Existence and uniqueness of solution in Sobolev space for an unsteady crystal growth problem with zero surface tension

Xuming Xie

Department of Mathematics, Morgan State University, Baltimore, MD 21251, United States

Abstract

We study an initial value problem for a two-dimensional dendritic crystal growth model with zero surface tension. If the initial data is in Sobolev space $H^2(\mathbb{R})$, it is proved that an unique local solution exists in proper Sobolev space.

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

Dendritic growth is one of the earliest and the most profound scientific problems in the area of interfacial pattern formation. This is not only due to its underlying vital technical importance in the material processing industries but also because dendritic growth represents a fascinating class example of nonlinear phenomena in nonequilibrium systems. The growth of a single crystal from an undercooled melt is a fundamental problem in material science. It is well known that a steadily moving front is unstable due to Mullins–Sekerka instability. Mullins–Sekerka instability leads to a wide variety of morphologies including compact shapes and dendrites. From a mathematical point of view, dendrite formation is a free boundary problem like the Stefan problem. This problem has been extensively studied based on various models. For review papers and books, we refer to Mullins and Sekerka [10], Langer [9], Kessler et al. [6], Pelce [11], Ball et al. [1] and Davis [3] and Xu [21].

The equations studied in this paper were derived in Kunka et al. [7,8] and Xie [18]. They are based on complex variable method which is a very effective technique in handling two-dimensional problems in fluid mechanics such as Hele–Shaw problem [4,12,14]. Based on this equations, Kunka et al. [7,8] studied the linear theory of localized disturbances and a class exact zero-surface-tension solutions if the initial conditions include only poles. They also studied the singular behavior of unsteady dendriticl crystal with surface tension. In those situations, a zero of the conformal map that describes the crystal gives birth to a daughter singularity that moves away from the zero and approaches the interface. For steady needle crystal, Xie [15,16] proved that in the limit of zero surface tension, these equations do not have any physically acceptable solutions when crystalline anisotropy is ignored even though the equations admit solutions (Ivantsov solutions [5]) when surface tension is zero. A discrete set of solution was found to exist when crystalline anisotropy is included. Linear stability of steady needle crystal was also studied [17]. Xie [18] established local existence and uniqueness of analytic solution for an unsteady crystal with zero surface tension if the initial data is analytic. In this paper, we study the existence and uniqueness of the same unsteady crystal problem in Sobolev space. We are going to use a similar method to that in Wu [13] for the vortex sheet problem.

✩ This work is supported by National Science Foundation grant DMS-0500642.

E-mail address: xuming.xie@morgan.edu.
2. Mathematical problem and notations

We are interested in the problem of a free dendrite growing in its undercooled melt. Temperature is measured in units of \( \frac{L}{c_p} \), where \( L \) is the latent heat, \( c_p \) the heat capacity. Lengths are measured in units of the tip radius of curvature \( a \) for the Ivantsov parabola which is the shape of Ivantsov solution [5]. \( U \) is the velocity of the advancing dendrite, \( D \) is the thermal diffusivity, \( P \) is the Peclet number defined by \( P = \frac{ua}{D} \). The dimensionless undercooling is defined as \( \Delta = \left( \frac{c_p}{L} \right) \left( T_m - T_\infty \right) \), where \( T_m \) is the melting temperature and \( T_\infty \) is the specific temperature an infinity.

If heat diffusion in the solid phase is neglected, in the frame where an Ivantsov parabolic interface would have been stationary, and then the dimensionless temperature \( T \) with the melting temperature subtracted, satisfies

\[
2P \frac{\partial T}{\partial t} = 2P \frac{\partial T}{\partial y} + \nabla^2 T. \tag{2.1}
\]

The condition at infinity that determines \( T \) for a specified undercooling is

\[
T \to -\Delta \quad \text{as } y \to \infty. \tag{2.2}
\]

The conservation of heat through the interface requires

\[
\frac{\partial T}{\partial n} = -2P(v_n + \cos \theta) \tag{2.3}
\]

where \( v_n \) is the normal component of the interface motion and \( \theta \) is the angle between the interface and \( y \) axis.

Since zero surface tension is assumed, we have on the free boundary:

\[
T = 0. \tag{2.4}
\]

We consider the conformal map \( z(\xi, t) \) with \( \xi = w + i s \) that maps the upper-half \( \xi \) plane into the exterior of the needle crystal in the \( z \) plane, where \( z = x + iy \). The real \( \xi \) axis \( s = 0 \) corresponds the unknown interface. It is clear that determination of the function \( z(\xi, t) \) yields the unknown interface. Under this transformation, (2.1) becomes

\[
2P|z_\xi|^2 \left[ \frac{\partial T}{\partial t} - \text{Re} \left( \frac{z_t}{z_\xi} \right) \frac{\partial T}{\partial w} - \text{Im} \left( \frac{z_t}{z_\xi} \right) \frac{\partial T}{\partial s} \right] = 2P \left( \text{Im}(z_\xi) \frac{\partial T}{\partial \xi} + \text{Re}(z_\xi) \frac{\partial T}{\partial \eta} \right) + \nabla^2 T, \tag{2.5}
\]

