Operations Research Perspectives 1 (2014) 42–49

Contents lists available at ScienceDirect

Operations Research Perspectives

journal homepage: www.elsevier.com/locate/orp

Power theories for multi-choice organizations and political rules: Rank-order equivalence

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ARTICLE INFO

Article history:
Received 29 January 2014
Received in revised form
17 June 2014
Accepted 17 June 2014
Available online 10 July 2014

Keywords:
Multi-choice organizations and political rules
(j, k) voting rules
Power theories
Rank-order equivalence

ABSTRACT

Voting power theories measure the ability of voters to influence the outcome of an election under a given voting rule. In general, each theory gives a different evaluation of power, raising the question of their appropriateness, and calling for the need to identify classes of rules for which different theories agree. We study the ordinal equivalence of the generalizations of the classical power concepts – the influence relation, the Banzhaf power index, and the Shapley–Shubik power index – to multi-choice organizations and political rules. Under such rules, each voter chooses a level of support for a social goal from a finite list of options, and these individual choices are aggregated to determine the collective level of support for this goal. We show that the power theories analyzed do not always yield the same power relationships among voters. Thanks to necessary and/or sufficient conditions, we identify a large class of rules for which ordinal equivalence obtains. Furthermore, we prove that ordinal equivalence obtains for all linear rules allowing a fixed number of individual approval levels if and only if that number does not exceed three. Our findings generalize all the previous results on the ordinal equivalence of the classical power theories, and show that the condition of linearity found to be necessary and sufficient for ordinal equivalence to obtain when voters have at most three options to choose from is no longer sufficient when they can choose from a list of four or more options.

1. Introduction

We study the ordinal equivalency of voting power theories for multi-choice organizations. In general, organizations have voting rules that are designed to guarantee a certain degree of fairness in the distribution of voting rights, where fairness means that more important stakeholders have a greater ability to affect group decisions. For instance, in international economic organizations including the International Monetary Fund and the World Bank, voting rights are apportioned among member countries on the basis of financial contribution. Similarly, an international political body such as the United Nations Security Council gives a greater amount of power to the victors of World War II, and political institutions (including the United States House of Representatives, the Parliament of Canada, and the European Union Council of Ministers) assign votes in proportion to the populations of the represented states, provinces, or countries. Still, in joint stock companies, each shareholder has a number of votes proportional to the number of shares he holds.

Although fairness in the distribution of political power in organizations is a well-accepted principle, there is little agreement among scholars on how power should be measured. Indeed, there exist several theories of power in the literature, the most prominent being the power indices proposed by Shapley and Shubik [1] and by Banzhaf [2], and the influence relation introduced by Isbell [3]. Both the Shapley–Shubik and Banzhaf power indices measure how likely a voter can turn a losing coalition into a winning coalition by joining it, but they differ in how these coalitions should be counted. The influence relation is an ordinal measure of power according to which a voter p is at least as influential as a voter q in an election if whenever q can turn a losing coalition into a winning coalition by joining it, p can achieve the same *ceteris paribus*.[1]

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[1] In this sense, individual influence is determined by the ability to affect the outcome of the vote; this notion of influence differs from the one introduced in [4],

http://dx.doi.org/10.1016/j.orp.2014.06.002

These power concepts have been analyzed and applied in a wide variety of contexts (see, e.g., [5–16], and the references therein). Also, most of them have compelling axiomatic properties, so that choosing one theory over the others in any theoretical or empirical analysis would be arbitrary. Any hope of reconciling them therefore lies in the characterization of political rules for which they ordinally agree.

In reaction to the inconsistent evaluation of political power by different theories, scholars have attempted to identify classes of voting rules for which different theories induce the same ordinal structure in the set of voters (see, e.g., [17,18,11,19,16] and the references therein). This question was initially examined within the class of voting rules under which each voter either supports or opposes a policy proposal, the final outcome being either the passage of the proposal or its failure.[2] Despite the popularity of these rules, it has been argued that they are restrictive and do not capture real-life political decision making, as more than two levels of individual and collective approval are generally observed [26,8,9,27–31]. This convincing argument motivates our analysis of the ordinal equivalence of power theories for multi-choice organizations and political rules.

1.1. Objective and findings

In this paper, we study the ordinal equivalence of the generalizations of the Shapley–Shubik and Banzhaf power indices and the influence relation for multi-choice organizations and political rules. Under such rules, each voter chooses from a finite ordered set of options his level of support for a given policy proposal, and these individual choices are aggregated to yield the collective level of support for this proposal [27,28]. Freixas and Zwicker [28] have introduced the concept of (j, k) voting rules to model such political rules, where j is the number of individual approval levels, and k the number of collective approval levels. Within this wide class of rules, simple games are $(2, 2)$ voting rules. In general, (j, k) voting rules can be used to model a wide variety of political and economic organizations. For instance, the rule governing the selection of papers in peer-reviewed journals can be modeled by a $(3, 3)$ voting rule wherein each referee chooses to accept, reject, or invite the resubmission of a paper, and an editor aggregates his own and referees' opinions into one of these three recommendations (see [32] for a thorough analysis of this class of rules). Similarly, in an election opposing an incumbent and a challenger, a voter might choose to support or oppose the incumbent or to abstain, the outcome of the election being either the victory or the defeat of either candidate ($j = 3$ and $k = 2$).[3] In certain legislatures like the United States Senate, a lawyer may support or oppose a bill or may abstain, or may carry out a *filibuster*, the outcome being the passage or failure of the bill, or its postponement ($j = 4$ and $k = 3$). Still, in a collaborative project, each participant might choose a high, medium or low level of effort, or may not contribute any effort at all, the outcome of the project being of excellent, good, fair, average or bad quality ($j = 4$ and $k = 5$).

