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Instanton algebras and quantum 4-spheres

Ludwik Dąbrowski [a,*], Giovanni Landi [b]

[a] *Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2-4, 34014 Trieste, Italy*
[b] *Dipartimento di Scienze Matematiche, Università di Trieste, Via Valerio 12/b, 34127 Trieste, Italy*

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Abstract

We study some generalized instanton algebras which are required to describe 'instantonic complex rank 2 bundles'. The spaces on which the bundles are defined are not prescribed from the beginning but rather are obtained from some natural requirements on the instantons. They turn out to be quantum 4-spheres S^4_q, with $q \in \mathbb{C}$, and the instantons are described by self-adjoint idempotents e. We shall also clarify some issues related to the vanishing of the first Chern–Connes class $ch_1(e)$ and on the use of the second Chern–Connes class $ch_2(e)$ as a volume form.

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1. Introduction

Recently there has been an intense activity on noncommutative [9] and quantum 4-spheres [3,4,10,13] and instanton bundles over them.

In this paper, by generalizing the methods presented in [7,9], we search for quantum instantons. Paralleling the classical situation [2,11,14], by this we just mean a complex rank 2 bundle, i.e., we require that the 0th Chern–Connes class vanishes, on some 'four-dimensional space' and with not trivial characteristic classes. Weakening the assumptions made in [7,9], we do not require from the beginning that the 1st Chern–Connes class of the bundle vanishes as well.

As we shall see, the spaces on which the instanton bundle are defined are not prescribed from the beginning but rather come *a posteriori*. We could say that first comes the bundle and then the space on

* Corresponding author.
E-mail addresses: dabrow@sissa.it (L. Dąbrowski), landi@dsm.univ.trieste.it (G. Landi).

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which the bundle is defined. While our procedure is completely general and could be used to produce other quantum spaces, in this paper the resulting spaces will be quantum 4-sphere S_q^4, with $q \in \mathbb{C}$. The quantum instantons will be described by self-adjoint idempotents $e \in \mathrm{Mat}_4(A_q)$, with A_q the noncommutative algebra of functions associated with (in fact, defining) the spheres S_q^4. These spheres and instantons interpolate between analogous objects recently found in [9] (for q such that $|q| = 1$) and [10] (for $q \in \mathbb{R}$).

We shall also clarify some issues related to the vanishing of the 1st Chern–Connes class and the use of the 2nd Chern–Connes class as a 'volume form'. It turns out that the first Chern–Connes class $ch_1(e)$ does vanish if and only if the deformation parameter q is such that $|q| = 1$. In contrast, the second Chern–Connes class $ch_2(e)$ does not vanish for any values of q. The couple $(ch_1(e), ch_2(e))$ defines a cycle in the reduced (b, B) bicomplex of cyclic homology and $ch_2(e)$ is closed, that is $b\,ch_2(e) = 0$, if and only if q is such that $|q| = 1$. It is only in the latter cases that the class $ch_2(e)$ is 'q-antisymmetric' and can be used as a volume form [9].

In the final section we shall make some remarks on alternative definitions of spheres.

2. The instanton projections

Consider first the free *-algebra with unity $F = \mathbb{C}[[\mathbb{I}, \alpha_j, \beta_j, \alpha_j^*, \beta_j^*\colon\ j = 1, 2, 3]]$ generated by elements α_j, β_j and their adjoints α_j^*, β_j^*. Then, take the following self-adjoint element $e = e^*$ in the algebra $\mathrm{Mat}_4(F) \simeq \mathrm{Mat}_4(\mathbb{C}) \otimes F$,

$$e = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^* & Q_3 \end{pmatrix}. \tag{1}$$

Each of the Q_j's is assumed to be a 2×2 matrix of 'generalized quaternions' that is,

$$Q_j = \begin{pmatrix} \alpha_j & \beta_j \\ -q\beta_j^* & \pi\alpha_j^* \end{pmatrix}, \quad j = 1, 2, 3, \tag{2}$$

and q and π are complex parameters for the time being. Being e self-adjoint also requires $Q_1 = Q_1^*$, $Q_3 = Q_3^*$, from which it follows that the parameter π must, in fact, be real, that $\alpha_1 = \alpha_1^*$ and $\alpha_3 = \alpha_3^*$, and that $\beta_1 = \beta_3 = 0$ (unless $q = -1$ which, for simplicity, we shall not consider here).

