = [P(\lambda, r)^\Gamma : V(\mu, s)] = \dim \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\mu, s)^\Gamma, P(\lambda, r)^\Gamma).$$

(iii) $\operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), P(\mu, s)) \cong \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\lambda, r)^\Gamma, P(\mu, s)^\Gamma)$.

(iv) *For all $j \in \mathbb{Z}_+$, $P_j(\mu, s)^\Gamma \in \mathcal{G}[\Gamma]$. The induced sequence*

$$\cdots \xrightarrow{d_3^\Gamma} P_2(\mu, s)^\Gamma \xrightarrow{d_2^\Gamma} P_1(\mu, s)^\Gamma \xrightarrow{d_1^\Gamma} P(\mu, s)^\Gamma \xrightarrow{d_0^\Gamma} V(\mu, s) \longrightarrow 0$$

is a finite projective resolution of $V(\mu, s) \in \mathcal{G}[\Gamma]$.

Proof. By Proposition 2.5(iii), we see that

$$(\nu, k) \succ (\lambda, r), \quad (\nu, k) \in \Lambda(P(\lambda, r)) \setminus \{(\lambda, r)\}.$$

By Proposition 3.3(ii) we see that $P(\lambda, r)^\Gamma \in \mathcal{G}[\Gamma]$ and maps onto $V(\lambda, r)$. Let $K = P(\lambda, r)_{\Lambda\setminus\Gamma}$; thus, in $\widehat{\mathcal{G}}$, we have a short exact sequence

$$0 \to K \to P(\lambda, r) \to P(\lambda, r)^\Gamma \to 0.$$

Applying $\operatorname{Hom}_{\widehat{\mathcal{G}}}(-, V(\mu, s))$ yields the long exact sequence

$$\cdots \to \operatorname{Hom}_{\widehat{\mathcal{G}}}(K, V(\mu, s)) \to \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(P(\lambda, r)^\Gamma, V(\mu, s)) \to 0.$$

If $(\mu, s) \in \Gamma$, we have,

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(K, V(\mu, s)) \cong \operatorname{Hom}_{\mathfrak{g}}(K[s], V(\mu)) = 0,$$

and hence we have

$$\operatorname{Ext}^1_{\widehat{\mathcal{G}}}(P(\lambda, r)^\Gamma, V(\mu, s)) = 0.$$

In particular, this proves that

$$\operatorname{Ext}^1_{\mathscr{G}[\Gamma]}(P(\lambda,r)^{\Gamma}, V(\mu,s)) = 0,$$

and hence $P(\lambda,r)^{\Gamma}$ is a projective object of $\mathscr{G}[\Gamma]$. The proof that $P(\lambda,r)^{\Gamma}$ is the projective cover of $V(\lambda,r)$ is similar to the proof given in Proposition 2.5.

For (ii), we again consider the short exact sequence

$$0 \to K \to P(\lambda,r) \to P(\lambda,r)^{\Gamma} \to 0.$$

Since $\mathscr{F}(\mathfrak{g})$ is a semisimple category, we have,

$$\dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), P(\lambda,r)[s]) = \dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), P(\lambda,r)^{\Gamma}[s]) + \dim \operatorname{Hom}_{\mathfrak{g}}(V(\mu), K[s]).$$

By the definition of K, we have,

$$\operatorname{Hom}_{\mathfrak{g}}(V(\mu), K[s]) = 0, \quad (\mu,s) \in \Gamma,$$

and hence we get

$$[P(\lambda,r) : V(\mu,s)] = [P(\lambda,r)^{\Gamma} : V(\mu,s)].$$

The second equality follows by imitating (in $\mathscr{G}[\Gamma]$) the proof of the first part of Proposition 2.5.

For (iii), choose a nonzero $f \in \operatorname{Hom}_{\widehat{\mathscr{G}}}(P(\lambda,r), P(\mu,s))$. Then, $f(1\otimes V(\lambda,r)) \notin P(\mu,s)_{\Lambda\setminus\Gamma}$, so $f^{\Gamma} \neq 0$. Thus, we have an injective map

$$\operatorname{Hom}_{\widehat{\mathscr{G}}}(P(\lambda,r), P(\mu,s)) \to \operatorname{Hom}_{\mathscr{G}[\Gamma]}(P(\lambda,r)^{\Gamma}, P(\mu,s)^{\Gamma}).$$

Since both of these spaces have the same dimension (by part (ii) and Proposition 2.5(i)), the map is an isomorphism.

For (iv), first note that $P_j(\lambda,r)^{\Gamma} \in \operatorname{Ob}\mathscr{G}[\Gamma]$ by Proposition 2.7(i) and Proposition 3.3(ii). Furthermore, a similar procedure as in part (i) shows that $P_j(\lambda,r)^{\Gamma}$ is projective in $\mathscr{G}[\Gamma]$. The fact that the resolution terminates after finitely many steps follows from the fact that Γ is finite, along with the fact that $P_j(\lambda,r)[k] = 0$ for all $k < r+j$. □

3.5

We recall the following definition from [7,15], where we define a length category in the sense of [9].