The Shapley–Shubik and Banzhaf power indices examined in this paper are the generalizations proposed by Freixas [35,36], and the influence relation is the global influence relation introduced in Tchantcho et al. [31] for $(3, 2)$ voting rules and generalized in Pongou et al. [30] to (j, k) voting rules. We find that the three power theories do not always ordinally coincide for linear multi-choice political rules (Example 2). We provide a sufficient condition on such rules for ordinal equivalence to obtain (Theorems 2 and 3). This partial characterization is inspired by a related work [37] in which we derive the global influence relation as a natural rule for ranking workers in an on-the-job trial tournament, and study the relationship between productivity rank and several compensation schemes related to the Shapley value (see Theorem 1). As a corollary of Theorem 3, we prove that all linear $(2, k)$ and $(3, k)$ voting rules satisfy this characterization (Corollary 1). Importantly, this implies that ordinal equivalence obtains for a large class of political rules strictly including the class of von Neumann–Morgenstern games.[4]

We also provide a necessary and sufficient condition on linear rules for the Banzhaf and the global influence relation to have the same ordinal structure (Theorem 4). Furthermore, we establish that the ordinal equivalence of the three power theories analyzed in this paper obtains for all multi-choice political rules allowing a fixed number of individual approval levels j if and only if $j \leq 3$ (Theorem 5). This result implies that if $j > 3$, one can always construct a linear political rule for which ordinal equivalence does not obtain. All these characterization theorems allow us to identify a large class of institutions for which the ordinal evaluation of power given by different theories is identical, which makes it irrelevant to argue about which power theory is more appropriate.

1.2. Contributions to the closely related literature

Our findings generalize previous analyses by Tomiyama [18] and Diffo Lambo and Moulen [11] for the class of $(2, 2)$ voting rules, and recent studies by Tchantcho et al. [31] and Parker [38] for the class of $(3, 2)$ voting rules. Tomiyama [18] proved that the Shapley–Shubik and Banzhaf power indices ordinally coincide for quota games. Diffo Lambo and Moulen [11] extend this result to the entire class of linear simple games, and Freixas et al. [16] to an even larger class of simple games. Tchantcho et al. [31] show that the Shapley–Shubik and Banzhaf power indices ordinally coincide with the global influence relation in the class of linear weakly equitable $(3, 2)$ voting rules. Parker [38] generalizes this result to the class of linear $(3, 2)$ voting rules. All these results are special cases of Corollary 1.

Although our findings generalize all the previous results on the ordinal equivalence of the classical power theories, they also show that the condition of "linearity" found to be necessary and sufficient for ordinal equivalence to obtain in the class of $(2, 2)$ and $(3, 2)$ voting rules is no longer sufficient when voters can choose from a list of four or more options. Indeed, our analysis is the first to establish that power theories do not always coincide under "linear" political rules in general.

Our analysis also identifies rules for which ordinal equivalence never obtains, suggesting that such rules should never be used to distribute power if one wants to avoid the debate over which power theory is more appropriate. Fortunately, the class of voting rules for which ordinal equivalence obtains is large enough to allow any rule designer to choose from among them. Under such rules, no matter which theory we choose to evaluate power in an organization, we will obtain the same power relationships.

Another distinctive feature of our paper comes from the fact that the proofs of our main results are new, as they are not based

where influence is measured in terms of the ability to change the opinion of other players.

2 These voting rules, commonly referred to as "Simple games", have been studied or used in a wide range of economic and political analyses (see, e.g., [20,21,15,22,23,13,24,25]).

3 See, e.g., [26,8] for alternative models of voting games with abstention, [33] for the study of their equilibrium properties, and [34] for the analysis of attainable hierarchies within this class of games.

4 The class of von Neumann–Morgenstern cooperative games is simply the class of $(2, k)$ voting rules where k is endogenously determined.

on the arguments usually employed in the literature. Some of these proofs are made easier thanks to two new reformulations of the generalized Banzhaf power index. We therefore make a methodological contribution in addition to the more substantive findings.

As noted by Freixas et al. [16], importance rankings are a critical topic in operational research. Cook [39] notes that power theories and ordinal rankings have not been sufficiently studied, although ordinal rankings are essential in aggregation theories, tournament, and multi-criteria decision-making.[5] By uncovering a large class of games for which the most prominent power theories coincide, this paper fills this gap in the literature. The results obtained in this study can also be useful in multi-criteria decision-making problems, where, as noted by Freixas et al. [16], voters can be treated as criteria, or even as device components depending on the context.

The remainder of this paper is organized as follows. Section 2 introduces the concept of multi-choice organizations and political rules. Section 3 recalls the notion of global influence relation. In Section 4, we show that power theories do not always coincide even under linear political rules, and prove that the Banzhaf rule weakly reflects global influence. In Section 5, we characterize rules for which ordinal equivalence obtains, and in Section 6, we conclude. For clarity, all the proofs are collected in the Appendix.

2. Multi-choice organizations and political rules

A *multi-choice organization or political rule* is defined as *a multi-choice game* (N, T, V) where $N = \{1, \ldots, n\}$ is a non-empty finite set of voters or players, $T = \{1, \ldots, j\}$ a finite set of individual approval levels ordered in increasing degree of support (j is the highest level of approval, $j-1$ the second highest level of approval, and so on), and V a characteristic function that maps each profile of approval levels into a collective approval level. Denote by $\{v_1, v_2, \ldots, v_k\}$ the range of V. We assume that v_1 is the highest level of collective approval, v_2 the second highest, and so on. Because one can always define a one-to-one correspondence between $\{v_1, v_2, \ldots, v_k\}$ and a set of real numbers of cardinality k, we shall assume, without loss of generality, that $v_i \in \mathbb{R}$ with $v_1 > \cdots > v_k$.[6]

Each profile of approval levels is a vote profile $x = (x_1, x_2, \ldots, x_n)$ of N, where x_i is the approval level of voter i. We denote by T^N the set of all vote profiles. We will assume that V is monotonic in the sense that a voter who increases his support for a policy proposal weakly increases the collective approval level of that proposal *ceteris paribus*. Formally, V is monotonic means that for any vote profiles x and y such that $x \geq y$, $V(x) \geq V(y)$.