The next requirement we make is that e be of rank 2. This we implement by requiring that its 0th Chern–Connes class $ch_0(e)$ vanish (see later for the definition of ch_0),

$$ch_0(e) = \left\langle \left(e - \frac{1}{2} \right) \right\rangle = 0. \tag{3}$$

As a consequence we get that

$$(1 + \pi)(\alpha_1 + \alpha_3) = 2, \tag{4}$$

which says that $\pi \neq -1$ and relates α_3 with α_1. Summing up (and denoting α_1 by t, α_2 by α and β_2 by β) up to now we have that

$$Q_1 = \begin{pmatrix} t & 0 \\ 0 & \pi t \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} \frac{2}{1+\pi} - t & 0 \\ 0 & \pi\left(\frac{2}{1+\pi} - t\right) \end{pmatrix},$$

$$Q_2 = Q = \begin{pmatrix} \alpha & \beta \\ -q\beta^* & \pi\alpha^* \end{pmatrix}, \qquad Q_2^* = Q^* = \begin{pmatrix} \alpha^* & -\bar{q}\beta \\ \beta^* & \pi\alpha \end{pmatrix}, \tag{5}$$

with $t = t^*$ (remember that $1 + \pi \neq 0$).

Finally we require that e is idempotent as well, that is $e^2 = e$.

One of the consequences is that

$$\begin{pmatrix} t & 0 \\ 0 & \pi t \end{pmatrix} Q + Q \begin{pmatrix} \frac{2}{1+\pi} - t & 0 \\ 0 & \pi\left(\frac{2}{1+\pi} - t\right) \end{pmatrix} = Q. \tag{6}$$

From the diagonal elements it follows that

$$t\alpha - \alpha t = \frac{\pi - 1}{\pi + 1}\alpha, \qquad t\alpha^* - \alpha^* t = \frac{1-\pi}{(\pi+1)\pi}\alpha^*, \tag{7}$$

the consistency of which requires that t commutes with α and α^*. Excluding the case $\alpha = 0$ we also get that $\pi = 1$. Then, the off-diagonal elements in (6) imply that t commutes with β and β^* as well.

From $e^2 = e$ it also follows that

$$QQ^* + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}^2 - \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = 0,$$

$$Q^*Q + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}^2 - \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = 0. \tag{8}$$

These constraints require that

$$\beta\alpha = \bar{q}\alpha\beta, \qquad \beta^*\alpha = q\alpha\beta^*, \tag{9}$$

$$\alpha^*\alpha + |q|^2\beta^*\beta + t^2 - t = 0, \qquad \alpha\alpha^* + \beta\beta^* + t^2 - t = 0, \tag{10}$$

$$\alpha^*\alpha + |q|^2\beta\beta^* + t^2 - t = 0, \qquad \alpha\alpha^* + \beta^*\beta + t^2 - t = 0. \tag{11}$$

Eqs. (10) and (11) in turn imply that

$$\left(|q|^2 - 1\right)(\beta\beta^* - \beta^*\beta) = 0. \tag{12}$$

Then, if $|q|^2 \neq 1$, it follows that β must commute with β^*. On the other hand, if $|q|^2 = 1$, from Eqs. (10) and (11) it follows directly that $\beta\beta^* = \beta^*\beta$ (and also $\alpha^*\alpha = \alpha\alpha^*$ in this case).