Definition. Suppose $\mathcal{C}$ is an abelian $\mathbb{C}$-linear length category. We say that $\mathcal{C}$ is *directed* if:

(i) The simple objects in $\mathcal{C}$ are parametrized by a poset $(\Pi,\le)$ such that the set $\{\xi \in \Pi : \xi < \tau\}$ is finite for all $\tau \in \Pi$.
(ii) For all simple objects $S(\xi), S(\tau) \in \mathcal{C}$, $\operatorname{Ext}^1_{\mathcal{C}}(S(\xi), S(\tau)) \neq 0 \implies \xi < \tau$.

In the case when $(\Pi,\le)$ is finite, a directed category is highest weight in the sense of [7].

We end this section by noting that we have established that for any subset Γ of Λ, the category $\mathscr{G}[\Gamma]$ is directed, and if Γ is finite and interval-closed, then $\mathscr{G}[\Gamma]$ is a directed highest weight category.

4. Undeformed infinitesimal Hecke algebras

For the rest of the paper, we restrict our attention to the case when $\mathfrak{a}_k = 0$ for all $k > 1$ and $\mathfrak{a}_1 = V$, where $V \in \mathscr{F}(\mathfrak{g})$ is such that $\operatorname{wt}(V) \neq \{0\}$. In this case, the algebra $\mathfrak{a} = \mathfrak{g} \ltimes V$ and we identify V with the abelian ideal $0 \ltimes V$ of $\mathfrak{a}$. In particular, this means that $\mathbf{U}(\mathfrak{a}_+) = \operatorname{Sym}(V)$, and it is immediate that

$$P_j(\lambda,r) = \mathbf{U}(\mathfrak{g}\ltimes V) \otimes_{\mathbf{U}(\mathfrak{g})} \wedge^j V \otimes V(\lambda,r)$$

is generated as an $\mathfrak{a}$-module by the component of degree $r+j$. This motivates our search for Koszulity in this picture.

4.1

We begin by computing extensions between the simple objects.

Proposition. *For all $j \in \mathbb{Z}_+$ and $(\mu,r), (\nu,s) \in \Lambda$,*

$$\operatorname{Ext}^j_{\widehat{\mathscr{G}}}(V(\mu,r), V(\nu,s)) \cong \begin{cases} \operatorname{Hom}_{\mathfrak{g}}(\wedge^j V \otimes V(\mu), V(\nu)), & \text{if } j = s-r; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Truncating the projective resolution from Proposition 2.7(ii) at
$\cdots \xrightarrow{d_j} P_{j-1}(\mu, r) \xrightarrow{d_{j-1}} \operatorname{im} d_{j-1} \longrightarrow 0$ yields

$$\operatorname{Ext}^j_{\widehat{\mathcal{G}}}(V(\mu, r), V(\nu, s)) \cong \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\operatorname{im} d_{j-1}, V(\nu, s)).$$

Applying $\operatorname{Hom}_{\widehat{\mathcal{G}}}(-, V(\nu, s))$ to the short exact sequence

$$0 \to \operatorname{im} d_j \to P_{j-1}(\mu, r) \to \operatorname{im} d_{j-1} \to 0$$

yields the exact sequence

$$0 \to \operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_{j-1}, V(\nu, s)) \to \operatorname{Hom}_{\widehat{\mathcal{G}}}(P_{j-1}(\mu, r), V(\nu, s)) \to$$
$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_j, V(\nu, s)) \to \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\operatorname{im} d_{j-1}, V(\nu, s)) \to 0.$$

The result follows if we prove that

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_j, V(\nu, s)) \neq 0 \implies j = s - r.$$

Suppose $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_j, V(\nu, s))$ is nonzero and choose $v \in \operatorname{im} d_j[s]$ with $f(v) \neq 0$. It is easily seen that we may write

$$v = \sum_p (u_p \otimes 1) d_j (1 \otimes w_p), \quad u_p \in \operatorname{Sym} V, \;\; w_p \in \wedge^j V \otimes V(\mu, r),$$

and hence we have

$$f(v) = \sum_p (u_p \otimes 1) f(d_j(1 \otimes w_p)).$$

Since $d_j(1 \otimes w_p) \in \operatorname{im} d_j[j + r]$ for all p, we see that $f(v) \in V(\nu, s)[r + j]$ and hence $s = r + j$.

