On Some New Nonlinear Delay Integral Inequalities

Qing-Hua Ma

Department of Mathematics, Huizhou University, Huizhou City, Guangdong 516015,
People’s Republic of China

and

En-Hao Yang

Department of Mathematics, Jinan University, Guangzhou 510632,
People’s Republic of China

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1. INTRODUCTION

In his study of boundedness of solutions to linear second order differential equations, Ou-Iang [2] established and used the following useful nonlinear integral inequality:

**THEOREM A.** Let x and h be real-valued, nonnegative, and continuous functions defined on $R_+ = [0, +\infty)$ and let $c \geq 0$ be a real constant. Then the nonlinear integral inequality

$$x^2(t) \leq c^2 + 2\int_0^t h(s)x(s) \, ds, \quad t \in R_+$$

(A)
implies
\[ x(t) \leq c + \int_0^t h(s) \, ds, \quad t \in R_+. \]

As indicated by Pachpatte in [4] the last result has been frequently used by authors to obtain global existence, uniqueness, and stability of solutions of various nonlinear differential equations. The last theorem has also been extend by authors; see, for example, the references [3–5, 7].

Recently, Pachpatte [5] discussed the following delay integral inequalities, which generalise the last result, by means of the same argument as that used by Tsamatos and Ntouyas [6],

\[ x^2(t) \leq c^2 + 2 \int_0^t x(\sigma(s)) \{ f(s) W[ x(\sigma(s)) ] + h(s) \} \, ds, \quad t \in R_+ \]  

\[ x^2(t) \leq c^2 + 2 \int_0^t x(\sigma(s)) \{ f(s) \left( \int_0^\sigma g(\tau) W[ x(\sigma(\tau)) ] \, d\tau \right) + h(s) \} \, ds, \quad t \in R_+ \]  

\[ x^2(t) \leq c^2 + 2 \int_0^t x^2(\sigma(s)) \left\{ f(s) \left( \int_0^\sigma g(\tau) W[ \log x(\sigma(\tau)) ] \, d\tau \right) 
+ h(s) \right\} \, ds, \quad t \in R_+ \]  

with the initial condition
\[ x(t) = \psi(t), \quad t \in [a, 0], \]
\[ \psi(\sigma(t)) \leq c \text{ for } t \in R_+ \text{ with } \sigma(t) \leq 0. \]  

Pachpatte [5] proved the following

**Theorem B.** Let \( f, g, h \) and \( W \in C(R_+, R_+), \sigma \in C(R_+, R) \) with \( \sigma(t) \leq t \) and \( -\infty < a := \inf\{\sigma(t) : t \in R_+\} \leq 0, \) \( \psi \in C([a, 0], R_+), \) and \( x \in C([a, \infty], R_+). \) Further let \( W \) be nondecreasing and let \( W(u) > 0 \) hold for \( u > 0. \) Then

(I) from (B)–(D) we have
\[ x(t) \leq G^{-1} \left[ G \left( c + \int_0^t h(s) \, ds \right) + \int_0^t f(s) \, ds \right], \quad 0 \leq t \leq v_1; \]

(II) from (C)–(D) we have
\[ x(t) \leq G^{-1} \left[ G \left( c + \int_0^t h(s) \, ds \right) + \int_0^t f(s) \left( \int_0^\sigma g(\tau) \, d\tau \right) \, ds \right], \quad 0 \leq t \leq v_2; \]
(III) from (L)–(D) we have

\[
x(t) \leq \exp \left( G^{-1} \left[ G \left( \log c + \int_0^t h(s) \, ds \right) + \int_0^t f(s) \left( \int_0^s g(\tau) \, d\tau \right) \, ds \right] \right), \quad 0 \leq t \leq \upsilon_3;
\]

where

\[
G(u) := \int_{u_0}^u \frac{ds}{W(s)}, \quad u \geq u_0 > 0,
\]

\(G^{-1}\) denotes the inverse function of \(G\), and the positive numbers \(\upsilon_1, \upsilon_2, \text{ and } \upsilon_3\) are chosen so that the quantity in the square brackets of (I), (II), and (III) is in the range of \(G\).