3. The algebra of the quantum spheres S_q^4

By slightly changing notations again, i.e., denote $t = \frac{1}{2}(\mathbb{I} - z)$ and replace $\alpha \to \frac{1}{2}\alpha$, $\beta \to \frac{1}{2}\beta$, the construction of the previous section amounts to the following. With $q \in \mathbb{C} \setminus \{0\}$, we consider first the free *-algebra with unity $F_q = \mathbb{C}[[\mathbb{I}, \alpha, \beta, z, \alpha^*, \beta^*, z^*]]$ generated by three elements α, β and z (and their adjoints α^*, β^*, z^*). Then, we take the following element e in the algebra $\mathrm{Mat}_4(F_q) \simeq \mathrm{Mat}_4(\mathbb{C}) \otimes F_q$,

$$e = \frac{1}{2}\begin{pmatrix} \mathbb{I} + z, & 0, & \alpha, & \beta \\ 0, & \mathbb{I} + z, & -q\beta^*, & \alpha^* \\ \alpha^*, & -\bar{q}\beta, & \mathbb{I} - z, & 0 \\ \beta^*, & \alpha, & 0, & \mathbb{I} - z \end{pmatrix}. \tag{13}$$

By construction e is self-adjoint, that is $e = e^*$. The algebra A_q of the quantum 4-sphere S_q^4 is thus the quotient of the free *-algebra F_q which results by requiring that e is idempotent as well, that is $e^2 = e$.

This is equivalent to the requirement that the generators satisfy the relations (and their $*$-adjoints)

$$\begin{aligned}
&z = z^*, \qquad z\alpha = \alpha z, \qquad z\beta = \beta z, \\
&\beta\alpha = \bar{q}\alpha\beta, \qquad \beta^*\alpha = q\alpha\beta^*, \qquad \beta\beta^* = \beta^*\beta, \\
&\alpha^*\alpha + |q|^2\beta^*\beta + z^2 = \mathbb{I}, \qquad \alpha\alpha^* + \beta\beta^* + z^2 = \mathbb{I}.
\end{aligned} \tag{14}$$

Thus the algebra A_q we are looking for is the unital *-algebra generated by the elements α, β and z satisfying the relations (14). Here the deformation parameter q could be restricted so that $|q| \in (0, 1]$; for q, such that $|q| > 1$, the transformation $q \mapsto 1/q$, $\alpha \mapsto \alpha^*$, $\beta \mapsto -q\beta$ and $z \mapsto z$ yields an isomorphic sphere.

By restricting to $q = \exp(2\pi \mathrm{i}\theta)$ we get the sphere S^4_θ introduced in [9]; while for $q \in \mathbb{R}$ the present S^4_q is the same as the one introduced in [10].

When $q = 1$ the algebra of the sphere S^4_q is commutative and coincides with the algebra of continuous functions on the 4-dimensional sphere S^4. Thus S^4_q provides a deformation of the classical sphere S^4.

The algebra A_q can be made into a C^*-algebra in the usual way. For $a \in F_q$ one defines $\|a\|$ as the supremum, over all representations π of F_q in $B(H)$ that are *admissible*, in the sense that the operators $\pi(\alpha)$, $\pi(\beta)$, $\pi(z)$ satisfy the relations (14), of the operator norms $\|\pi(a)\|$. Then $\mathcal{J} := \{a \in F_q\colon |a| = 0\}$ is a two-sided ideal and one obtains a C^*-norm on $F_q/\mathcal{J}$. The completion of this quotient algebra defines a C^*-algebra, which we shall denote by the same symbol A_q.

By using relations (14) it can be seen that the elements $a_{kmn\ell}$, with $k \in \mathbb{Z}$ and m, n, ℓ nonnegative integers, of the form

$$a_{kmn\ell} = \begin{cases} \alpha^{*k}\beta^{*m}\beta^n z^\ell & \text{for } k = 0, 1, 2, \ldots, \\ \alpha^{-k}\beta^{*m}\beta^n z^\ell & \text{for } k = -1, -2, \ldots \end{cases} \tag{15}$$

provides a linear basis for A_q.

We note that for the generic situation when $0 < |q| < 1$ any character χ, besides

$$\chi(\alpha^*) = \overline{\chi(\alpha)}, \qquad \chi(\beta^*) = \overline{\chi(\beta)}, \qquad \chi(z^*) = \chi(z), \tag{16}$$

has to satisfy the equations

$$\chi(\beta) = 0 \quad \text{and} \quad \bigl|\chi(\alpha)\bigr|^2 + \bigl(\chi(z)\bigr)^2 = 1. \tag{17}$$

Thus the space of all (nonzero) characters, which can be thought of as the space of ‘classical points’ of S^4_q, is homeomorphic to the 2-dimensional sphere S^2.