If $j = s - r$, then since $\wedge^{j-1} V \otimes V(\mu, r)$ is concentrated in degree $s - 1$ and $P_{j-1}(\mu, r)$ is the projective cover of $\wedge^{j-1} V \otimes V(\mu, r)$, we have

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(P_{s-r-1}(\mu, r), V(\nu, s)) = 0,$$

and:

$$\begin{aligned} \operatorname{Ext}^{s-r}_{\widehat{\mathcal{G}}}(V(\mu, r), V(\nu, s)) &\cong \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\operatorname{im} d_{s-r-1}, V(\nu, s)) \\ &\cong \operatorname{Hom}_{\widehat{\mathcal{G}}}(\operatorname{im} d_{s-r}, V(\nu, s)) \cong \operatorname{Hom}_{\mathfrak{g}}((\operatorname{im} d_{s-r})[s], V(\nu)) \\ &\cong \operatorname{Hom}_{\mathfrak{g}}(\wedge^{s-r} V \otimes V(\mu), V(\nu)). \end{aligned}$$

This completes the proof of the Proposition. □

4.2

If $\Gamma \subset \Lambda$ is finite and interval-closed, then we can make the following observation regarding Ext -groups in the truncated subcategory $\mathcal{G}[\Gamma]$.

Proposition. *Let Γ be finite and interval-closed. For all $(\mu, r), (\nu, s) \in \Gamma$, we have*

$$\operatorname{Ext}^j_{\mathcal{G}[\Gamma]}(V(\mu, r), V(\nu, s)) \cong \operatorname{Ext}^j_{\widehat{\mathcal{G}}}(V(\mu, r), V(\nu, s)) \quad \forall j \in \mathbb{Z}_+.$$

Proof. By Proposition 2.7 and Proposition 3.4, we have a projective resolution $P_\bullet(\mu, s)$ of each simple object $V(\mu, s)$ in $\mathcal{G}[\Gamma]$ and $\widehat{\mathcal{G}}$. Then one shows as in [5, Proposition 3.3], that for all $(\lambda, r) \in \Gamma$,

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(P_\bullet(\mu, s), V(\lambda, r)) \to \operatorname{Hom}_{\mathcal{G}[\Gamma]}(P_\bullet(\mu, s)^\Gamma, V(\lambda, r))$$

is an isomorphism. □

4.3

Define $P(\Gamma) := \bigoplus_{(\lambda, r) \in \Gamma} P(\lambda, r)$, and set

$$\mathfrak{B}(\Gamma) := \operatorname{End}_{\widehat{\mathcal{G}}} P(\Gamma), \quad \text{and} \quad \mathfrak{B}^\Gamma(\Gamma) := \operatorname{End}_{\mathcal{G}[\Gamma]}(P(\Gamma)^\Gamma).$$

Notice that $\mathfrak{B}(\Gamma)$ is graded via

$$\mathfrak{B}(\Gamma)[k] = \bigoplus_{(\lambda, r), (\mu, r-k) \in \Gamma} \operatorname{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), P(\mu, r - k)).$$

In particular, $\mathfrak{B}(\Gamma)[0] = \bigoplus_{(\lambda, r) \in \Gamma} \operatorname{End}_{\widehat{\mathcal{G}}}(P(\lambda, r))$.

Proposition. *If $\Gamma \subset \Lambda$ is finite and interval-closed, then the category $\mathcal{G}[\Gamma]$ is equivalent to the category of right modules over $\mathfrak{B}(\Gamma)$.*

Proof. By Proposition 3.4(iii), $\mathfrak{B}(\Gamma) \cong \mathfrak{B}^\Gamma(\Gamma)$ if Γ is finite and interval-closed; thus, it is standard ([1, Theorem II.1.3]) that

$$\mathrm{Hom}_{\mathcal{G}[\Gamma]}(P(\Gamma)^\Gamma, -) : \mathcal{G}[\Gamma] \to \mathrm{Mod} - \mathfrak{B}(\Gamma)$$

is an equivalence of categories. □

5. Faces of polytopes and Koszul algebras

This section is devoted to proving the main theorem.

5.1

We begin with a key technical observation about the set of weights which lie on a face of the weight polytope of V. Namely, we wish to consider the subsets $\Psi \subset \mathrm{wt}(V)$ that satisfy the following property:

$$\text{If } \sum_{\alpha\in\Psi} m_\alpha \alpha = \sum_{\beta\in\mathrm{wt}(V)} r_\beta \beta, \quad \text{for } m_\alpha, r_\beta \in \mathbb{Z}_+, \tag{5.1}$$
$$\text{then } \sum_\alpha m_\alpha \le \sum_\beta r_\beta, \text{ with equality if and only if } \beta \in \Psi \text{ whenever } r_\beta > 0.$$

The main result of [12] states that Ψ satisfies (5.1) if and only if the set Ψ lies on a proper face of the weight polytope of V.