The aim of this paper is to generalise the conclusions (I)–(III) of Theorem B. A delay integral inequality similar to inequality (L) is also discussed.

2. INTEGRAL INEQUALITIES WITH DELAY

Throughout, we define \(R = (-\infty, \infty), R_+ = (0, \infty), R_0 = (0, \infty), R_1 = [1, \infty)\), and denote by \(C(M, S)\) the class of all continuous functions defined on set \(M\) with range in the set \(S\). The basic assumption in the paper is as follows:

**Assumption (H).** (i) \(f, g, h, \text{ and } n \in C(R_+, R_+), \text{ with } n(t) \text{ nondecreasing.}\)

(ii) \(W \in C(R_+, R_+)\) is nondecreasing with \(W(u) > 0 \text{ for } u > 0\),

(iii) \(\sigma \in C(R_+, R), \quad \sigma(t) \leq t \text{ for } t \in R_+, \text{ with } -\infty < a := \inf\{\sigma(t) : t \in R_+\} \leq 0,\)

(iv) \(\psi \in C([a, 0], R_+)\) and \(x \in C([a, \infty), R_+),\)

(v) \(\phi \in C^1(R_+, R_+)\) with \(\phi'\) nondecreasing and \(\phi'(u) > 0 \text{ for } u > 0\).

Consider first the next generalization of inequality (B),

\[
\phi(x(t)) \leq n(t) + \int_0^t \phi' \left[ x(\sigma(s)) \right] \left[ f(s)W[x(\sigma(s))] + g(s)x(\sigma(s)) + h(s) \right] \, ds, \quad t \in R_+ \quad (1)
\]
with the initial condition
\[ x(t) = \psi(t), \quad t \in [a, 0] \]
\[ \psi(\sigma(t)) \leq \phi^{-1}(n(t)) \text{ for every } t \geq 0 \text{ with } \sigma(t) \leq 0 \]  \hspace{1cm} (D')

**Theorem 1.** Let Assumption (H) hold. Then inequality (1) with condition (D') implies

\[
x(t) \leq G^{-1}\left\{ G\left[ \exp \int_0^t g(s) \, ds \right] \left( \phi^{-1}(n(s)) + \int_0^t h(s) \, ds \right) \right. \\
+ \left. \left( \exp \int_0^t g(s) \, ds \right) \int_0^t f(s) \, ds \right\}, \\
0 \leq t \leq \alpha ,
\]

where \( G \) and \( G^{-1} \) are as defined in Theorem B and the positive number \( \alpha \) is chosen so that the quantity in the curly brackets of (2) is in the range of \( G \).

**Proof.** Let \( \varepsilon > 0 \) be an arbitrary small constant. Fixing any positive number \( T(\leq \alpha) \), we define a positive nondecreasing function \( u(t) \) by

\[
\phi(u(t)) = n(T) + \varepsilon + \int_0^t \phi'\left[ x(\sigma(s)) \right] \left\{ f(s) W[x(\sigma(s))] \right. \\
+ g(s) x(\sigma(s)) + h(s) \right\} ds, \\
t \in J = [0, \alpha].
\]

Then \( u(t) \geq \phi^{-1}(n(T) + \varepsilon) > 0 \) for \( t \in J \) and

\[
x(t) < u(t), \quad t \in J.
\]

Thus, for every \( t \geq 0 \) with \( \sigma(t) \geq 0 \), we have

\[
x(\sigma(t)) < u(\sigma(t)) \leq u(t)
\]

since \( u(t) \) is nondecreasing and \( \sigma(t) \leq t \). By condition (D'), for every \( t \geq 0 \) with \( \sigma(t) \leq 0 \) we have

\[
x(\sigma(t)) = \psi(\sigma(t)) \leq \phi^{-1}(n(t)) \leq \phi^{-1}(n(T)) \leq \phi^{-1}(n(\tau) + \varepsilon) \\
\leq u(t), \quad t \in J,
\]

since \( \phi^{-1} \) is nondecreasing. Hence we always have the relation

\[
x(\sigma(t)) \leq u(t), \quad t \in J.
\]
By differentiation, we derive from (3) that
\[
\phi'(u(t)) \frac{du}{dt} = \phi'(x(\sigma(t))) \left[ f(t)W(x(\sigma(t))) + g(t)x(\sigma(t)) + h(t) \right] \\
\leq \phi'(u(t)) \left[ f(t)W[u(t)] + g(t)u(t) + h(t) \right];
\]
i.e.,
\[
\frac{du}{dt} \leq f(t)W[u(t)] + g(t)u(t) + h(t)
\]
since \(u(t) > 0\) for \(t \in J\), \(\phi'\) is nondecreasing with \(\phi'(u) > 0\) for \(u > 0\), and (5) holds.