(2.2) becomes

\[
T \to -\Delta \quad \text{as } s \to \infty, \tag{2.6}
\]

(2.3) becomes

\[
\frac{\partial T}{\partial s} = -2P|z_\xi|^2 \text{Im} \left( \frac{z_t + i}{z_\xi} \right). \tag{2.7}
\]

The Ivantsov steady solution corresponds to

\[
z_I(\xi) = -i\xi^2/2 + \xi \tag{2.8}
\]

and

\[
T_I = -\Delta + \sqrt{\pi Pe} e^p \text{erfc} \left[ \sqrt{P}(1 + \eta) \right] \tag{2.9}
\]

where

\[
\Delta = \sqrt{\pi Pe} e^p \text{erfc} \left[ \sqrt{P} \right]. \tag{2.10}
\]

We then assume regular perturbation expansion

\[
T = PT_0 + O(P^2), \quad z(\xi, t) = O(P). \tag{2.11}
\]

Then, to \( O(P) \), (2.5)–(2.7) become:

\[
\nabla^2 T_0 = 0, \tag{2.12}
\]

\[
\frac{\partial T_0}{\partial s} = -2|z_\xi|^2 \text{Im} \left( \frac{z_t + i}{z_\xi} \right), \quad \text{for real } \xi, \tag{2.13}
\]

\[
T_0 = 0 \quad \text{for real } \xi. \tag{2.14}
\]

We further assume that

\[
T = T_I + o(P). \tag{2.15}
\]
Using (2.9), (2.10) and (2.12), we have
\[ \frac{\partial T_0}{\partial s} = -2. \]
Plugging into (2.13), we have
\[ \text{Im} \left( \frac{z_t + i}{z_t} \right) = \frac{1}{|z_t|^2}, \quad \text{for real } \xi, \] (2.15)
which implies from Plemelj formula (see [2])
\[ z_t + i = z_\xi \left[ H(Q)(\xi, t) + i Q[z](\xi, t) \right], \quad \text{for real } \xi, \] (2.16)
where
\[ H(Q)(\xi, t) = \frac{1}{\pi} \left( P \right) \int_{-\infty}^{\infty} \frac{Q[z](\xi', t)}{(\xi' - \xi)} d\xi'. \] (2.17)
is the Hilbert transform, and
\[ Q[z](\xi, t) = \frac{1}{|z_\xi|^2}. \] (2.18)
Where \( z_\xi \) is the partial derivative with respect to \( \xi \), \( z_t \) is the partial derivative with respect to \( t \).

The initial condition is
\[ z(\xi, 0) = z_0(\xi). \] (2.19)
We introduce
\[ \log z_\xi = \ln |z_\xi| + i \arg z_\xi = u(\xi) + i v(\xi), \] (2.20)
then
\[ Q[z] = Q[u] = \frac{1}{e^{2u}}. \] (2.21)
Taking derivative with respect to \( \xi \) in Eq. (2.16) and taking real part, we have
\[ u_t = u_\xi H(Q) - v_\xi Q + \partial_\xi \left[ H(Q) \right], \] (2.22)
where \( \partial_\xi \) means the partial derivative with respect to \( \xi \), and we will use \( \partial_\xi^2 = \partial_\xi \partial_\xi \) for the second derivative.

Since \( z(\xi, t) \) is conformal, \( z_\xi \neq 0 \); hence \( \log z_\xi \) is analytic in the upper-half plane \( \text{Im} \xi > 0 \). \( u \) and \( v \) are boundary values of real and imaginary part of \( \log z_\xi \) on real \( \xi \) axis, so (see [2])
\[ v = -H(u). \] (2.23)
The initial condition is
\[ u|_{t=0} = u_0. \] (2.24)

In this paper, \( H^k(R) \) denotes the Sobolev space over the real line \( R \), with the norm \( \|f\|_{H^k(R)} = \left( \int (1 + |\lambda|^2)^k |\hat{f}(\lambda)|^2 d\lambda \right)^{1/2} \), where \( \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int f(\xi) e^{-i\xi\lambda} d\xi \) is the Fourier transform of \( f(\xi) \). If \( s \) is a nonnegative integer \( k \), we also use \( \|f\|_{H^k(R)} = \sum_{j=0}^{\infty} \|\partial_\xi^j f\|_{L^2(R)} \), \( P(D) \) denotes the pseudodifferential operator defined by
\[ P(D) f(\xi, t) = \frac{1}{\sqrt{2\pi}} \int P(\lambda) \hat{f}(\lambda) e^{i\xi\lambda} d\lambda. \]
Note that \( |D| = H \partial_\xi = \partial_\xi H \).

We are going to prove the main theorem in this paper:

**Theorem 2.1.** Assume that \( u_0 \in H^2(R) \), \( M_0 = \|u_0\|_{H^2(R)} \), then there is \( T = T(M_0) > 0 \) and \( \delta > 0 \) such that if \( M_0 < \delta \), the initial value problem (2.22), (2.24) has a unique solution \( u \in C([0, T], H^{3/2}(R)) \cap C^1([0, T], H^{1/2}(R)) \cap L^2([0, T], H^2(R)) \).