We note that a multi-choice game (N, T, V) may also be interpreted as a market organization in which N is the set of employees or workers assigned to fixed tasks, T the different levels of effort a worker may exert, and V a production function mapping each effort profile into a level of collective output. Although all our results can be interpreted in the context of a market organization, we will nevertheless interpret our results mostly in the context of voting.

5 See, e.g., [40–44].

6 We note that any tuple (N, T, V) where V is a real-valued function is also a multi-choice game where $\{v_1, v_2, \ldots, v_k\}$ is endogenously determined by V. In this sense, this model of a multi-choice organization, which is essentially the one introduced by Hsiao and Raghavan [27] and Freixas and Zwicker [28], generalizes the traditional model of transferable utility cooperative games introduced by von Neumann and Morgenstern [45], where $j = 2$.

3. Global and local influence relations

In this section, we recall the notions of global and local influence relations introduced in [30] to evaluate the ordinal structure of influence among voters in a multi-choice game. Let x be a vote profile and p a player. Denote by $e^p = (0, \ldots, 0, 1, 0, \ldots, 0)$ the pth unit n-component vector. We have the following definition.

Definition 1. Let (N, T, V) be a monotonic multi-choice game, and p and q two voters.

(1) Let s and r be two individual approval levels such that $r > s$. p is said to be more (r, s)-influential than q, denoted $p \succeq_{(r,s)} q$, if: $\forall x \in T^N$ such that $x_p = x_q = s$, $V(x + (r-s)e^p) \geq V(x + (r-s)e^q)$.
(2) p is said to be globally more influential than q, denoted $p \succeq q$, if: $p \succeq_{(r,s)} q$ for all $s, r \in T$ such that $r > s$.

The symmetric and asymmetric components of each of these relations are defined as usual.

(N, T, V) is said to be linear if the global influence relation $\succeq$ is transitive and complete.

Literally, a voter p is more (r, s)-influential than a voter q in a monotonic multi-choice game if the collective support for a proposal increases more whenever p increases his support for that proposal from level s to level r in a vote profile than if q does the same. p is globally more influential than q if an increase in the support of p always has more effect than a similar increase in the support of q. The following example illustrates this definition.

Example 1. A family is comprised of a father (1), a mother (2), and a son (3). The collective decision-making rule in the household is as follows: the opinion of the father on any pass–fail issue always prevails unless he abstains; if the father abstains, the opinion of the mother always prevails unless she abstains; if the father and the mother abstain, the opinion of the son prevails unless he abstains; if everybody abstains, the issue fails. Such a decision rule can be modeled as a function V defined from T^N ($N = \{1, 2, 3\}$ and $T = \{1$ (no), 2 (abstain), 3 (yes)$\}$) to $\{0$ (fail), 1 (pass)$\}$ as follows: $V(x) = 1$ if (i) $x_1 = 3$ or (ii) $x_1 = 2$ and $x_2 = 3$ or (iii) $x_1 = 2$ and $x_2 = 2$ and $x_3 = 3$; and $V(x) = 0$ for all other profiles.

We derive the following structure of influence within the family:

$1 \sim_{(1,2)} 2$, $1 \succ_{(1,2)} 3$, $2 \sim_{(1,2)} 3$;
$1 \succ_{(1,3)} 2$, $1 \succ_{(1,3)} 3$, $2 \succ_{(1,3)} 3$;
$1 \sim_{(2,3)} 2$, $1 \succ_{(2,3)} 3$, $2 \sim_{(2,3)} 3$; and
$1 \succ 2$, $1 \succ 3$, $2 \succ 3$.

We note that if voter influence is evaluated on the basis of how the outcome of an election would change if an individual shifts his/her support from "abstention" to a "yes" vote or from a "no" vote to "abstention", then the father and the mother are equivalent ($1 \sim_{(1,2)} 2$ and $1 \sim_{(2,3)} 2$), the father is more influential than the son ($1 \succ_{(1,2)} 3$ and $1 \succ_{(2,3)} 3$), and the son is equivalent to the mother ($2 \sim_{(1,2)} 3$ and $2 \sim_{(2,3)} 3$). If influence is evaluated on the basis of a shift from a "no" vote to a "yes" vote, then the father is more influential than the mother and the son ($1 \succ_{(1,3)} 2$ and $1 \succ_{(1,3)} 3$), and the mother is more influential than the son ($2 \succ_{(1,3)} 3$). In terms of global influence, we note that the father is more influential than the mother and the son ($1 \succ 2$ and $1 \succ 3$), and the mother is more influential than the son ($2 \succ 3$).

4. The global influence relation and the Banzhaf value

In this section, we study the relationship between the global influence relation and the generalized Banzhaf power index proposed by Freixas [36]. The definition of this power concept is recalled below.

4.1. Definition of the Banzhaf value

Let (N, T, V) be a monotonic multi-choice game, x a vote profile, and p a voter such that $x_p > 1$. Let $l, m \in \{1, 2, \ldots, k\}$ be such that $1 \leq l < m \leq k$. p is said to be (l, m)-*critical* in x if $v_l = V(x) > V(x - e^p) = v_m$. We denote by $\eta_p^{l,m}(V)$ the number of profiles in which p is (l, m)-critical.

The Banzhaf power index is defined below.

Definition 2. Let (N, T, V) be a monotonic multi-choice game, and $p \in N$ a voter.

(1) The Banzhaf score of p is:

$$\eta_p(V) = \sum_{m=2}^{k} \sum_{l=1}^{m-1} (v_l - v_m)\eta_p^{l,m}(V).$$

(2) The normalized Banzhaf index of p is:

$$\beta_p(V) = \frac{\eta_p(V)}{\sum_{i \in N} \eta_i(V)}.$$

For our purpose, we will need the following equivalent reformulation of the Banzhaf score shown in [46]:

$$\eta_p(V) = \frac{1}{j} \sum_{x \in T^N} (V(x + (j - x_p)e^p) - V(x + (1 - x_p)e^p)).$$

Pongou et al. [46] also introduce the following binary relation, useful in this paper. Let x and y be two vote profiles, p a voter. We say y is p-equivalent to x if $\forall q \neq p, x_q = y_q$. There are exactly j vote profiles in any equivalence class, and there are j^{n-1} classes. Denote by $cl(p, x)$ the equivalence class of a given vote profile x, and by N_p^j the set of all p-equivalence classes. Also denote by T_{pq}^N the set of all vote profiles x such that $x_p = x_q$.