Integrating the both sides of the last inequality from 0 to \(t\), then we obtain
\[
u(t) \leq n_1(t) + \int_0^t g(s)u(s) \, ds, \quad t \in J,
\]
where
\[
n_1(t) = \phi^{-1}(n(T) + \varepsilon) + \int_0^T h(s) \, ds + \int_0^T f(s)W[u(s)] \, ds.
\]
From the last inequality and the well-known Gronwall inequality it follows that
\[
u(t) \leq \left[ \phi^{-1}(n(T) + \varepsilon) + \int_0^T h(s) \, ds + \int_0^T f(s)W[u(s)] \, ds \right] \\
\times \exp \int_0^T g(s) \, ds \\
\leq \left[ \phi^{-1}(n(T) + \varepsilon) + \int_0^T h(s) \, ds + \int_0^T f(s)W[u(s)] \, ds \right] \\
\times \exp \int_0^T g(s) \, ds, \quad t \in J.
\]
Setting
\[
v(t) := \left[ \phi^{-1}(n(T) + \varepsilon) + \int_0^T h(s) \, ds + \int_0^T f(s)W[u(s)] \, ds \right] \\
\times \exp \int_0^T g(s) \, ds, \quad t \in J,
\]
then by (6) we have
\[ u(t) \leq v(t), \quad t \in J. \]  

(8)

Differentiating (7) and using (8), we derive
\[ \frac{dv(t)}{dt} \leq \left( \exp \int_0^T g(s) \, ds \right) f(t) W[v(t)] \]

or
\[ dG[v(t)] = \frac{dv(t)}{W[v(t)]} \leq \left( \exp \int_0^T g(s) \, ds \right) f(t) \, dt, \quad t \in J, \]

since by (7) \( v(t) > 0 \) for \( t \in J \) and the condition (ii) in Assumption (H). Integrating the both sides of the last relation from 0 to \( t \), and in view of
\[ v(0) = \left[ \phi^{-1}(n(T) + \varepsilon) + \int_0^T h(s) \, ds \right] \times \exp \int_0^T g(s) \, ds \]  
from (7), we have
\[ \begin{align*} 
G[v(t)] & \leq G \left( \left( \exp \int_0^T g(s) \, ds \right) \left( \phi^{-1}(n(T) + \varepsilon) + \int_0^T h(s) \, ds \right) \right) \\
& \quad + \left( \exp \int_0^T g(s) \, ds \right) \int_0^T f(s) \, ds, \quad t \in J. 
\end{align*} \]

Taking \( t = T \) in the last inequality and then letting \( \varepsilon \to 0 \), we obtain
\[ \begin{align*} 
G[v(T)] & \leq G \left( \left( \exp \int_0^T g(s) \, ds \right) \left( \phi^{-1}(n(T)) + \int_0^T h(s) \, ds \right) \right) \\
& \quad + \left( \exp \int_0^T g(s) \, ds \right) \int_0^T f(s) \, ds. 
\end{align*} \]

Since \( T \in (0, \alpha] \) is arbitrary, from the last relation we have
\[ \begin{align*} 
G[v(t)] & \leq G \left[ \left( \exp \int_0^t g(s) \, ds \right) \left( \phi^{-1}(n(t)) + \int_0^t h(s) \, ds \right) \right] \\
& \quad + \left( \exp \int_0^t g(s) \, ds \right) \int_0^t f(s) \, ds, \quad t \in R_0. 
\end{align*} \]
or

\[ v(t) \leq G^{-1}\left\{ G\left[ \exp\int_0^t g(s) \, ds \left( \phi^{-1}(n(t)) + \int_0^t h(s) \, ds \right) \right] + \left( \exp\int_0^t g(s) \, ds \right) \int_0^t f(s) \, ds \right\}, \]

\[ 0 < t \leq \alpha. \]