**Remark.** We are going to prove the main theorem by first rewriting the equation as a quasi-linear equation, and then design an iteration scheme in appropriate Sobolev spaces. The smallness of the initial data in the above theorem is necessary for the contraction argument for the convergence of the iteration. It is possible to remove the smallness of the initial data using a different approach.
Lemma 2.2.
(1) For any \( u \in H^{1/2}(R) \), \( \|u\|_{L^4(R)} \leq K_0 \|D^{1/4}u\|_{L^2(R)} \).
(2) Let \( f \in H^1(R) \) and \( g \in H^{1/2}(R) \), then \( fg \in H^{1/2}(R) \) and \( \|fg\|_{H^{1/2}(R)} \leq K_0 \|f\|_{H^1(R)} \|g\|_{H^{1/2}(R)} \), where \( K_0 \) is some positive constant.

Proof. See Lemma 7.1 in [13].

Lemma 2.3. Let \( u \in H^s(R) \), then \( H(u) \in H^1(R) \) and \( \|H(u)\|_{H^1(R)} \leq \|u\|_{H^s(R)} \).

Proof. The lemma follows from the fact \( H(\hat{u}(\lambda)) = -i \text{sgn}(\lambda)u \), where \( \text{sgn}(\lambda) = 1 \) if \( \lambda > 0 \) and \( \text{sgn}(\lambda) = -1 \) if \( \lambda < 0 \).

The Sobolev embedding theorem implies

Lemma 2.4. Let \( u \in H^{3/2}(R) \), then \( u \in H^s(R) \cap C(R) \) for any \( s < \frac{3}{2} \), and \( \|u\|_{H^s(R)} \leq \|u\|_{H^{3/2}(R)} \). \( \|u\|_{L^\infty(R)} \leq K_1 \|u\|_{H^{3/2}(R)} \), where \( K_1 \) is a positive constant.

The following lemma is well known.

Lemma 2.5. Let \( f \in H^s(R) \), \( g \in H^s(R) \) with \( s > \frac{1}{2} \), then \( fg \in H^s(R) \) and \( \|fg\|_{H^s(R)} \leq K_2 \|f\|_{H^s(R)} \|g\|_{H^s(R)} \), where \( K_2 > 1 \) is a constant.

3. Proof of the main theorem

We define \( R_1[u] \) as
\[
R_1[u] = e^{-2u} - 1 + 2u. \tag{3.1}
\]
Linearizing \( Q[u] \) gives
\[
Q[u] = 1 - 2u + R_1[u], \tag{3.2}
\]
then Eq. (2.22) can be written as
\[
u_t + H(u_t) = N[u], \tag{3.3}
\]
where
\[
N[u] := -2u_t H(u) + u_t H(R_1) + H(\delta_t R_1) - 2uH(u_t) + R_1 H(u_t). \tag{3.4}
\]

Lemma 3.1.
(1) Let \( u \in H^{3/2}(R) \), then \( \|u^n\|_{H^{3/2}(R)} \leq nK_0K_2^{n-1}\|u\|_{H^{1/2}(R)}^{n-1}\|u\|_{H^{3/2}(R)} \) for positive integer \( n \).
(2) Let \( u \in H^{3/2}(R) \), then \( \|R_1\|_{H^{3/2}(R)} \leq 2K_0\|u\|_{H^{1/2}(R)}(e^{2K_2\|u\|_{H^{1/2}(R)}} - 1) \).

Proof. (1) \( \|u^n\|_{H^{3/2}(R)} \leq \|nu^n u^{n-1}\|_{H^{3/2}(R)} \leq nK_0\|u^n\|_{H^{1/2}(R)}\|u^{n-1}\|_{H^{3/2}(R)} \leq nK_0K_2^{n-1}\|u\|_{H^{1/2}(R)}^{n-1}\|u\|_{H^{3/2}(R)} \).

(2) \( \|R_1\|_{H^{3/2}(R)} \leq \sum_{n=2}^{\infty} \frac{|-2u|^n}{n!} \|\|_{H^{3/2}(R)} \leq \sum_{n=2}^{\infty} \frac{|-2u|^n}{n!} \|\|_{H^{3/2}(R)} \leq \frac{K_02^nK_2^{n-1}\|u\|_{H^1(R)}\|u\|_{H^{3/2}(R)}}{(n-1)!} \leq 2K_0\|u\|_{H^{3/2}(R)}(e^{2K_2\|u\|_{H^{1/2}(R)}} - 1). \)
Lemma 3.2. Let $T > 0, u \in C([0, T], H^{3/2}(R))$ and let $M$ be such that $\sup_{t \in [0, T]} \|u\|_{H^{3/2}(R)} \leq M$, then $N[u] \in C([0, T], H^{1/2}(R))$ and for every $t \in [0, T]$,

\[
\|N[u](\cdot, t)\|_{L^2(R)}^2 \leq C_1(M),
\]
\[
\|N[u](\cdot, t)\|_{H^{1/2}(R)} \leq C_2(M) \left( \|u(\cdot, t)\|_{H^1(R)} + 1 \right) \|u(\cdot, t)\|_{H^{3/2}(R)};
\]

where

\[
C_1(M) = 24K_1^2 M^4 + 12K_2^2 K_1^2 M^4 (e^{2K_1 M} - 1)^2 + 24(e^{2M K_1} - 1)^2 M^2,
\]
\[
C_2(M) = \max \{ 4K_0, 4K_0^2 K_1 (e^{2K_1 M} - 1), (e^{2K_1 M} - 1) \}.
\]