5.2

For our next result, recall d_Ψ and $\le_\Psi$ defined in Section 1.4.

Proposition. *Suppose $\Psi \subset \mathrm{wt}(V)$ satisfies (5.1).*

(i) *$\le_\Psi$ is a partial order on $\mathfrak{h}^*$. Moreover,*

$$d_\Psi(\eta, \mu) + d_\Psi(\mu, \nu) = d_\Psi(\eta, \nu) \quad \forall \eta \le_\Psi \mu \le_\Psi \nu \in \mathfrak{h}^*.$$

(ii) *Ψ induces a refinement $\preccurlyeq_\Psi$ of the partial order $\preccurlyeq$ on $\Lambda = P^+ \times \mathbb{Z}$ via: $(\mu, r) \preccurlyeq_\Psi (\lambda, s)$ if and only if $\mu \le_\Psi \lambda$ and $d_\Psi(\mu, \lambda) = s - r$. If the interval $[(\nu, r), (\mu, s)]_{\preccurlyeq_\Psi}$ is nonempty, then $[(\nu, r), (\mu, s)]_{\preccurlyeq_\Psi} = [(\nu, r), (\mu, s)]_{\preccurlyeq}$.*

Proof. By definition, $\le_\Psi$ is reflexive and transitive. To see that $\le_\Psi$ is anti-symmetric, let

$$\nu - \mu = \sum_{\beta\in\Psi} r_\beta \beta, \qquad \mu - \nu = \sum_{\beta\in\Psi} m_\beta \beta, \quad r_\beta, m_\beta \in \mathbb{Z}_+ \ \forall\, \beta \in \Psi.$$

Then,

$$0 = \sum_{\beta\in\Psi} (r_\beta + m_\beta)\beta,$$

which gives $r_\beta + m_\beta = 0$ for all $\beta \in \Psi$ using condition (5.1). In particular, $r_\beta = 0$ for all $\beta \in \Psi$, so $\mu = \nu$. This shows that $\le_\Psi$ is a partial order on $\mathfrak{h}^*$.

Suppose that $\mu \le_\Psi \nu$, and let

$$\nu - \mu = \sum_{\beta\in\Psi} r_\beta \beta = \sum_{\beta\in\Psi} m_\beta \beta, \quad r_\beta, m_\beta \in \mathbb{Z}_+ \ \forall\, \beta \in \Psi.$$

Applying condition (5.1) to each sum gives

$$\sum_{\beta\in\Psi} r_\beta \le \sum_{\beta\in\Psi} m_\beta \le \sum_{\beta\in\Psi} r_\beta,$$

which shows that $d_\Psi(\mu, \nu)$ is taken over a singleton set. This uniqueness and the fact that $\lambda - \mu = (\lambda - \nu) + (\nu - \mu)$ show that $d_\Psi(\mu, \nu) + d_\Psi(\nu, \lambda) = d_\Psi(\mu, \lambda)$.

The fact that $\preccurlyeq_\Psi$ is a partial order follows immediately from part (i).

Notice that

$$(\mu, r) \preccurlyeq (\lambda, s) \iff \lambda - \mu = \sum_{\nu\in\mathrm{wt}(V)} m_\nu \nu, \quad m_\nu \in \mathbb{Z}_+ \forall \nu \in \mathrm{wt}(V), \qquad \sum_{\nu\in\mathrm{wt}(V)} m_\nu = s - r.$$

It follows immediately that $\preccurlyeq_\Psi$ is a refinement of $\preccurlyeq$ and that the interval $[(\nu, r), (\mu, s)]_{\preccurlyeq_\Psi}$ is a subset of $[(\nu, r), (\mu, s)]_{\preccurlyeq}$ for all $(\nu, r) \preccurlyeq_\Psi (\mu, s)$.

Now, suppose that

$$(\nu, r) \preccurlyeq (\eta, t) \preccurlyeq (\mu, s), \quad (\nu, r) \preccurlyeq_\Psi (\mu, s) \in \Gamma$$

Then, we can write

$$\mu - \eta = \sum_{i=1}^{t-s} \beta_i, \qquad \eta - \nu = \sum_{j=1}^{s-r} \gamma_j, \quad \beta_i, \gamma_j \in \mathrm{wt}\,(V)\ \forall\, i, j.$$

Since

$$\mu - \nu = (\mu - \eta) + (\eta - \nu), \quad \text{and} \quad (s - t) + (t - r) = s - r = d_\Psi(\mu, \lambda),$$

it follows that $\beta_i, \gamma_j \in \Psi$ for all i, j by condition (5.1). This gives

$$(\nu, r) \preccurlyeq_\Psi (\eta, t) \preccurlyeq_\Psi (\mu, s),$$

which proves (ii). □

Remark. In [5], the authors work with $V = \mathfrak{g}_{\mathrm{ad}}$ and $\Psi \subset R^+$. However, they use the partial order $\preccurlyeq'_\Psi$ on Λ given by $(\lambda, r) \preccurlyeq'_\Psi (\mu, s)$ if and only if $\mu \leq_\Psi \lambda$ and $d_\Psi(\mu, \lambda) = s - r$. We use $\preccurlyeq_\Psi$ instead, because $\preccurlyeq'_\Psi$ is not a refinement of the standard partial order $\preccurlyeq$ on Λ.