Hence by (4), (8), and (9), we have

\[ x(t) \leq G^{-1}\left\{ G\left[ \exp\int_0^t g(s) \, ds \left( \phi^{-1}(n(t)) + \int_0^t h(s) \, ds \right) \right] + \left( \exp\int_0^t g(s) \, ds \right) \int_0^t f(s) \, ds \right\}, \]

\[ 0 < t \leq \alpha. \quad (10) \]

By (1), (10) holds also when \( t = 0. \) Q.E.D.

Now we consider the next generalization of inequality (C):

\[ \phi(x(t)) \leq n(t) + \int_0^t \phi'(x(\sigma(s))) \left( f(s) \left( \int_0^s g(\tau)W[x(\sigma(\tau))] \, d\tau \right) + h(s)x(\sigma(s)) + k(s) \right) \, ds, \]

\[ t \in R_+. \quad (11) \]

**THEOREM 2.** Let \( k(t) \in C(R_+, R_+) \) and Assumption (H) holds. Then inequality (11) with condition (D') implies

\[ x(t) \leq G^{-1}\left\{ G\left[ \exp\int_0^t h(s) \, ds \left( \phi^{-1}(n(t)) + \int_0^t k(s) \, ds \right) \right] + \left( \exp\int_0^t h(s) \, ds \right) \int_0^t f(s) \, ds \right\}, \]

\[ 0 < t \leq \beta, \quad (12) \]

where \( G \) and \( G^{-1} \) are as defined in Theorem B, and the positive number \( \beta \) is chosen so that the quantity in the curly brackets of (12) is in the range of \( G. \)
Proof. Fixing any positive number $T(\leq \beta)$ and taking an arbitrary positive small constant $\epsilon$, we define on interval $[0, T]$ a function $v(t)$ by

$$
\phi(v(t)) := n(T) + \epsilon + \int_0^t \phi'(x(\sigma(s))) f(s) \left( \int_0^s g(\tau) W[x(\sigma(\tau))] d\tau \right) + h(s) x(\sigma(s)) + k(s) ds,
$$

$$
t \in J. \quad (13)
$$

From (11) and (13) we observe

$$
x(t) \leq v(t), \quad t \in J. \quad (14)
$$

Using the same argument as used in the proof of Theorem 1, we obtain

$$
x(\sigma(t)) \leq v(t), \quad t \in J.
$$

Differentiating (13) and using the last relation we derive

$$
\frac{dv(t)}{dt} \leq k(t) + h(t) v(t) + f(t) \int_0^t g(\tau) W[v(t)] d\tau, \quad t \in J.
$$

Integrating the both sides of the last inequality from 0 to $t$, and using $v(0) = \phi^{-1}(n(T) + \epsilon)$ then we derive

$$
v(t) \leq \left[ \phi^{-1}(n(T) + \epsilon) + \int_0^t k(s) ds \right] + \int_0^t f(s) \left( \int_0^s g(\tau) W[v(\tau)] d\tau \right) ds + \int_0^t h(s) v(s) ds, \quad t \in J.
$$

Using the Gronwall inequality to the last inequality, we get

$$
u(t) \leq \left[ \phi^{-1}(n(T) + \epsilon) + \int_0^t k(s) ds \right] + \int_0^t f(s) \left( \int_0^s g(\tau) W[v(\tau)] d\tau \right) ds \exp \int_0^t h(s) ds
$$

$$
\leq H(T) \theta_x(T) + H(T) \int_0^t f(s) \left( \int_0^s g(\tau) W[v(\tau)] d\tau \right) ds,
$$

$$
t \in J, \quad (15)
$$

where $H(t) = \exp \int_0^t h(s) ds$, $\theta_x(t) = \phi^{-1}(n(T) + \epsilon) + \int_0^t k(s) ds$. 