Proof. Applying Lemmas 2.2–2.4, we see that each term in (3.4) is in $C([0, T], H^{1/2}(R))$; therefore $N \in C([0, T], H^{1/2}(R))$. Using Schwarz inequality, we obtain

\[
\int N^2 \, d\xi \leq 12 \int u^2_\xi (H(u))^2 \, d\xi + 6 \int u^2_\xi (H(R_1))^2 \, d\xi + 6 \int (\partial_\xi R_1)^2 \, d\xi
\]

\[
\leq 12 \left( K_0 \|u\|_{H^{3/2}(R)} \right)^2 \int u^2_\xi \, d\xi + 6 \left( K_1 \|H(R_1)\|_{L^\infty(R)} \right)^2 \int u^2_\xi \, d\xi + 6 \int (\partial_\xi R_1)^2 \, d\xi
\]

From Sobolev embedding theorem, we have

\[
\|H(u)\|_{L^\infty(R)} \leq K_1 \|u\|_{H^{3/2}(R)} \leq K_1 \|u\|_{H^{1/2}(R)} \leq K_1 M;
\]
\[
\|H(R_1)\|_{L^\infty(R)} \leq K_1 \|H(\cdot)\|_{H^{1/2}(R)} \leq K_1 \|R_1\|_{H^{3/2}(R)} \leq 2K_1K_0 \|u\|_{H^{3/2}(R)} (e^{2\|u\|_{H^1(R)}} - 1) \leq 2K_1K_0 M (e^{2K_1 M} - 1);
\]
\[
\|u\|_{L^\infty(R)} \leq K_1 \|u\|_{H^{3/2}(R)} \leq K_1 M;
\]
\[
\|R_1\|_{L^\infty(R)} \leq K_1 \|R_1\|_{H^{3/2}(R)} \leq 2K_1K_0 \|u\|_{H^{3/2}(R)} (e^{2\|u\|_{H^1(R)}} - 1) \leq 2K_1K_0 M (e^{2K_1 M} - 1).
\]

From (3.1), we have $\partial_\xi R_1 \equiv 2(1 - e^{-2u})u_\xi$, so

\[
\int (\partial_\xi R_1)^2 \, d\xi = 4 \int (1 - e^{-2u})^2 u_\xi^2 \, d\xi \leq 4 \left( 1 - e^{-2u} \right)^2 \|u\|_{L^\infty(R)} \int u_\xi^2 \, d\xi \leq 4(e^{2K_1 M} - 1)^2 M^2.
\]

From the above and (3.9), we obtain (3.5) with $C_1(M)$ given by (3.7).

Now applying Lemma 2.2, we obtain

\[
\|N\|_{H^{1/2}(R)} \leq 2 \|u_\xi H(u)\|_{H^{1/2}(R)} + \|u_\xi H(R_1)\|_{H^{1/2}(R)} + \|H(\partial_\xi R_1)\|_{H^{1/2}(R)}
\]

\[
+ 2 \|uH(u_\xi)\|_{H^{1/2}(R)} + \|R_1 H(u_\xi)\|_{H^{1/2}(R)}
\]

\[
\leq 2K_0 \|u_\xi\|_{H^{3/2}(R)} \|H(u)\|_{H^1(R)} + K_0 \|u_\xi\|_{H^{1/2}(R)} \|H(\cdot)\|_{H^1(R)} + \|R_1\|_{H^{3/2}(R)}
\]

\[
+ 2K_0 \|u_\xi\|_{H^{1/2}(R)} \|H(u)\|_{H^1(R)} + K_0 \|R_1\|_{H^1(R)} \|u_\xi\|_{H^{1/2}(R)}
\]

\[
\leq 4K_0 \|u_\xi\|_{H^{3/2}(R)} \|u\|_{H^1(R)} + 2K_0 \|u\|_{H^{3/2}(R)} \|R_1\|_{H^1(R)} + \|R_1\|_{H^{3/2}(R)}
\]

\[
\leq 4K_0 \|u_\xi\|_{H^{3/2}(R)} \|u\|_{H^1(R)} + 2K_0 \|u\|_{H^{3/2}(R)} \|R_1\|_{H^1(R)} + \|R_1\|_{H^{3/2}(R)}
\]

\[
\leq 4K_0 \|u_\xi\|_{H^{3/2}(R)} \|u\|_{H^1(R)} + 4K_0^2 K_1 \|u\|_{H^{3/2}(R)} M (e^{2K_1 M} - 1) + 2K_0 \|u\|_{H^{3/2}(R)} (e^{2K_1 M} - 1)
\]

\[
\leq C_2(M) \left( \|u\|_{H^1(R)} + 1 \right) \|u\|_{H^{3/2}(R)}. \quad \square
\]