5.3

We need the following well-known result.

Lemma. *Suppose $\mathfrak{g}$ is a complex semisimple Lie algebra, $V \in \mathcal{F}(\mathfrak{g})$, and $\lambda, \mu \in P^+$. Define $V^+ := \{v \in V : \mathfrak{n}^+ v = 0\}$.*

(i) $\dim \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), V) = \dim(V^+ \cap V_\lambda)$.
(ii) *As vector spaces,*

$$\mathrm{Hom}_{\mathfrak{g}}(V \otimes V(\mu), V(\lambda)) \cong \{v \in V_{\lambda-\mu} : (x^+_{\alpha_i})^{\mu(h_i)+1} v = (x^-_{\alpha_i})^{\lambda(h_i)+1} v = 0\}. \quad \square$$

5.4

We now discuss some results on specific sets of $\mathfrak{g}$-module homomorphisms which will be useful later. Recall that $\lambda \in P^+$ is said to be *regular* if $\lambda(h_i) > 0$ for all $i \in I$.

Lemma. *Suppose $\Psi \subset \mathrm{wt}\,(V)$ satisfies condition (5.1). Define $\lambda_\Psi := \sum_{\mu \in \Psi} (\dim V_\mu)\mu \in P$ and $N_\Psi := \sum_{\mu \in \Psi} (\dim V_\mu)$.*

(i) *If $\nu, \nu + \lambda_\Psi \in P^+$, then* $\dim \mathrm{Hom}_{\mathfrak{g}}(\wedge^{N_\Psi} V \otimes V(\nu), V(\nu + \lambda_\Psi)) \leq 1$.
(ii) *Given $\eta \in P^+$, there exists $\nu \in P^+$ such that $\nu, \nu + \lambda_\Psi \in P^+$ are both regular, $\eta \leq \nu$, and*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\wedge^{N_\Psi} V \otimes V(\nu), V(\nu + \lambda_\Psi)) = 1.$$

Proof. Suppose $v_{\mu_1} \wedge \cdots \wedge v_{\mu_{N_\Psi}} \in (\wedge^{N_\Psi} V)_{\lambda_\Psi}$, where each $v_{\mu_i} \in V_{\mu_i}$. Then

$$\mu_1 + \cdots + \mu_{N_\Psi} = \lambda_\Psi = \sum_{\mu \in \Psi} (\dim V_\mu)\mu, \quad N_\Psi = \sum_{\mu \in \Psi} \dim V_\mu,$$

so $\mu_i \in \Psi\ \forall i$ by condition (5.1). In particular, $\dim(\wedge^{N_\Psi} V)_{\lambda_\Psi} = 1$. Hence (i) follows by Lemma 5.3.

Now suppose that $(\wedge^{N_\Psi} V)_{\lambda_\Psi} = \mathbb{C}\mathbf{v}$. Let

$$\lambda_\Psi = \sum_{i \in I} d_i \omega_i, \qquad \eta = \sum_{i \in I} c_i \omega_i.$$

Let $2\rho = \sum_{\alpha \in R^+} \alpha = 2 \sum_{i \in I} \omega_i$. Choose $k \in \mathbb{Z}_+$ sufficiently large such that

$$c_i + 2k, c_i + d_i + 2k \in \mathbb{N}$$

and

$$(x^+_{\alpha_i})^{c_i+2k+1}\mathbf{v} = (x^-_{\alpha_i})^{c_i+d_i+2k+1}\mathbf{v} = 0 \quad \forall\, i \in I.$$

Let $\nu = \eta + 2k\rho$. Then, $\eta \leq \nu$, and it follows from Lemma 5.3 that

$$\mathrm{Hom}_{\mathfrak{g}}(\wedge^{N_\Psi} V \otimes V(\nu), V(\nu + \lambda_\Psi)) \cong \mathbb{C}\mathbf{v},$$

which proves (ii). □

5.5

Lemma. *Fix $(\mu, r) \in \Lambda$. Suppose $\Gamma \subset \Lambda$ is finite and interval-closed with respect to $\preccurlyeq_\Psi$. Also, assume that $(\mu, r) \preccurlyeq_\Psi (\nu, s)\ \forall\, (\nu, s) \in \Gamma$.*

(i) *If* $\mathrm{Hom}_{\mathcal{G}[\Gamma]}(P(\nu, s)^\Gamma, P(\nu', s')^\Gamma) \neq 0$, *then* $(\nu', s') \preccurlyeq_\Psi (\nu, s)$.
(ii) *If* $\mathrm{Ext}^j_{\mathcal{G}[\Gamma]}(V(\nu', s'), V(\nu, s)) \neq 0$, *then* $(\nu', s') \preccurlyeq_\Psi (\nu, s),\ j = d_\Psi(\nu', \nu)$.
(iii) gldim $\mathcal{G}[\Gamma] \leq N_\Psi$, *and equality holds for some* $(\mu, r) \in \Lambda$ *and some* Γ.