Next, we describe infinite dimensional irreducible representations of the algebra A_q (for $0 < |q| < 1$) in $B(H)$, the algebra of bounded operators on a Hilbert space H. Let $\{\psi_n,\ n = 0, 1, 2, \ldots\}$ be an orthonormal basis for the Hilbert space H. With $\zeta \in \mathbb{C}$, $|\zeta| \leqslant 1$, we get two families of representations $\pi_{\zeta,\pm}: A_q \to B(H)$ given by

$$\begin{aligned}
&\pi_{\zeta,\pm}(z)\psi_n = \pi_{\zeta,\pm}(z^*)\psi_n = \pm\sqrt{1 - |\zeta|^2}\,\psi_n, \\
&\pi_{\zeta,\pm}(\alpha)\psi_n = \zeta\sqrt{1 - |q|^{2(n+1)}}\,\psi_{n+1}, \qquad \pi_{\zeta,\pm}(\alpha^*)\psi_n = \bar{\zeta}\sqrt{1 - |q|^{2n}}\,\psi_{n-1}, \\
&\pi_{\zeta,\pm}(\beta)\psi_n = \zeta\bar{q}^n\psi_n, \qquad \pi_{\zeta,\pm}(\beta^*)\psi_n = \bar{\zeta}q^n\psi_n.
\end{aligned} \tag{18}$$

In fact, for ζ such that $|\zeta| = 1$, the two representations $\pi_{\zeta,+}$ and $\pi_{\zeta,-}$ are identical so that the representations are parametrized by points on a classical sphere S^2, similarly to what happens for one-dimensional representations (characters) as described before.

4. Chern–Connes classes

The self-adjoint idempotent e given by (13) is clearly an element in the matrix algebra $\mathrm{Mat}_4(A_q) \simeq \mathrm{Mat}_4(\mathbb{C}) \otimes A_q$. It naturally acts on the right free A_q-module $A_q^4 = A_q \otimes \mathbb{C}^4$, and one gets as its range a projective module of finite type which may be thought of as the module of 'sections of a vector bundle over S_q^4'. The module eA_q^4 is a deformation of the classical instanton bundle over S^4: for $q = 1$, the module eA_q^4 is the module of sections of the complex rank two, instanton bundle over S^4 [1].

We compute now the Chern–Connes Character of the module eA_q^4 for a generic value of the deformation parameter q. If $\langle\,\rangle$ is the projection on the commutant of 4×4 matrices, up to normalization the component of the (reduced) Chern–Connes Character are given by the formulae

$$ch_n(e) = \Big\langle \Big(e - \tfrac{1}{2}\Big) \underbrace{e \otimes \cdots \otimes e}_{2n} \Big\rangle, \tag{19}$$

and they are elements of the tensor product

$$A_q \otimes \underbrace{\overline{A}_q \otimes \cdots \otimes \overline{A}_q}_{2n}, \tag{20}$$

where $\overline{A}_q = A_q/\mathbb{C}\mathbb{I}$ is the quotient of the algebra A_q by the scalar multiples of the unit $\mathbb{I}$.

The crucial property of the components $ch_n(e)$ is that they define a *cycle* in the (b, B) bicomplex of cyclic homology [5,6,8,12],

$$B\, ch_n(e) = b\, ch_{n+1}(e). \tag{21}$$

The operator b is defined by

$$\begin{aligned} b(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = {} & \sum_{j=0}^{m-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_m \\ & + (-1)^m a_m a_0 \otimes a_1 \otimes \cdots \otimes a_{m-1} \end{aligned} \tag{22}$$

while the operator B is written as

$$B = AB_0, \tag{23}$$

where

$$B_0(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = \mathbb{I} \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_m, \tag{24}$$

$$A(a_0 \otimes a_1 \otimes \cdots \otimes a_m) = \frac{1}{m} \sum_{j=0}^{m} (-1)^{mj} a_j \otimes a_{j+1} \otimes \cdots \otimes a_{j-1}, \tag{25}$$

with the obvious cyclic identification $m + 1 = 0$. To be precise, in formulae (22), (24) and (25), all elements in the tensor products but the first one should be taken modulo complex multiples of the unit $\mathbb{I}$, that is one has to project onto $\overline{A}_q = A_q/\mathbb{C}\mathbb{I}$.