For any player p, denote by $T^{N \setminus \{p\}}$ the set of vote profiles of $N \setminus \{p\}$. For any $x \in T^{N \setminus \{p\}}$ and $q \in N \setminus \{p\}$, let x_{pq} be the corresponding vote profile of N in which p and q have the same level of approval. We have the following useful reformulation of the Banzhaf score.

Proposition 1. *Let $G = (N, T, V)$ be a multi-choice game, and p a player.*

For any $q \in N$,

$$\eta_p(V) = \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p)).$$

We compute the normalized Banzhaf value of each voter in Example 1, finding that this value is $\frac{9}{13}$ for the father, $\frac{3}{13}$ for the mother, and $\frac{1}{13}$ for the son. We note that these values agree with the fact that the father is globally more influential than the mother, who is globally more influential than the son, as shown in Example 1. In the next subsection, we answer the question of whether the Banzhaf value strictly reflects the global influence relation.

4.2. The Banzhaf value only weakly reflects global influence

Does a globally more influential voter always have a greater Banzhaf value in all monotonic multi-choice games? The answer is "no" in general, especially if the number of individual approval levels is more than three. We illustrate this point in the following example.

Table 1
Voting rule.

x	$V(x)$
(1, 1)	0
(2, 1)	0
(3, 1)	0
(4, 1)	0
(1, 2)	0
(1, 3)	0
(1, 4)	0
(2, 2)	0
(3, 2)	1
(4, 2)	1
(2, 3)	0
(2, 4)	1
(3, 3)	1
(4, 3)	1
(3, 4)	1
(4, 4)	1

Example 2. A monotonic multi-choice game (N, T, V) involves two players and four approval levels. Its voting rule is defined in Table 1. It can be easily checked that 1 is strictly globally more influential than 2: $1 \succ 2$. But both players have the same Banzhaf value: $\beta_1(V) = \beta_2(V) = \frac{1}{2}$. To see this, we note that 1 is $(1, 0)$-critical in $(3, 2)$, $(2, 4)$ and $(3, 3)$, and 2 is $(1, 0)$-critical in $(3, 2)$, $(4, 2)$ and $(2, 4)$. Therefore, the Banzhaf score of 1 and 2 is 3, from which it follows that the Banzhaf value is $\frac{1}{2}$ for each.

It follows from Example 2 that two players who differ in terms of their global influence may have the same Banzhaf value in a linear organization. However, we show below that two equally influential players always have the same Banzhaf value.

Proposition 2. *Let (N, T, V) be a monotonic multi-choice game, and $p, q \in N$ two players. Then,*

$$p \sim q \Longrightarrow \beta_p(V) = \beta_q(V).$$

This result implies that the global influence relation and the Banzhaf value coincide for anonymous multi-choice political rules, as players always play interchangeable roles under such rules [29]. We also show that the Banzhaf value is a weakly increasing function of global influence.

Proposition 3. *Let (N, T, V) be a monotonic multi-choice game. Then,*

$$\forall p, q \in N, \quad p \succeq q \Longrightarrow \beta_p(V) \geq \beta_q(V).$$

Propositions 2 and 3 are encouraging, but the possibility of two unequally influential players having the same Banzhaf value is puzzling. Our goal is to characterize voting rules under which the global influence relation is ordinally equivalent to the Banzhaf value. We carry out this analysis in the next section.

5. Ordinal equivalence of power theories

In this section, we study the rank-order equivalence of the global influence relation, the Banzhaf power index, and the Shapley–Shubik power index. We provide necessary and/or sufficient conditions for ordinal equivalence to obtain.

5.1. A sufficient condition for the ordinal equivalence of power theories

We provide a sufficient condition for the global influence relation and the Banzhaf power index to coincide. This condition is inspired by Pongou, Tchantcho and Tedjeugang [37] where it is shown to be sufficient for the global influence relation and the generalized Shapley value proposed by Freixas [35] to be ordinally equivalent. Although the latter result was obtained in the context of market organizations, we reinterpret it in the context of voting in this section. These results therefore allow us to identify a large class of multi-choice political rules for which the global influence relation, the Banzhaf value, and the Shapley–Shubik value yield the same rankings of voters.

The underlying condition of our partial characterization result is introduced in the definition below.

Definition 3. Let (N, T, V) be a monotonic multi-choice game, and $p, q \in N$ two players.

(1) Let $x, y \in T^N$ be two vote profiles. y is said to harmonize the views of p and q in x if:

$$\forall z \in N \setminus \{p, q\}, \quad x_z = y_z \quad \text{and} \quad y_p = y_q.$$

We denote by $T_x^N(p, q)$ the set of all vote profiles which harmonize the views of p and q in x.

(2) (N, T, V) is said to satisfy condition C_1 if for any vote profile x such that $x_p = x_q = r$:

$$\forall s \neq r, \quad V(x + (s - r)e^p) \neq V(x + (s - r)e^q)$$
$$\Longrightarrow \begin{cases} \exists y \in T_x^N(p, q) \quad \text{such that,} \\ V(y + (j - y_p)e^p) \neq V(y + (j - y_q)e^q) \quad \text{or} \\ V(y + (1 - y_p)e^p) \neq V(y + (1 - y_q)e^q). \end{cases}$$

Condition C_1 identifies voting rules that discriminate among voters at the top level or at the bottom level of approval. Denote by $\succeq_B$ and $\succeq_S$ the preorders induced on the set of voters by the Banzhaf and Shapley–Shubik values, respectively. The following result in [37] states that in linear monotonic multi-choice games satisfying condition C_1, the global influence relation and the preorder induced by the Shapley–Shubik value coincide.