Proof. (i) Suppose that $\mathrm{Hom}_{\mathcal{G}[\Gamma]}(P(\nu, s)^\Gamma, P(\nu', s')^\Gamma) \neq 0$. Then, by Propositions 3.4 and 2.5,

$$\mathrm{Hom}_{\mathfrak{g}}(V(\nu), \mathrm{Sym}^{s-s'} V \otimes V(\nu')) \neq 0.$$

Using Lemma 5.3 and Steinberg's formula [10, Section 24],

$$\nu - \nu' = \sum_{i=1}^{s-s'} \xi_i, \quad \xi_i \in \mathrm{wt}\,(V).$$

On the other hand, since $(\mu, r) \preccurlyeq_\Psi (\nu, s), (\nu', s') \in \Gamma$,

$$\nu - \mu = \sum_{j=1}^{s-r} \eta_j, \qquad \nu' - \mu = \sum_{k=1}^{s'-r} \eta'_k, \quad \eta_j, \eta'_k \in \Psi\ \forall j, k.$$

Combining these gives

$$\nu - \mu = \sum_{j=1}^{s-r} \eta_j = \sum_{i=1}^{s-s'} \xi_i + \sum_{k=1}^{s'-r} \eta'_k.$$

Finally, since Ψ satisfies (5.1) and $s - r = (s - s') + (s' - r)$, we get $\xi_i \in \Psi\ \forall i$, whence $(\nu', s') \preccurlyeq_\Psi (\nu, s)$.

(ii) Suppose $\mathrm{Ext}^j_{\mathcal{G}[\Gamma]}(V(\nu', s'), V(\nu, s)) \neq 0$. By Propositions 4.1 and 4.2, $j = s - s'$ and

$$\mathrm{Hom}_{\mathfrak{g}}(\wedge^j V \otimes V(\nu), V(\nu')) \neq 0.$$

Using Lemma 5.3, $\nu' - \nu = \xi_1 + \cdots + \xi_j$ for some $\xi_i \in \mathrm{wt}\,(V)$. Since Ψ satisfies (5.1), an argument similar to part (i) shows that $\xi_i \in \Psi\ \forall\, i$ and $\nu \leq_\Psi \nu'$. Finally, by Proposition 5.2, $j = d_\Psi(\nu, \nu')$ and, therefore, $(\nu, s) \preccurlyeq_\Psi (\nu', s')$.

(iii) Since $\mathcal{G}[\Gamma]$ is a length category, it suffices to work with extensions between simple objects. By Propositions 4.1 and 4.2 again, we have

$$\mathrm{Ext}^j_{\mathcal{G}[\Gamma]}(V(\nu, s), V(\nu', s')) \neq 0 \implies \mathrm{Hom}_{\mathfrak{g}}(\wedge^j V \otimes V(\nu), V(\nu')) \neq 0,$$

so gldim $\mathcal{G}[\Gamma] \leq N_\Psi$.

Using Lemma 5.4,

$$\mathrm{Hom}_{\mathcal{G}[\Gamma]}(\wedge^{N_\Psi} V \otimes V(\mu), V(\mu + \lambda_\Psi)) \neq 0$$

for some $\mu, \mu + \lambda_\Psi \in P^+$. Let $r \in \mathbb{Z}$ and define

$$\Gamma := [(\mu, r), (\mu + \lambda_\Psi, r + N_\Psi)]_{\preccurlyeq_\Psi}.$$

Then, gldim $\mathcal{G}[\Gamma] = N_\Psi$. □

5.6

Theorem. *Assume that $\Gamma \subset \Lambda$ is finite and interval-closed under $\preccurlyeq_\Psi$. Then, the algebra $\mathfrak{B}(\Gamma)^{\mathrm{op}}$ is Koszul.*

Proof. We use the numerical condition from [2, Theorem 2.11.1] to show Koszulity. Let $B = \mathfrak{B}(\Gamma)^{\mathrm{op}}$. We note that the $|\Gamma| \times |\Gamma|$-Hilbert matrices $H(B, t)$ of B and $H(E(B), t)$ of its Yoneda algebra $E(B)$ are lower triangular in this case.