Setting $X(t) := H(T)\theta(t) + H(T)[\int_0^t f(s)\{\int_0^s g(\tau)W[v(\tau)]d\tau\}ds$, by (15), we have
\begin{equation}
\nu(t) \leq X(t), \quad t \in J. \tag{16}
\end{equation}
Differentiating $X(t)$ and using (16) we obtain
\begin{align*}
\frac{dX(t)}{dt} &= H(T)f(t)\int_0^t g(\tau)W[\nu(\tau)]d\tau \\
&\leq H(T)f(t)\int_0^t g(\tau)W[X(\tau)]d\tau \\
&\leq H(T)f(t)\left(\int_0^t g(\tau) d\tau\right)W[X(t)], \quad t \in J.
\end{align*}
Because $X(t)$ is positive and $W(u) > 0$ for $u > 0$, the last relation can be rewritten in the form
\begin{equation*}
dG[X(t)] = \frac{dX(t)}{W[X(t)]} \leq H(T)f(t)\left(\int_0^t g(\tau) d\tau\right)dt, \quad t \in J.
\end{equation*}
Integrating the both sides of the last inequality from 0 to $t$, then we obtain
\begin{equation*}
G[X(t)] \leq G[H(T)\theta(T)] + H(T)\int_0^t f(s)\left(\int_0^s g(\tau) d\tau\right)ds, \quad t \in J.
\end{equation*}
Since $T \in (0, \beta]$, taking $t = T$ and letting $\varepsilon \to 0$ then we derive from the last inequality that
\begin{equation*}
X(T) \leq G^{-1}\left(G[H(T)\theta(T)] + H(T)\int_0^T f(s)\left(\int_0^s g(\tau) d\tau\right)ds\right).
\end{equation*}
Because $T$ is any number from $(0, \beta]$, by (14), (16), and the last inequality, we obtain the validity of (12) on $(0, \beta]$. By (11), inequality (12) holds also when $t = 0$. Q.E.D.

Now, we consider the following nonlinear delay inequality which is a variant of the inequality (L):
\begin{equation}
x'(t) \leq c' + \int_0^t x'(\sigma(s))\{f(s)x(\sigma(s)) + g(s)\}ds, \quad t \in R. \tag{17}
\end{equation}

**THEOREM 3.** Let $c > 0$, $q > 0$, $r > 0$ be constants, and $f$, $h$, $x$, and $\sigma$ are defined as in Theorem B. Then inequality (17) with condition (D) implies
\begin{equation}
x(t) \leq \left(\exp\int_0^t \frac{g(s)}{r}ds\right)\left\{\frac{1}{c''} - \int_0^t \frac{qf(s)}{r}\left(\exp\int_0^s \frac{g(\tau)}{r}d\tau\right)ds\right\}^{-1/q} \tag{18}
\end{equation}
for $0 \leq t \leq v$, where $v$ is a positive number satisfying
\[
\frac{1}{e^q} > \int_0^v \frac{qf(s)}{r} \left( \exp \int_0^s \frac{qg(t)}{r} \, dt \right) \, ds.
\]

Proof. Define
\[
w'(t) := c^t + \int_0^t x'(\sigma(s)) \{ f(s)x^q(\sigma(s)) + g(s) \} \, ds, \quad t \in R_+.
\]
(19)

By (17) we have
\[
x(t) \leq w(t), \quad t \in R_+.
\]
(20)

Applying the same argument as used in the proof of Theorem 1, we obtain
\[
x(\sigma(t)) \leq w(t), \quad t \in R_+.
\]

Differentiating (19) and using the last relation, we derive
\[
\frac{dw(t)}{dt} \leq \frac{g(t)}{r}w(t) + \frac{f(t)}{r}w^{1+q}(t), \quad t \in R_+.
\]
(21)

In view of $w(0) = c$, by a well-known comparison theorem for ODEs, from (21) we infer that
\[
w(t) \leq y(t) \quad \text{for } t \in I,
\]
(22)

where $I = (0, \rho)$ is the maximal existence interval of the solution $y(t)$ to the following initial value problem of the Bernoulli equation:
\[
\frac{dy(t)}{dt} = \frac{g(t)}{r}y + \frac{f(t)}{r}y^{1+q}, \quad t \in R_+, \quad y(0) = c.
\]

The unique solution of the last equation is
\[
y(t) = \left( \exp \int_0^t \frac{g(s)}{r} \, ds \right) \left( c^{-q} - \int_0^t \frac{qf(s)}{r} \left( \exp \int_0^s \frac{qg(t)}{r} \, dt \right) \, ds \right)^{-1/q},
\]

\[0 \leq t \leq v.
\]

Hence the desired inequality (18) follows from (20), (22), and the last relation immediately.