Lemma 3.3. If $u \in L^2([0, T], H^2(R)) \cap C([0, T], H^{3/2}(R))$, let $M$ be such that $\sup_{t \in [0, T]} \|u\|_{H^{3/2}(R)} \leq M$, then for a.e. $t \in [0, T]$.

\[
\|\partial_\xi N[u](\cdot, t)\|_{L^2(R)}^2 \leq \epsilon \int u_\xi^2(\xi, t) \, d\xi + \frac{C_3(M)}{\epsilon} \|u(\cdot, t)\|_{H^{3/2}(R)}^2 + C_4(M) \|u(\cdot, t)\|_{H^{3/2}(R)}^4,
\]

where $\epsilon$ is any positive constant,

\[
C_3(M) = 26K_1^2 + 2K_1 K_0 (e^{2M K_1} - 1)^2 + 144 \left( \frac{e^{2M K_1} - 1}{M} \right)^2 + 26K_1 + 4K_0^2 K_1^2 (e^{2MK_1} - 1)^2
\]
and
\[ C_4(M) = 52K_0^4K_1^4 + 24K_0^4(e^{2MK_1} - 1)^2 + 288K_0^4K_1^4e^{4K_1M} + 48K_0^4K_1^4. \]

**Proof.** Taking derivative in (3.4) gives
\[
\partial_t N = -2u_{t\xi} H(u) - 2u_{t\xi} H(u_{\xi}) + u_{t\xi} H(R_1) + u_{t\xi} H(\partial_t R_1) + H(\partial_t R_1) - 2u_{t\xi} H(u_{\xi}) - 2u H(u_{\xi}) + (\partial_t R_1)H(u_{\xi}) + R_1 H(u_{\xi}).
\] (3.11)

Applying Cauchy–Schwarz to (3.11), we obtain
\[
\begin{align*}
\int (\partial_t N)^2 \, d\xi &\leq 26 \int [u_{t\xi} H(u)]^2 \, d\xi + 26 \int [u_{t\xi} H(u_{\xi})]^2 \, d\xi + \int [u_{t\xi} H(R_1)]^2 \, d\xi \\
&\quad + 12 \int [u_{t\xi} H(\partial_t R_1)]^2 \, d\xi + 12 \int [H(\partial_t R_1)]^2 \, d\xi + 26 \int [u_{t\xi} H(u_{\xi})]^2 \, d\xi \\
&\quad + 26 \int [u_{t\xi} H(u_{\xi})]^2 \, d\xi + 12 \int (\partial_t R_1)H(u_{\xi}) \, d\xi + 12 \int R_1 H(u_{\xi}) \, d\xi.
\end{align*}
\] (3.12)

We are going to estimate each term in the right-hand side of (3.12).

The first term is estimated as
\[
\begin{align*}
\int [u_{t\xi} H(u)]^2 \, d\xi &\leq \|H(u)\|_{L^\infty(R)}^2 \int u_{t\xi}^2 \, d\xi \\
&\leq K_1\|H(u)\|_{H^{1/2}(R)}^2 \int u_{t\xi}^2 \, d\xi \leq K_1^2\|u\|_{H^{1/2}(R)}^2 \int u_{t\xi}^2 \, d\xi;
\end{align*}
\]
applying Lemma 2.2 to second term, we obtain
\[
\begin{align*}
\int [u_{t\xi} H(u_{\xi})]^2 \, d\xi &\leq \left( \int u_{t\xi}^4 \, d\xi \right)^{1/2} \left( \int [H(u_{\xi})]^4 \, d\xi \right)^{1/2} \\
&\leq K_0^4 \int [\|D\|^{1/4}u_{t\xi}]^2 \, d\xi \int [\|D\|^{1/4}H(u_{\xi})]^2 \, d\xi \leq K_0^4\|u\|_{H^{1/4}(R)}^4 \leq K_0^4K_1^4\|u\|_{H^{1/2}(R)}^4.
\end{align*}
\]