Note from the definition of the grading that $B[0]$ is semisimple, commutative, and spanned by pairwise orthogonal idempotents $\{1_{(\nu,s)} : (\nu, s) \in \Gamma\}$. For each $(\nu', s') \preccurlyeq_\Psi (\nu, s) \in \Gamma$, we compute:

$$\begin{aligned}
&(H(E(B), -t)H(B, t))_{(\nu,s),(\nu',s')} \\
&= \sum_{(\xi,l)\in\Gamma} H(E(B), -t)_{(\nu,s),(\xi,l)} H(B, t)_{(\xi,l),(\nu',s')} \\
&= \sum_{\nu' \leq_\Psi \xi \leq_\Psi \nu} (-t)^{d_\Psi(\xi,\nu)} \dim \mathrm{Ext}^{d_\Psi(\xi,\nu)}_{\mathcal{G}[\Gamma]}(V(\xi, l), V(\nu, s)) \cdot t^{d_\Psi(\nu',\xi)} [P(\nu', s')^\Gamma : V(\xi, l)] \\
&= \sum_{j\geq 0} \sum_{\nu' \leq_\Psi \xi \leq_\Psi \nu} (-1)^j t^{d_\Psi(\nu',\nu)} [P(\nu', s')^\Gamma : V(\xi, l)] \dim \mathrm{Ext}^j_{\mathcal{G}[\Gamma]}(V(\xi, l), V(\nu, s)),
\end{aligned}$$

where the first equality is by definition, the second uses the definitions of the Hilbert matrices, and the third uses Proposition 5.2 and Lemma 5.5. Now use the long exact sequence of Ext groups and a Jordan–Holder series for $P(\nu', s')^{\Gamma}$, along with the fact that all $P(\nu', s')^{\Gamma}$ are projective, to obtain:

$$= t^{d_\Psi(\nu',\nu)} \sum_{j\geq 0} (-1)^j \dim \mathrm{Ext}^j_{\mathfrak{g}[\Gamma]}(P(\nu', s')^{\Gamma}, V(\nu, s))$$
$$= t^{d_\Psi(\nu',\nu)} \dim \mathrm{Hom}_{\mathfrak{g}[\Gamma]}(P(\nu', s')^{\Gamma}, V(\nu, s))$$
$$= t^{d_\Psi(\nu',\nu)} \delta_{(\nu',s'),(\nu,s)} = \delta_{(\nu',s'),(\nu,s)}.$$

Thus, $H(E(B), -t)H(B, t)$ is the identity matrix, so $B = \mathfrak{B}(\Gamma)^{\mathrm{op}}$ is Koszul by [2, Theorem 2.11.1]. □

5.7

We are now ready to approach the proof of Theorem 1.6. The following Lemma will provide a major component of the proof. Let $\pi_1 : \Lambda \to P^+$ be the projection map onto the first coordinate. Recall that $A = \mathrm{Sym}\, V$.

Lemma. *Fix $(\mu, r) \in \Lambda$. Let $\Gamma \subset \Lambda$ be finite and interval-closed with $(\mu, r) \preccurlyeq_\Psi (\nu, s)\ \forall\ (\nu, s) \in \Gamma$. Then, $\mathfrak{B}(\Gamma)^{\mathrm{op}}$ has global dimension at most N_Ψ, and*

$$\mathfrak{B}(\Gamma)^{\mathrm{op}} \cong \mathbf{A}_\Psi(\pi_1(\Gamma))^{\mathfrak{g}}$$

as $\mathbb{Z}_+$-graded algebras.

Proof. By definition,

$$1_\nu \mathbf{A}^{\mathfrak{g}}_\Psi[d_\Psi(\mu, \nu)]1_\mu = (V(\nu)^* \otimes \mathrm{Sym}^{\,d_\Psi(\mu,\nu)} V \otimes V(\mu))^{\mathfrak{g}}.$$

For any finite-dimensional $\mathfrak{g}$-modules, V, W, the map

$$\sum_i (f_i \otimes w_i) \to (v \mapsto \sum_i f_i(v)w_i)$$

gives an isomorphism $(V^* \otimes W)^{\mathfrak{g}} \cong \mathrm{Hom}_{\mathfrak{g}}(V, W)$. In particular,

$$(V(\nu)^* \otimes \mathrm{Sym}^{\,d_\Psi(\mu,\nu)} V \otimes V(\mu))^{\mathfrak{g}} \cong \mathrm{Hom}_{\mathfrak{g}}(V(\nu), \mathrm{Sym}^{\,d_\Psi(\mu,\nu)} V \otimes V(\mu)).$$

Finally,

$$P(\mu, r)[r + d_\Psi(\mu, \nu)] = (\mathbf{U}(\mathfrak{a}) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda, r))[r + d_\Psi(\mu, \nu)] = \mathrm{Sym}^{\,d_\Psi(\mu,\nu)} V \otimes V(\lambda, r)$$

by Proposition 2.4, and

$$\mathrm{Hom}_{\mathfrak{g}}(V(\nu), \mathrm{Sym}^{\,d_\Psi(\mu,\nu)} V \otimes V(\mu)) \cong \mathrm{Hom}_{\widehat{\mathfrak{g}}}(P(\nu, r + d_\Psi(\mu, \nu)), P(\mu, r))$$

by Proposition 2.5.