The fact that $ch_0(e) = 0$ has been imposed from the very beginning and was one of the conditions that lead to the projections (13). As already remarked, this could be interpreted as saying that the idempotent and the corresponding module (the 'vector bundle') has complex rank equal to 2.

Next one finds,

$$
\begin{aligned}
ch_1(e) &= \big\langle \big(e - \tfrac{1}{2}\big) \otimes e \otimes e \big\rangle \\
&= \tfrac{1}{8}\big(1 - |q|^2\big)\big\{z \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) + \beta^* \otimes (z \otimes \beta - \beta \otimes z) + \beta \otimes (\beta^* \otimes z - z \otimes \beta^*)\big\}.
\end{aligned} \tag{26}
$$

It is straightforward to check that

$$
b\, ch_1(e) = 0 = B\, ch_0(e). \tag{27}
$$

Furthermore, we have the following

Proposition 1. *Given the projections* (13) *its first Chern–Connes class vanishes, $ch_1(e) = 0$, if and only if the deformation parameter q is such that $|q| = 1$.*

This result matches the analogous one found in [9].

Finally,

$$
ch_2(e) = \big\langle \big(e - \tfrac{1}{2}\big) \otimes e \otimes e \otimes e \otimes e \big\rangle \tag{28}
$$

is the sum of five terms

$$
ch_2(e) = \tfrac{1}{32}(z \otimes c_z + \alpha \otimes c_\alpha + \alpha^* \otimes c_{\alpha^*} + \beta \otimes c_\beta + \beta^* \otimes c_{\beta^*}), \tag{29}
$$

with

$$
\begin{aligned}
c_z = {} & \big(1 - |q|^4\big)(\beta \otimes \beta^* \otimes \beta \otimes \beta^* - \beta^* \otimes \beta \otimes \beta^* \otimes \beta) \\
& + \big(1 - |q|^2\big)\big\{z \otimes z \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) + (\beta \otimes z \otimes z \otimes \beta^* - \beta^* \otimes z \otimes z \otimes \beta) \\
& \qquad + (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes z \otimes z + z \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes z \\
& \qquad - z \otimes (\beta \otimes z \otimes \beta^* - \beta^* \otimes z \otimes \beta) - (\beta \otimes z \otimes \beta^* - \beta^* \otimes z \otimes \beta) \otimes z\big\} \\
& + \big(\alpha \otimes \alpha^* - |q|^2 \alpha^* \otimes \alpha\big) \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) + (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes \big(\alpha \otimes \alpha^* - |q|^2 \alpha^* \otimes \alpha\big) \\
& + (\beta \otimes \alpha - \bar{q}\alpha \otimes \beta) \otimes (\alpha^* \otimes \beta^* - q\beta^* \otimes \alpha^*) + (\alpha^* \otimes \beta^* - q\beta^* \otimes \alpha^*) \otimes (\beta \otimes \alpha - \bar{q}\alpha \otimes \beta) \\
& + (\alpha^* \otimes \beta - \bar{q}\beta \otimes \alpha^*) \otimes (q\alpha \otimes \beta^* - \beta^* \otimes \alpha) + (q\alpha \otimes \beta^* - \beta^* \otimes \alpha) \otimes (\alpha^* \otimes \beta - \bar{q}\beta \otimes \alpha^*);
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
c_\alpha = {} & (z \otimes \alpha^* - \alpha^* \otimes z) \otimes (\beta^* \otimes \beta - \beta \otimes \beta^*) + |q|^2(\beta^* \otimes \beta - \beta \otimes \beta^*) \otimes (z \otimes \alpha^* - \alpha^* \otimes z) \\
& + \bar{q}(z \otimes \beta - \beta \otimes z) \otimes (\alpha^* \otimes \beta^* - q\beta^* \otimes \alpha^*) + (\alpha^* \otimes \beta^* - q\beta^* \otimes \alpha^*) \otimes (z \otimes \beta - \beta \otimes z) \\
& + q(\beta^* \otimes z - z \otimes \beta^*) \otimes (\alpha^* \otimes \beta - \bar{q}\beta \otimes \alpha^*) + (\alpha^* \otimes \beta - \bar{q}\beta \otimes \alpha^*) \otimes (\beta^* \otimes z - z \otimes \beta^*);
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
c_{\alpha^*} = {} & |q|^2(z \otimes \alpha - \alpha \otimes z) \otimes (\beta \otimes \beta^* - \beta^* \otimes \beta) + (\beta \otimes \beta^* - \beta^* \otimes \beta) \otimes (z \otimes \alpha - \alpha \otimes z) \\
& + (\beta^* \otimes z - z \otimes \beta^*) \otimes (\beta \otimes \alpha - \bar{q}\alpha \otimes \beta) + q(\beta \otimes \alpha - \bar{q}\alpha \otimes \beta) \otimes (\beta^* \otimes z - z \otimes \beta^*) \\
& + (z \otimes \beta - \beta \otimes z) \otimes (\beta^* \otimes \alpha - q\alpha \otimes \beta^*) + \bar{q}(\beta^* \otimes \alpha - q\alpha \otimes \beta^*) \otimes (z \otimes \beta - \beta \otimes z);
\end{aligned} \tag{32}
$$