Q.E.D.
3. COROLLARIES AND REMARKS

Letting $\phi(u) = u^p$ $(p \geq 1)$ in Theorem 1 then we obtain the following

**COROLLARY 1.** Let $p \geq 1$ be a constant and Assumption (H) holds. Then the nonlinear delay inequality

$$x^p(t) \leq n(t) + \int_0^t x^{p-1}(\sigma(s)) \{f(s)W[x(\sigma(s))]$$

$$+ g(s)x(\sigma(s)) + h(s)\} ds, \quad t \in \mathbb{R}_+ \quad (23)$$

with condition (D') implies

$$x(t) \leq G^{-1}\left[G\left(\xi_p(t)\right) + \left(\exp\int_0^t \frac{g(s)}{p} ds\right)\int_0^t \frac{f(s)}{p} ds\right], \quad 0 \leq t \leq \gamma,$$

(24)

where $\xi_p(t) = (\exp\int_0^t \frac{f(s)}{p} ds)\frac{(n^1/p)(t) + \int_0^t f(s) ds}{p}$, and the positive number $\gamma$ is chosen so that the quantity in the curly brackets of (24) is in the range of $G$.

**Remark 1.** (i) In Corollary 1, letting $p = 2$, $n(t) = c^2$, $f(t) = 2a(t)$, $g(t) = 0$, and $h(t) = 2b(t)$ then it follows conclusion (I) of Theorem B. (ii) The special case of inequality (23) when $W(u) = u$, $g(t) = 0$, and $\sigma(t) = t$ was studied by Yang in [7].

**COROLLARY 2.** Let $x(t) \in C([a, \infty), \mathbb{R}_1)$, $n(t) \in C(\mathbb{R}_+, \mathbb{R}_0)$, $p > 0$, be a constant and Assumption (H) holds. Then the nonlinear delay integral inequality

$$x^p(t) \leq n(t) + \int_0^t x^{p-1}(\sigma(s)) \{f(s)W[\log x(\sigma(s))]$$

$$+ g(s) \log x(\sigma(s)) + h(s)\} ds, \quad t \in \mathbb{R}_+ \quad (25)$$

with condition (D') implies

$$x(t) \leq \exp\left[G^{-1}\left(G\left(\xi_p(t)\right) + \left(\exp\int_0^t \frac{g(s)}{p} ds\right)\int_0^t \frac{f(s)}{p} ds\right)\right], \quad 0 \leq t \leq \delta,$$

(26)

where $\xi_p(t) = (\exp\int_0^t \frac{f(s)}{p} ds)\frac{1}{2} \log n(t) + \int_0^t \frac{f(s)}{p} ds$, $G$ and $G^{-1}$ are defined as in Theorem B, and the positive number $\delta$ is chosen so that the quantity in the curly brackets of (26) is in the range of $G$. 
Proof. Taking \( u(t) = \log x(t) \), then inequality (25) reduces to
\[
e^{pu(t)} \leq n(t) + \int_0^t e^{pu(s)} \left[ f(s) W[u(\sigma(s))] + g(s) u(\sigma(s)) + h(s) \right] ds, \quad t \in R_+,
\]
which is a special case of inequality (1) when \( \phi(u) = \exp(pu) \). By Theorem 1, we get the desired inequality (26) directly.

Remark 2. The inequality (25) with \( p = 2 \) is different from the inequality (L) of Theorem B, and we believe it is new.