Theorem 1. *Let (N, T, V) be a linear monotonic multi-choice game satisfying condition C_1. $\succeq$ and $\succeq_S$ coincide.*

We show below that global influence is strictly reflected by the Banzhaf value in monotonic multi-choice games satisfying condition C_1.

Proposition 4. *Let (N, T, V) be a monotonic multi-choice game satisfying condition C_1, and $p, q \in N$ two players. Then,*

$$\forall p, q \in N, p \succ q \Longrightarrow \beta_p(V) > \beta_q(V).$$

The following result states that condition C_1 is a sufficient condition for the global influence relation and the preorder induced by the Banzhaf value to ordinally coincide.

Theorem 2. *Let (N, T, V) be a linear monotonic multi-choice game satisfying condition C_1. $\succeq$ and $\succeq_B$ coincide.*

A straightforward implication of Theorems 1 and 2 is that the global influence relation and the Banzhaf and Shapley–Shubik preorderings coincide in any linear monotonic multi-choice game satisfying condition C_1.

Theorem 3. *Let (N, T, V) be a linear monotonic multi-choice game satisfying condition C_1. $\succeq$, $\succeq_B$ and $\succeq_S$ coincide.*

A corollary of Theorem 3 is that under voting rules with at most three levels of individual approval and any possible number of collective approval levels, $\succeq$, $\succeq_B$ and $\succeq_S$ coincide. This is because all such voting rules automatically satisfy condition C_1, as stated below.

Corollary 1. *Let (N, T, V) be a linear monotonic multi-choice game with at most three levels of individual approval and any possible number of collective approval levels. Then, $\succeq$, $\succeq_B$ and $\succeq_S$ coincide.*

We note that the ordinal equivalence of the influence relation and the Banzhaf and Shapley–Shubik preorderings obtained in [18,11] for linear $(2, 2)$ voting rules, and in Tchantcho et al. [31] and Parker [38] for linear $(3, 2)$ voting rules are special cases of Corollary 1. Indeed, linear $(2, 2)$ voting rules are a special case of linear monotonic multi-choice games with two levels of individual approval. Also, Tchantcho et al. [31] show that ordinal equivalence obtains for the class of linear weakly equitable $(3, 2)$ voting rules. Parker [38] extends this result to the entire class of linear $(3, 2)$ voting rules. According to Corollary 1, ordinal equivalence obtains for the entire class of $(2, k)$ and $(3, k)$ voting rules where k is any natural number greater than or equal to 2. Corollary 1 is therefore a more general result. Furthermore, Theorem 3 provides a general characterization of multi-choice political rules for which ordinal equivalence obtains even when the number of individual approval levels allowed by these rules is strictly greater than 3.

5.2. Two necessary and sufficient conditions for ordinal equivalence

In this section, we provide two necessary and sufficient conditions for ordinal equivalence. The first condition is a condition on political rules for the global influence relation and the Banzhaf preordering to coincide. The second condition is a condition on the number of individual approval levels for the global influence relation and the Banzhaf and Shapley–Shubik preorderings to coincide for all rules allowing that number of individual approval levels.

The first condition defines political rules under which whenever a voter who, by increasing his support for a social option, has a greater impact than another voter who does the same still has a greater impact if he had increased his support from the bottom level of approval to the top level of approval. This condition is formalized below.

Definition 4. Let (N, T, V) be a monotonic multi-choice game. (N, T, V) is said to satisfy condition C_2 if for any players p and q, and any vote profile x such that $x_p = x_q = s$:

$$\forall r > s, \quad V(x + (r - s)e^p) > V(x + (r - s)e^q)$$
$$\Longrightarrow \begin{cases} \exists y \in T_{pq}^N \quad \text{such that,} \\ V(y + (j - y_p)e^p) - V(y + (1 - y_p)e^p) \\ \quad > V(y + (j - y_q)e^q) - V(y + (1 - y_q)e^q). \end{cases}$$

We have the following characterization result:

Theorem 4. *Let (N, T, V) be a linear monotonic multi-choice game. $\succeq$ and $\succeq_B$ coincide if and only if (N, T, V) satisfies C_2.*

A natural extension of this analysis would be to determine whether condition C_2 is a necessary and sufficient condition for the Shapley–Shubik value and the global influence relation to have the same ordinal structure.

We state below our second characterization result. This result is inspired by Example 2, an illustration of a linear monotonic multi-choice game allowing four individual approval levels for which the influence relation and the Banzhaf and Shapley–Shubik preorderings do not have the same ordinal structure. In general, we show that the ordinal equivalence of these three power theories obtains for all voting rules allowing a fixed number of input approval levels j if and only if $j \leq 3$.

Theorem 5. *$\succeq$, $\succeq_B$ and $\succeq_S$ coincide for all linear monotonic multi-choice games allowing a fixed number of individual approval levels j if and only if $j \leq 3$.*

6. Conclusion

Which voting power theory should we use to evaluate the ability of individuals to affect the outcome of a vote in an organization? Although each theory gives a different evaluation of power in general, our study shows that the generalizations of three classical power concepts – the global influence relation, the Banzhaf power index, and the Shapley–Shubik power index – ordinally coincide for a large class of multi-choice organizations and political rules. The analysis therefore implies that one can avoid the debate over the appropriateness of each particular theory if the voting rule of an organization is chosen from this class, as different power theories would yield the same power relationships among its members.

Two characterizations of voting rules for which ordinal equivalence obtains are given, and it is shown that all voting rules that allow up to three levels of individual approval and any possible number of collective approval levels satisfy the condition underlying these characterizations. Although our findings generalize all the previous results on the ordinal equivalence of the classical voting power theories, they also show that the condition of linearity found to be necessary and sufficient for ordinal equivalence to obtain when voters have at most three options to choose from is no longer sufficient when they can choose from a list of four or more options.

Acknowledgments

We thank three anonymous referees for very useful comments and suggestions. Roland Pongou acknowledges a research grant from the University of Ottawa. This paper was written when Bertrand Tchantcho was visiting the Department of Economics at the University of Ottawa. The hospitality of this institution is gratefully acknowledged.