$$\begin{aligned} c_\beta = {} & \big(1-|q|^4\big)\big[(\beta^*\otimes z - z\otimes\beta^*)\otimes\beta\otimes\beta^* + \beta^*\otimes\beta\otimes(\beta^*\otimes z - z\otimes\beta^*)\big] \\ & + \big(1-|q|^2\big)\big\{\beta^*\otimes z\otimes z\otimes z - z\otimes\beta^*\otimes z\otimes z + z\otimes z\otimes\beta^*\otimes z - z\otimes z\otimes z\otimes\beta^*\big\} \\ & + (\beta^*\otimes z - z\otimes\beta^*)\otimes\big(\alpha\otimes\alpha^* - |q|^2\alpha^*\otimes\alpha\big) + \big(\alpha\otimes\alpha^* - |q|^2\alpha^*\otimes\alpha\big)\otimes(\beta^*\otimes z - z\otimes\beta^*) \\ & + (\alpha\otimes z - z\otimes\alpha)\otimes(\alpha^*\otimes\beta^* - q\beta^*\otimes\alpha^*) + \bar q(\alpha^*\otimes\beta^* - q\beta^*\otimes\alpha^*)\otimes(\alpha\otimes z - z\otimes\alpha) \\ & + (\beta^*\otimes\alpha - q\alpha\otimes\beta^*)\otimes(\alpha^*\otimes z - z\otimes\alpha^*) + \bar q(\alpha^*\otimes z - z\otimes\alpha^*)\otimes(\beta^*\otimes\alpha - q\alpha\otimes\beta^*); \end{aligned} \tag{33}$$