The third term is estimated as
\[
\int [u_{t\xi} H(R_1)]^2 \, d\xi \leq \|H(R_1)\|_{L^\infty(R)}^2 \int u_{t\xi}^2 \, d\xi \\
\leq K_1\|H(R_1)\|_{H^{1/2}(R)}^2 \int u_{t\xi}^2 \, d\xi \leq 2K_0K_1\|u\|_{H^{1/2}(R)}^2 (e^{2K_1M} - 1)^2 \int u_{t\xi}^2 \, d\xi;
\]
applying Lemma 2.2 to fourth term, we obtain
\[
\begin{align*}
\int [u_{t\xi} H(\partial_t R_1)]^2 \, d\xi &\leq \left( \int u_{t\xi}^4 \, d\xi \right)^{1/2} \left( \int [H(\partial_t R_1)]^4 \, d\xi \right)^{1/2} \\
&\leq K_0^4 \int [\|D\|^{1/4}u_{t\xi}]^2 \, d\xi \int [\|D\|^{1/4}H(\partial_t R_1)]^2 \, d\xi \leq K_0^4\|u\|_{H^{1/4}(R)}^2 \|H(R_1)\|_{H^{1/4}(R)}^2 \\
&\leq K_0^4K_1^4\|u\|_{H^{1/2}(R)}^2 \|R_1\|_{H^{1/2}(R)}^2 \leq 4K_0^4K_1^4\|u\|_{H^{1/2}(R)}^4 (e^{2K_1M} - 1)^2
\end{align*}
\]
and
\[
\begin{align*}
\int [H(\partial_t R_1)]^2 \, d\xi &= \int (\partial_t R_1)^2 \, d\xi = 4 \int (\partial_t \xi \xi (1 - e^{-2u}) + 2u_{t\xi} e^{-2u})^2 \, d\xi \\
&\leq 12 \int u_{t\xi}^2 (1 - e^{-2u})^2 \, d\xi + 24 \int u_{t\xi} e^{-4u} \, d\xi \\
&\leq 12 \|1 - e^{-2u}\|_{L^\infty(R)}^2 \int (\partial_t \xi \xi u) \, d\xi + 24 \|e^{-4u}\|_{L^\infty(R)}^2 \int u_{t\xi} \, d\xi \\
&\leq 12(e^{2MK_1} - 1)^2 \int (\partial_t \xi \xi u)^2 \, d\xi + 24K_0^4e^{4K_1M} \left( \int [\|D\|^{1/4}u_{t\xi}]^2 \, d\xi \right)^2 \\
&\leq 12(e^{2K_1M} - 1)^2 \int (\partial_t \xi \xi u)^2 \, d\xi + 24K_0^4e^{4K_1M} \|u\|_{H^{1/2}(R)}^4.
\end{align*}
\]
Now we can estimate the second and fifth terms of (3.15) as follows:

\[ \int [H(u \xi) u]^2 \, d\xi \leq \|u\|^2_{L^2(R)} \int [H(u \xi)]^2 \, d\xi \leq K_1 \|u\|^2_{H^{3/2}(R)} \int u_{\xi}^2 \, d\xi; \]

applying Lemma 2.2, we obtain

\[
\int [\partial_\xi R_1 H(u \xi)]^2 \, d\xi \leq \left( \int (\partial_\xi R_1)^4 \, d\xi \right)^{1/2} \left( \int [H(u \xi)]^4 \, d\xi \right)^{1/2} \\
\leq K_0^4 \int |D|^{1/4} \partial_\xi R_1 \, d\xi \int [D|^{1/4} H(u \xi)]^2 \, d\xi \leq K_0^4 \|u\|^2_{H^{3/4}(R)} \|R_1\|^2_{H^{3/4}(R)} \\
\leq K_0^4 K_1^4 \|u\|_{H^{3/2}(R)} \|R_1\|^2_{H^{3/2}(R)} \leq 4K_1^4 K_0^2 (e^{2K_1 M} - 1)^2 \|u\|^4_{H^{3/2}(R)};
\]

and

\[
\int [H(u \xi) R_1]^2 \, d\xi \leq \|R_1\|^2_{H^{3/2}(R)} \int [H(u \xi)]^2 \, d\xi \\
\leq K_2^2 \|R_1\|^2_{H^{3/2}(R)} \int u_{\xi}^2 \, d\xi \leq 4K_1^2 K_0^2 \|u\|_{H^{3/2}(R)} (e^{2K_1 M} - 1)^2 \int u_{\xi}^2 \, d\xi.
\]

Adding above inequality up and applying Young's inequality, we obtain the lemma. \( \square \)

**Lemma 3.4.** Assume that \( u^k \in L^\infty([0, T], H^{3/2}(R)), k = 1, 2, \) and \( \sup_{t \in [0, T]} \|u^k(\cdot, t)\|_{H^{3/2}(R)} \leq M \). Let \( N^k = N[u^k], R_1^k = e^{-2u^k} - 1 + 2u^k, \) then for a.e. \( t \in [0, T], \)

\[
\|N^1(\cdot, t) - N^2(\cdot, t)\|^2_{L^2(R)} \leq K_2 M^2 \|u^1(\cdot, t) - u^2(\cdot, t)\|_{H^1(R)}^2,
\]

where \( K_2 > 0 \) is a constant independent of \( M \).

**Proof.** From (3.4), we have

\[
N^1 - N^2 = -2(u_{\xi}^1 - u_{\xi}^2) H(u^1) - 2u_{\xi}^2 H(u^1 - u^2) + (u_{\xi}^1 - u_{\xi}^2) H(R_1^1) \\
+ u_{\xi}^1 H(R_1^1 - R_1^2) + H(\partial_\xi (R_1^1 - R_1^2)) - 2(u^1 - u^2) H(u_{\xi}^1) \\
- 2u_{\xi}^2 H(u_{\xi}^1 - u_{\xi}^2) + (R_1^1 - R_1^2) H(u_{\xi}^1) + R_1^2 H(u_{\xi}^1 - u_{\xi}^2).
\]

(3.14)