Appendix. Proofs

Proof of Proposition 1. Let $G = (N, T, V)$ be a multi-choice game, and $p, q \in N$ two players. We know that:

$$\eta_p(V) = \frac{1}{j}\sum_{x\in T^N}(V(x + (j - x_p)e^p) - V(x + (1 - x_p)e^p)).$$

But $T^N = \bigcup_{\overline{X}\in N_p^j}\overline{X}$, and there are exactly j^{n-1} equivalence classes and j vote profiles in each class. Moreover, each class is uniquely associated with a vote profile of $N\setminus\{p\}$. It follows that:

$$\eta_p(V) = \frac{1}{j}\sum_{\overline{X}\in N_p^j}\sum_{x\in\overline{X}}(V(x + (j - x_p)e^p) - V(x + (1 - x_p)e^p)).$$

Since for all $x, y \in \overline{X}$,

$$V(x + (j - x_p)e^p) - V(x + (1 - x_p)e^p) \\ = V(y + (j - y_p)e^p) - V(y + (1 - y_p)e^p),$$

we have:

$$\sum_{x\in\overline{X}}(V(x + (j - x_p)e^p) - V(x + (1 - x_p)e^p)) \\ = j(V(z + (j - z_p)e^p) - V(z + (1 - z_p)e^p))$$

for any representative vote profile z in $\overline{X}$.

Finally,

$$\eta_p(V) = \frac{1}{j}\sum_{\overline{z}\in N_p^j} j(V(z + (j - z_p)e^p) - V(z + (1 - z_p)e^p)) \\ = \sum_{\overline{z}\in N_p^j}(V(z + (j - z_p)e^p) - V(z + (1 - z_p)e^p)).$$

As noted earlier, each class is uniquely associated with a vote profile of $N\setminus\{p\}$. Therefore, given a vote profile x of $N\setminus\{p\}$, by considering the representative x_{pq}, which is a vote profile of N in which p and q have the same level of approval and the other players have the same level of approval as in x, we can write:

$$\eta_p(V) = \sum_{x\in T^{N\setminus\{p\}}}(V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p))$$

where $q \in N\setminus\{p\}$. ■

Proof of Proposition 2. If $p \sim q$, then p and q are interchangeable in any vote profile, hence p and q have the same Banzhaf value. ■

Proof of Proposition 3. Let (N, T, V) be a multi-choice game, and p and q two players such that $p \succeq q$.

$p \succeq q$ implies that for all vote profiles x such that $x_p = x_q = s$, and for all $r \in \{1, 2, \ldots, j\}$, the following holds:

$$\begin{cases}\text{if } r < s, \quad \text{then } V(x + (r - s)e^p) \le V(x + (r - s)e^q);\\ \text{and if } r > s, \quad \text{then } V(x + (r - s)e^p) \ge V(x + (r - s)e^q).\end{cases}$$

Thus, for $r = 1$ and $r = j$, respectively, $V(x + (1 - s)e^p) \le V(x + (1 - s)e^q)$ and $V(x + (j - s)e^p) \ge V(x + (j - s)e^q)$ for any vote profile x such that $x_p = x_q$.

We also have:

$$\begin{cases}\eta_p(V) = \sum_{x\in T^{N\setminus\{p\}}}(V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p))\\ \text{and}\\ \eta_q(V) = \sum_{x\in j^{N\setminus\{q\}}}(V(x_{pq} + (j - x_p)e^q) - V(x_{pq} + (1 - x_p)e^q)).\end{cases}$$

Note that $T^{N\setminus\{p\}}$ and $T^{N\setminus\{q\}}$ have exactly j^{n-1} elements each. There is a natural bijection b from $T^{N\setminus\{p\}}$ to $T^{N\setminus\{q\}}$ that maps each $x \in T^{N\setminus\{p\}}$ into the corresponding vote profile $b(x) \in T^{N\setminus\{q\}}$ in which q is merely replaced with p thus implying that $x_q = b(x)_p$. It is obvious that $x_{pq} = b(x)_{pq}$, which implies that:

$$\begin{cases}\eta_p(V) = \sum_{x\in T^{N\setminus\{p\}}}(V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p))\\ \text{and}\\ \eta_q(V) = \sum_{x\in T^{N\setminus\{p\}}}(V(b(x)_{pq} + (j - b(x)_p)e^q) - V(b(x)_{pq}\\ \quad + (1 - b(x)_p)e^q))\\ \quad = \sum_{x\in T^{N\setminus\{p\}}}(V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q)).\end{cases}$$

Now consider a vote profile x in $T^{N\setminus\{p\}}$; then x_{pq} is the corresponding vote profile in T^N in which p and q have the same level of approval.

Therefore:

$$V(x_{pq} + (j - x_q)e^p) \ge V(x_{pq} + (j - x_q)e^q) \quad \text{and} \\ V(x_{pq} + (1 - x_q)e^p) \le V(x_{pq} + (1 - x_q)e^q);$$

which implies:

$$V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p) \\ \ge V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q).$$

Given the fact that this inequality holds for any vote profile x of $N\setminus\{p\}$, we have:

$$\begin{cases}\sum_{x\in T^{N\setminus\{p\}}}(V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p))\\ \quad \ge \sum_{x\in T^{N\setminus\{p\}}}(V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q));\end{cases}$$

which in turn implies $\eta_p(V) \ge \eta_q(V)$ and $\beta_p(V) \ge \beta_q(V)$. ■

Proof of Proposition 4. Let $p, q \in N$ such that $p \succ q$. It suffices to prove that $\eta_p(V) > \eta_q(V)$.

It is the case that $p \succ q \Longrightarrow p \succeq q$ and not $(q \succeq p)$. But, not $(q \succeq p)$ implies:

$$\begin{cases} \exists x \in T^N,\ x_p = x_q = s \in \{2, \ldots, , j\},\ \exists r > s \quad \text{such that} \\ V(x + (r-s)e^p) = v_l > v_m = V(x) \quad \text{and} \\ V(x + (r-s)e^q) < v_l. \end{cases}$$

This in turn implies $V(x + (r-s)e^q) < V(x + (r-s)e^p)$.