$$\begin{aligned} c_{\beta^*} = {} & \big(1-|q|^4\big)\big[(z\otimes\beta - \beta\otimes z)\otimes\beta^*\otimes\beta + \beta\otimes\beta^*\otimes(z\otimes\beta - \beta\otimes z)\big] \\ & + \big(1-|q|^2\big)\big\{-\beta\otimes z\otimes z\otimes z + z\otimes\beta\otimes z\otimes z - z\otimes z\otimes\beta\otimes z + z\otimes z\otimes z\otimes\beta\big\} \\ & + (z\otimes\beta - \beta\otimes z)\otimes\big(\alpha\otimes\alpha^* - |q|^2\alpha^*\otimes\alpha\big) + \big(\alpha\otimes\alpha^* - |q|^2\alpha^*\otimes\alpha\big)\otimes(z\otimes\beta - \beta\otimes z) \\ & + q(z\otimes\alpha^* - \alpha^*\otimes z)\otimes(\beta\otimes\alpha - \bar q\alpha\otimes\beta) + (\beta\otimes\alpha - \bar q\alpha\otimes\beta)\otimes(z\otimes\alpha^* - \alpha^*\otimes z) \\ & + q(\alpha^*\otimes\beta - \bar q\beta\otimes\alpha^*)\otimes(z\otimes\alpha - \alpha\otimes z) + (z\otimes\alpha - \alpha\otimes z)\otimes(\alpha^*\otimes\beta - \bar q\beta\otimes\alpha^*). \end{aligned} \tag{34}$$

By using the relations (14) for our algebra, and remembering that we need to project on $\overline{A_q}$ in all terms of the tensor product but the first one, a long (one needs to compute 750 terms) but straightforward computation gives

$$\begin{aligned} b\,ch_2(e) = {} & \tfrac{1}{16}\big(1-|q|^2\big)\big\{\mathbb{I}\otimes z\otimes(\beta\otimes\beta^* - \beta^*\otimes\beta) + \mathbb{I}\otimes\beta\otimes(\beta^*\otimes z - z\otimes\beta^*) \\ & + \mathbb{I}\otimes\beta^*\otimes(z\otimes\beta - \beta\otimes z)\big\} \end{aligned} \tag{35}$$

and this is exactly equal to $B\,ch_1(e)$.

Again as in [9], we shall have the following

Proposition 2. *Given the projection in* (13), *the cycle $ch_2(e)$ is closed, i.e., $b\,ch_2(e) = 0$, if and only if the deformation parameter q is such that $|q| = 1$. Then, the resulting class $ch_2(e)$ is 'q-antisymmetric' and can be used as a 'volume form'.*

5. Final remarks

In the present paper, we have searched for a general algebra A for which the element $e \in \mathrm{Mat}_4(A)$ is an hermitian rank 2 projection of the form (1). As a result we have obtained a family of quantum 4-spheres S^4_q, $q \in \mathbb{C}$, which turn out to be a suspension of a family of quantum 3-spheres S^3_q. In the two special cases $q \in \mathbb{R}$ and $|q| = 1$, we obtain the two families found in [10] and [9], respectively. Moreover, our instanton projections e also specialize at the same time to the projections presented therein. This explains in which sense these two special families are related by analytic continuation of the deformation parameter.

While this work was being completed, there has been the papers [13] and its generalization [4], where other families $S^4_{q,\theta}$, $q,\theta \in \mathbb{R}$, and $S^4_{p,q,s}$, $|p| = 1$, $q,s \in \mathbb{R}$, of noncommutative 4-spheres have been introduced. In the appropriate limits (when $|q| = 1$) they exactly reduce to the quantum 4-sphere S^4_θ of [9]. However, the projections defined therein do not specialize correspondingly to those of [9].

Provided that we could perform the polar decomposition $\beta = |\beta|\,\mathrm{phase}(\beta)$ of β, the change of variables $q' \mapsto |q|$, $e^{2i\theta} \mapsto \bar q/|q|$, $\alpha' \mapsto \alpha$, $\beta' \mapsto |q|\ |\beta|$ and $U \mapsto z\,\mathrm{phase}(\beta)$, seems to suggest that the family $S^4_{q,\theta}$ (and then $S^4_{p,q,s}$) should not be too much different from our S^4_q. This is, however, not the case and these families are not equivalent as it is clear, e.g., from the fact that the related spaces of characters are different.

We also remark that the form of the projections e in [13] (and in [4]) is different from ours. The same holds for e' of [13] but e', being independent of phase(β), is clearly even nonequivalent. In fact e', rather than on a noncommutative 4-sphere, lives on a 3-sphere and thus, not surprisingly, corresponds to some trivial bundle (with vanishing $ch_k(e),\ k = 0, 1, 2$). (The same observations apply to $\tilde{e}$ in [4].)

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