Applying Cauchy–Schwartz inequality to (3.14), we have

\[
\int (N^1 - N^2)^2 \, d\xi \leq 22 \int [(u_{\xi}^1 - u_{\xi}^2) H(u^1)]^2 \, d\xi + 22 \int [u_{\xi}^2 H(u^1 - u^2)]^2 \, d\xi + 13 \int [(u_{\xi}^1 - u_{\xi}^2) H(R_1^1)]^2 \, d\xi \\
+ 13 \int [u_{\xi}^1 H(R_1^1 - R_1^2)]^2 \, d\xi + 13 \int [H(\partial_\xi (R_1^1 - R_1^2))]^2 \, d\xi + 22 \int [(u^1 - u^2) H(u_{\xi}^1)]^2 \, d\xi \\
+ 22 \int [u_{\xi}^2 H(u_{\xi}^1 - u_{\xi}^2)]^2 \, d\xi + 13 \int [(R_1^1 - R_1^2) H(u_{\xi}^1)]^2 \, d\xi + 13 \int [R_1^2 H(u_{\xi}^1 - u_{\xi}^2)]^2 \, d\xi.
\]

(3.15)

Note that we can write

\[
R_1^1 - R_1^2 = e^{-2u^2} \left( \frac{e^{-2(u^1-u^2)} - 1}{u^1-u^2} \right) (u^1-u^2) + 2(u^1 - u^2),
\]

and

\[
\partial_\xi (R_1^1 - R_1^2) = -2u_{\xi}^1 e^{-2u^2} \left( \frac{e^{-2(u^1-u^2)} - 1}{u^1-u^2} \right) (u^1-u^2) - 2(u_{\xi}^1 - u_{\xi}^2)(e^{-2u^1} - 1).
\]

Now we can estimate the second and fifth terms of (3.15) as follows:

\[
\int [u_{\xi}^2 H(R_1^1 - R_1^2)]^2 \, d\xi \leq \|u_{\xi}^2\|^2_{L^2(R)} \int [H(R_1^1 - R_1^2)]^2 \, d\xi \leq \|u_{\xi}^2\|^2_{L^2(R)} \int [R_1^1 - R_1^2]^2 \, d\xi \\
\leq \|u_{\xi}^2\|^2_{L^2(R)} \left( \int e^{-4u^2} \left( \frac{e^{-2(u^1-u^2)} - 1}{u^1-u^2} \right)^2 (u^1 - u^2)^2 \, d\xi + 6 \int (u^1 - u^2)^2 \, d\xi \right) \\
= \|u_{\xi}^2\|^2_{L^2(R)} \left( \int e^{-4u^2} \left( \frac{e^{-2(u^1-u^2)} - 1}{u^1-u^2} \right)^2 \right) \|u_{\xi}^2\|^2_{L^2(R)} + 6 \int (u^1 - u^2)^2 \, d\xi.
\]
Lemma 3.6. See Lemma 7.6 and Lemma 7.7 in [13].

Proof. We use induction.

Proof of Theorem 2.1. Let \(f \in L^2([0, T], H^1(R))\) and \(u_0 \in H^{3/2}(R)\). If \(u \in L^\infty([0, T], H^{3/2}(R)) \cap C^1([0, T], H^1(R)) \cap L^2([0, T], H^2(R))\) (\(s\) is any real number) is a solution of
\[
u_t + |D|u = f, \quad u|_{t=0} = u_0,
\] then \(u \in C([0, T], H^{3/2}(R))\) and for every \(t \in [0, T]\) and \(j = 0, 1,\)
\[
\|u(\cdot, t)\|_{H^{3/2}(R)}^2 + \int_0^t e^{\beta(t-s)} \|\partial_x^j u(\cdot, s)\|_{L^2(R)}^2 ds 
\leq e^{\beta t} \|u_0(\cdot, t)\|_{H^{3/2}(R)}^2 + 3 \int_0^t e^{\beta(t-s)} \|f(\cdot, s)\|_{H^1(R)}^2 ds,
\]
where \(\beta\) is a positive constant.

(2) Let \(f \in L^2([0, T], H^1(R)) \cap C([0, T], H^{1/2}(R))\) and \(u_0 \in H^{3/2}(R)\), then the initial value problem (3.16) has a unique solution \(u \in C([0, T], H^{3/2}(R)) \cap C^1([0, T], H^{1/2}(R)) \cap L^2([0, T], H^2(R))\).

Proof. See Lemma 7.6 and Lemma 7.7 in [13].

Proof of Theorem 2.1. We construct a sequence of functions \(\{u^k\}\) by solving
\[
\partial_t u^{k+1} + |D|u^{k+1} = N^k, \quad u^{k+1}|_{t=0} = u_0, \quad k = 0, 1, 2, \ldots
\]
and \(u^0 = u_0, N^k = N[u^k]\). From Lemma 3.5, we have \(u^k \in C([0, T], H^{3/2}(R)) \cap C^1([0, T], H^{1/2}(R)) \cap L^2([0, T], H^2(R))\) for any \(T > 0\).