We have $V(x+(r-s)e^q) \neq V(x+(r-s)e^p)$. Given that condition C_1 is satisfied, it is the case that there exists $y \in T_x^N(p, q)$ such that $V(y + (1 - y_p)e^p) \neq V(y + (1 - y_q)e^q)$ or $V(y + (j - y_p)e^p) \neq V(y + (j - y_q)e^q)$.

Without loss of generality, suppose that $V(y + (j - y_p)e^p) \neq V(y + (j - y_q)e^q)$. It follows from $p \succeq q$ that:

$$V(y + (1 - y_p)e^p) \le V(y + (1 - y_q)e^q) \quad \text{and}$$
$$V(y + (j - y_p)e^p) > V(y + (j - y_q)e^q).$$

Following Proposition 3, $p \succeq q$ also implies that for all $z \in T^{N \setminus \{p\}}$, $V(z_{pq} + (j - z_q)e^p) \ge V(z_{pq} + (j - z_q)e^q)$ and $V(z_{pq} + (1 - z_q)e^p) \le V(z_{pq} + (1 - z_q)e^q)$.

This implies:

$$\begin{cases} \sum_{z \in T^{N \setminus \{p\}}} (V(z_{pq} + (j - z_q)e^p) - V(z_{pq} + (1 - z_q)e^p)) \\ \quad \ge \sum_{z \in T^{N \setminus \{p\}}} (V(z_{pq} + (j - z_q)e^q) - V(z_{pq} + (1 - z_q)e^q)). \end{cases} \tag{1}$$

Because the vote profile y considered above belongs to $T_x^N(p, q)$, there exists another vote profile $z \in T^{N \setminus \{p\}}$ such that $y = z_{pq}$. And since $V(y + (j - y_p)e^p) > V(y + (j - y_q)e^q)$ and $V(y + (1 - y_p)e^p) \le V(y + (1 - y_q)e^q)$, we have:

$$V(y + (j - y_p)e^p) - V(y + (1 - y_p)e^p)$$
$$\quad > V(y + (j - y_q)e^q) - V(y + (1 - y_q)e^q). \tag{2}$$

From (1) and (2), we deduce:

$$\begin{cases} \sum_{z \in T^{N \setminus \{p\}}} (V(z_{pq} + (j - z_q)e^p) - V(z_{pq} + (1 - z_q)e^p)) \\ \quad > \sum_{z \in T^{N \setminus \{p\}}} (V(z_{pq} + (j - z_q)e^q) - V(z_{pq} + (1 - z_q)e^q)), \end{cases}$$

that is, $\eta_p(V) > \eta_q(V)$. ■

Proof of Theorem 2. Considering Propositions 1, 2 and 4 and the fact that $\succeq$ is complete, it is obvious that $\succeq$ and $\succeq_B$ coincide. ■

Proof of Corollary 1. Given Theorem 2, it suffices to show that any linear monotonic multi-choice game (N, T, V) such that $|T| = 2$ or $|T| = 3$ satisfies condition C_1.

(1) The proof is straightforward for the case of $|T| = 2$.

(2) Now assume that $|T| = 3$.

Let $x \in T^N$ and $p, q \in N$ such that $x_p = x_q = s$.

Let $r \neq s$ such that $V(x + (r-s)e^p) \neq V(x + (r-s)e^q)$. Find $y \in T_x^N(p, q)$ such that $V(y + (1 - y_p)e^p) \neq V(y + (1 - y_q)e^q)$ or $V(y + (j - y_p)e^p) \neq V(y + (j - y_q)e^q)$.

Case 1: Suppose $s = 1$ and $r \in \{2, 3\}$.

If $r = 2$, then $V(x + e^p) \neq V(x + e^q)$. Set $y \in T^N$ such that $y_p = y_q = 2$; $y_i = x_i$ for any $i \in N \setminus \{p, q\}$. We have $y \in T_x^N(p, q)$, $y - e^p \neq x + e^q$; $y - e^q = x + e^p$. Given that $V(x + e^p) \neq V(x + e^q)$, it follows that $V(y - e^p) \neq V(x + e^q)$.

If $r = 3$, then $V(x + 2e^p) \neq V(x + 2e^q)$. We can take $y = x$.

Case 2: Suppose $s = 2$ and $r \in \{1, 3\}$.

If $r = 1$, then $V(x - e^p) \neq V(x - e^q)$, and we take $y = x$.

If $r = 3$, then $V(x + e^p) \neq V(x + e^q)$, and we take $y = x$.

Case 3: Suppose that $s = 3$ and $r \in \{1, 2\}$.

If $r = 1$, then we take $y = x$.

If $r = 2$, then set $y \in T^N$ such that $y_p = y_q = 2$ and $y_i = x_i$ for any $i \in N \setminus \{p, q\}$. We have $y \in T_x^N(p, q)$, $y + e^p = x - e^q$, and $y + e^q = x - e^p$. Given that $V(x - e^q) \neq V(x - e^p)$, it follows that $V(y + e^p) \neq V(y + e^q)$. We conclude that any linear monotonic multi-choice game (N, T, V) such that $|T| = 3$ satisfies condition C_1. ■

Proof of Theorem 4. (1) Let us show that if for any multi-choice game (N, T, V), $[\forall p, q \in N, p \succeq q \Leftrightarrow \beta_p(V) \ge \beta_q(V)]$, then (N, T, V) satisfies condition C_2. Assume by contradiction that (N, T, V) does not satisfy C_2. Then, there exist a vote profile x, two players p and q such that $x_p = x_q = s$, and $r > s$ such that $V(x+(r-s)e^p) > V(x+(r-s)e^q)$ and $V(y+(j-y_p)e^p) - V(y+(1-y_p)e^p) \le V(y + (j - y_q)e^q) - V(y + (1 - y_q)e^q)$ for all $y \in T_{pq}^N$ (**).