Notice that the product of the terms $1_\nu \mathbf{A}^{\mathfrak{g}}_\Psi[d_\Psi(\mu, \nu)]1_\mu$ is from left to right, whereas the composition of the Hom-spaces is from right to left. The bound on global dimensions follows from Lemma 5.5. □

5.8

We are now able to prove our main result:

Proof of Theorem 1.6. Notice first that $\leq_\Psi \nu$ and $[\mu, \nu]_\Psi$ are finite and interval-closed, so $\mathbf{A}_\Psi(\leq_\Psi \nu)^{\mathfrak{g}}$ and $\mathbf{A}_\Psi([\mu, \nu]_\Psi)^{\mathfrak{g}}$ are Koszul and have finite global dimension by Lemma 5.7 and Theorem 5.6.

It remains to show the results for $\mathbf{A}_\Psi(\mu \leq_\Psi)^{\mathfrak{g}}$ and $\mathbf{A}^{\mathfrak{g}}_\Psi$. We begin by showing that the algebras in question have finite global dimension. We only show the proof for $\mathbf{A}^{\mathfrak{g}}_\Psi$; the proof is similar for $\mathbf{A}_\Psi(\mu \leq_\Psi)^{\mathfrak{g}}$.

Suppose $\mu \in P^+$, and let S_μ be the simple left $\mathbf{A}^{\mathfrak{g}}_\Psi$-module corresponding to the idempotent 1_μ. Recall that $1_\nu \mathbf{A}^{\mathfrak{g}}_\Psi 1_\mu \neq 0$ only if $\mu \leq_\Psi \nu$ by Lemma 5.7. Thus, the projective cover of S_μ in the category of finite-dimensional left $\mathbf{A}^{\mathfrak{g}}_\Psi$-modules is

$$P_\mu := \mathbf{A}^{\mathfrak{g}}_\Psi 1_\mu = \bigoplus_{\mu \leq_\Psi \nu} 1_\nu \mathbf{A}^{\mathfrak{g}}_\Psi 1_\mu = \mathbf{A}_\Psi(\mu \leq_\Psi)^{\mathfrak{g}} 1_\mu.$$

As in Proposition 2.5, this yields that

$$[P_\nu : S_\mu] > 0 \Longrightarrow \mu \leq_\Psi \nu,$$

so we obtain a projective resolution of S_μ in the category of finite-dimensional left $\mathbf{A}_\Psi(\mu \leq_\Psi)^{\mathfrak{g}}$-modules. Applying $\mathrm{Hom}_{\mathbf{A}^{\mathfrak{g}}_\Psi}(-, S_\nu)$ to this projective resolution and using Lemma 5.7 and Lemma 5.4, the statements on the global dimension follow from the result for $\mathbf{A}_\Psi(\leq_\Psi \nu)^{\mathfrak{g}}$ and $\mathbf{A}_\Psi([\mu, \nu]_\Psi)^{\mathfrak{g}}$.

It remains to show that $\mathbf{A}^{\mathfrak{g}}_\Psi$ and $\mathbf{A}_\Psi(\mu \leq_\Psi)^{\mathfrak{g}}$ are Koszul. The proof is finished by adapting the proof of the analogous theorem in [5] while keeping in mind that we use a different definition for $\leq_\Psi$ and the reverse ordering on the summands of $\mathbf{A}$. A brief summary of the proof for $\mathbf{A}^{\mathfrak{g}}_\Psi$ is provided for the reader (see [16] for more details):

Let $\mathbf{T}V = \mathbf{V}^{\circledast} \otimes T(V) \otimes \mathbf{V}$. Use the kernel of the canonical projection $\Pi : \mathbf{T}V \to \mathbf{A}$ to show that $\mathbf{A}_\Psi^{\mathfrak{g}}$ is quadratic. Finally, show that the Koszul resolution of $\mathbf{A}_\Psi^{\mathfrak{g}}$ is exact, which shows that $\mathbf{A}_\Psi^{\mathfrak{g}}$ is Koszul by [2, Theorem 2.6.1]. $\square$

We conclude by remarking that it is possible to construct a linear graded resolution for the algebras addressed in Theorem 1.6. More generally, such a resolution has been constructed for every Koszul algebra in [2, Theorem 2.6.1].

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