Let \(M_0 = \|u_0\|_{H^2(R)}\) and we are going to show

Lemma 3.6. There exists \(T = T(M_0) > 0\) such that for all \(k\),
\[
\sup_{t \in [0, T]} \|u^k(\cdot, t)\|_{H^{3/2}(R)}^2 \leq 8M_0^2, \quad \text{and} \quad \int_0^t \int |\partial_x^2 u^k(\xi, t)|^2 d\xi dt \leq 8M_0^2.
\]

Proof. We use induction. \(u^0 = u_0\) satisfies (3.19) obviously. Assume (3.19) hold for \(u^k\). From Lemma 3.5, we have
\[
\|u^{k+1}(\cdot, t)\|_{H^{3/2}(R)}^2 + \int_0^t e^{\beta(t-s)} \|\partial_x^2 u^{k+1}(\cdot, s)\|_{L^2(R)}^2 ds 
\leq e^{\beta t} \|u_0\|_{H^{3/2}(R)}^2 + 3 \int_0^t e^{\beta(t-s)} \|N^k\|_{H^1(R)}^2 ds 
\leq e^{\beta t} M_0^2 + 3e^{\beta t} \int_0^t \|N^k\|_{H^1(R)}^2 ds.
\]

and all \(L^\infty\) norms above can be bounded by \(K_1M\) from Sobolev embedding theorem.

The other terms in (3.15) can be estimated in the same fashion. \(\square\)
From Lemma 3.3, we have
\[
\int_0^t \|N^k\|_{H^1(R)}^2 \, ds \leq \frac{8}{\epsilon} C_3 M_0^2 T + \epsilon \int_0^t \left\| \partial_\xi u^k \right\|_{L^2(R)}^2 \, dt + 64 C_4 M_0^4 T. \tag{3.21}
\]

Now choose \( \epsilon = \frac{1}{4 C_4 M_0^4} \) in (3.21) and using (3.20), we have
\[
\sup_t \left\| u^{k+1}(\cdot, t) \right\|_{H^{3/2}(R)}^2 + \int_0^t \left\| \partial_\xi u^{k+1}(\cdot, s) \right\|_{L^2(R)}^2 \, ds \\
\leq e^{\beta T} M_0^2 + M_0^2 + 576 C_3 e^{2 \beta T} M_0^2 + 172 e^{\beta T} C_4 M_0^4 T. \tag{3.22}
\]

Now choose \( T \) so that
\[ e^{\beta T} \leq 3, 576 C_3 e^{2 \beta T} T \leq 2, 172 e^{\beta T} C_4 M_0^4 T \leq 2. \]

we have
\[
\sup_t \left\| u^{k+1}(\cdot, t) \right\|_{H^{3/2}(R)}^2 + \int_0^T \left\| \partial_\xi u^{k+1}(\cdot, s) \right\|_{L^2(R)}^2 \, ds \leq 8 M_0^2.
\]

Then the lemma follows from induction. \( \Box \)

Now we consider the convergence of \( u^k \). From (3.18), we obtain
\[
\partial_t (u^{k+1} - u^k) + |D|(u^{k+1} - u^k) = N^{k+1} - N^k,
\]
\[
(u^{k+1} - u^k)|_{t=0} = 0.
\]

Assume that \( M_0 \leq 1 \), applying Lemma 3.4 and Lemma 3.5, we obtain
\[
\left\| u^{k+1} - u^k \right\|_{H^{3/2}(R)}^2 + \int_0^T \left\| \partial_\xi (u^{k+1} - u^k) \right\|_{L^2(R)}^2 \, ds \\
\leq 3 e^{\beta T} \int_0^T \left\| N^k - N^{k-1} \right\|_{H^1(R)}^2 \, ds \leq 72 K_2 M_0^2 \int_0^T \left\| u^k - u^{k-1} \right\|_{H^1(R)}^2 \, ds. \tag{3.23}
\]

Now choose \( \delta = \min\{1, \frac{1}{\sqrt{2} K_2^2} \} \). If \( M_0 < \delta \), (3.23) implies that \( u^k \) is a Cauchy sequence in \( C([0, T], H^{1/2}(R)) \cap L^2([0, T], H^1(R)) \) and converges to a limit \( u \in C([0, T], H^{1/2}(R)) \cap L^2([0, T], H^1(R)). u \) is a solution of (3.3), and Eq. (3.3) implies \( u \in C^1([0, T], H^{-1}(R)). \) Since \( u^k \) is bounded in \( L^\infty([0, T], H^{3/2}(R)) \cap L^2([0, T], H^2(R)) \) from Lemma 3.6 and \( u \) is also the weak limit of \( u^k \) in \( L^\infty([0, T], H^{3/2}(R)) \cap L^2([0, T], H^2(R)) \), we have \( u \in C^1([0, T], H^{1/2}(R)) \cap L^2([0, T], H^2(R)). \) Lemma 3.5 and (3.3) imply that \( u \in C([0, T], H^{1/2}(R)) \cap C^1([0, T], H^{3/2}(R)). \) We complete the proof of Theorem 2.1. \( \Box \)

4. Conclusions

We have obtained local existence and uniqueness of solution in Sobolev space to a needle crystal growth problem with zero surface tension if the initial data is small in some Sobolev space. We remark that it is possible to remove the smallness restriction on the initial data in Theorem 2.1 using a different approach. The approach in this paper is generalized to deal with the needle crystal growth problem with nonzero surface tension [20]. It is an interesting open question to show linear and nonlinear instability of Ivantsov solution in appropriate Sobolev space.

Acknowledgments

The author thanks the referee for many helpful suggestions and comments. This work is supported by National Science Foundation under grant DMS-0500642.
References