Since (N, T, V) is linear, either $p \succeq q$ or $q \succeq p$. But thanks to the vote profile x (of N) for which $x_p = x_q = s$, $r > s$ and $V(x + (r-s)e^p) > V(x + (r-s)e^q)$, $q \succeq p$ is impossible, thus $p \succeq q$. At the same time, the existence of x above implies $p \succ q$ and therefore $\beta_p(V) \ge \beta_q(V)$.

As shown in Proposition 1, we have:

$$\begin{cases} \eta_p(V) = \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p)) \\ \text{and} \\ \eta_q(V) = \sum_{x \in T^{N \setminus \{p\}}} (V(b(x)_{pq} + (j - b(x)_p)e^q) - V(b(x)_{pq} \\ \quad + (1 - b(x)_p)e^q)) \\ \quad = \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q)); \end{cases}$$

and since $p \succeq q$, for all $x \in T^{N \setminus \{p\}}$,

$$V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p)$$
$$\quad \ge V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q).$$

This implies:

$$\begin{cases} \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p)) \\ \ge \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q)). \end{cases}$$

Now, since condition C_2 is not satisfied, (**) implies:

$$\forall y \in T_{pq}^N, \quad V(y + (j - y_p)e^p) - V(y + (1 - y_p)e^p)$$
$$\quad \le V(y + (j - y_q)e^q) - V(y + (1 - y_q)e^q).$$

That is,

$$\forall x \in T^{N \setminus \{p\}}, \quad V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p)$$
$$\quad \le V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q).$$

Summing up over $T^{N \setminus \{p\}}$, we get:

$$\sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq} + (j - x_q)e^p) - V(x_{pq} + (1 - x_q)e^p))$$
$$\quad \le \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq} + (j - x_q)e^q) - V(x_{pq} + (1 - x_q)e^q)),$$

which implies that $\eta_p(V) \le \eta_p(V)$. This clearly shows that $\beta_p(V) = \beta_q(V)$.

Finally, since p and q are such that $p \succ q$ and $\beta_p(V) = \beta_q(V)$, the equivalence $[\forall p, q \in N, p \succeq q \Leftrightarrow \beta_p(V) \ge \beta_q(V)]$ is not true (in fact, the implication $[\forall p, q \in N, \beta_p(V) \ge \beta_q(V) \Rightarrow p \succeq q]$ is not true).

(2) Conversely, let us assume that (N, T, V) satisfies condition C_2. Let $p, q \in N$ be two players.

We know from Proposition 3 that $p \succeq q \Rightarrow \beta_p(V) \geq \beta_q(V)$.

Now assume that $not(p \succeq q)$ and show that $not(\beta_p(V) \geq \beta_q(V))$, that is, $\beta_p(V) < \beta_q(V)$.

$not\ (p \succeq q)$ implies that $q \succ p$. Also,

$$q \succ p \Rightarrow \begin{cases} q \succeq p \quad \text{and} \\ \exists y \in T^N : y_p = y_q = s, \exists r > s : V(y+(r-s)e^q) \\ \quad > V(y+(r-s)e^p) \end{cases}$$

$$\Rightarrow \begin{cases} \forall x \in T^N_{pq}, V(x+(j-x_q)e^q) - V(x+(1-x_q)e^q) \\ \quad \geq V(x+(j-x_p)e^p) - V(x+(1-x_p)e^p) \\ \text{and} \\ \exists y \in T^N_{pq} : V(y+(j-y_q)e^q) - V(y+(1-y_q)e^q) \\ \quad > V(y+(j-y_p)e^p) - V(y+(1-y_p)e^p) \end{cases}$$

$$(\text{thanks to } C_2) \Rightarrow \begin{cases} \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq}+(j-x_q)e^q) \\ \quad - V(x_{pq}+(1-x_q)e^q)) \\ \geq \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq}+(j-x_q)e^p) \\ \quad - V(x_{pq}+(1-x_q)e^p)) \\ \text{and} \\ \exists y \in T^{N \setminus \{p\}} : V(y_{pq}+(j-y_q)e^q) \\ \quad - V(y_{pq}+(1-y_q)e^q) > \\ V(y_{pq}+(j-y_q)e^p) - V(y_{pq}+(1-y_q)e^p) \end{cases}$$

$$(\text{thanks to } C_2) \Rightarrow \begin{cases} \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq}+(j-x_q)e^q) \\ \quad - V(x_{pq}+(1-x_q)e^q)) \\ > \sum_{x \in T^{N \setminus \{p\}}} (V(x_{pq}+(j-x_q)e^p) \\ \quad - V(x_{pq}+(1-x_q)e^p)) \end{cases}$$

$$\Rightarrow \beta_q(V) > \beta_p(V). \quad \blacksquare$$

Proof of Theorem 5. ($\Longleftarrow$) Let (N, T, V) be a linear monotonic multi-choice game such that $|T| = 2$ or $|T| = 3$. The fact that $\succeq$, $\succeq_B$ and $\succeq_S$ coincide simply follows from Corollary 1.

($\Longrightarrow$) It suffices to show that of all linear monotonic multi-choice games allowing a fixed number of input approval levels $j > 3$, there exists one for which $\succeq$, $\succeq_B$ and $\succeq_S$ do not coincide.

If $j = 4$, then the proof follows from Example 2 where $1 \succ 2$, but the Banzhaf value is $\frac{1}{2}$ for each of the two players, implying that $1 \sim_B 2$. It can also be shown that the Shapley–Shubik value is $\frac{1}{2}$ for each of the two players, implying that $1 \sim_S 2$.

If $j > 4$, let $N = \{1, 2\}$ and the voting rule be defined by $V(j-1, j-2) = 1$; $V(x) = 0$ if $x \leq (j-1, j-2)$ and $x \neq (j-1, j-2)$; $V(x) = 0$ if x is such that $x_2 < j-2$; and $V(x) = 1$ if x is such that $(j-1, j-2) \leq x$. As in Example 2, we can show that $1 \succ 2$, but that the Banzhaf and the Shapley–Shubik value is $\frac{1}{2}$ for each of the two players, implying that $1 \sim_B 2$ and $1 \sim_S 2$. $\blacksquare$

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