Letting \( \phi(u) = u^p \) in Theorem 2 then we obtain the next

**Corollary 3.** Let \( p \geq 1 \) be a constant and Assumption (H) holds; then the inequality
\[
x^p(t) \leq n(t) + \int_0^t x^{p-1}(\sigma(s)) \left( f(s) \left( \int_0^s g(\tau) W[x(\sigma(\tau))] d\tau \right) + h(s) x(\sigma(s)) + k(s) \right) ds, \quad t \in R_+ \tag{27}
\]
with condition (D') implies
\[
x(t) \leq G^{-1} \left[ G[H_p(t) \theta_p(t)] + H_p(t) \int_0^t f(s) \left( \int_0^s g(\tau) d\tau \right) ds \right], \quad 0 \leq t \leq \eta, \tag{28}
\]
where \( H_p(t) = \exp \int_0^t \frac{h(s)}{p} ds \), \( \theta_p(t) = n^1/p(t) + \int_0^t \frac{k(s)}{p} ds \), and the positive number \( \eta \) is chosen so that the quantity of the curly brackets of (28) is in the range of \( G \).

Remark 3. In Corollary 3, letting \( p = 2 \), \( n(t) = c^2 \), \( f(t) = 2a(t) \), \( h(t) \equiv 0 \), and \( k(t) = 2b(t) \), then we derive the conclusion (II) of Theorem B.

**Corollary 4.** Let \( x(t) \in C([a, \infty), R_1) \), \( n(t) \in C(R_+, R_0) \), \( p > 0 \), be a constant, and Assumption (H) holds. Then the nonlinear delay inequality
\[
x^p(t) \leq n(t) + \int_0^t x^p(\sigma(s)) \left( f(s) \left( \int_0^s g(\tau) W[\log x(\sigma(\tau))] d\tau \right) + h(s) \log x(\sigma(s)) + k(s) \right) ds, \quad t \in R_+ \tag{29}
\]
with condition (D') implies

\[ x(t) \leq \exp \left[ G^{-1} \left( G \left[ H_p(t) \hat{\theta}_p(t) \right] + H_p(t) \int_0^t \frac{f(s)}{p} \left( \int_0^s g(\tau) \, d\tau \right) \, ds \right) \right], \]

\[ 0 \leq t \leq \mu, \tag{30} \]

where \( \hat{\theta}_p(t) = \frac{1}{p} \log n(t) + \int_0^t \frac{x(s)}{p} \, ds \), \( H_p(t) \) is defined as in Corollary 3, and the positive number \( \mu \) is chosen so that the quantity of the curly brackets of (30) is in the range of \( G \).

**Proof.** Taking \( u(t) = \log x(t) \), then inequality (30) reduces to

\[ e^{pu(t)} \leq n(t) + \int_0^t e^{p\sigma(s)} \left( f(s) \left( \int_0^s g(\tau) W(u(\sigma(\tau))) \, d\tau \right) + h(s) u(\sigma(s)) + k(s) \right) \, ds, \quad t \in R_+. \]

This is a special case of inequality (12) when \( \phi(u) = \exp(pu) \). An application of Theorem 2 to the inequality yields the desired inequality (30).

**Remark 4.** In Corollary 4, letting \( p = 2 \), \( n(t) = e^c \), \( f(t) = 2a(t) \), \( h(t) = 0 \), and \( k(t) = 2b(t) \), then it follows conclusion (III) of Theorem B.

**Remark 5.** In Theorem 3, if \( r \) obeys the more restrictive condition \( r \geq 1 \), then (17) can be considered as a particular case of inequality (23) when \( p = r \), \( h(t) = 0 \), \( n(t) = e^c \), and \( W(\xi) = \xi^{q+1} \); i.e.,

\[ x'(t) \leq c^c + \int_0^t x^{c-1}(\sigma(s)) \left( f(s)x^{q+1}(\sigma(s)) + g(s) \right) \, ds, \quad t \in R_+. \]

By definition, we have \( G(u) = \frac{1}{q}(u^{-q} - u_0^{-q}) \) and hence

\[ G^{-1}(v) = \left[ u_0^{-q} - qv \right]^{-1/q}. \]

An application of Corollary 1 to the last inequality yields

\[ x(t) \leq \exp \left( \int_0^t \frac{g(s)}{r} \, ds \right) \times \left\{ \frac{1}{c^q} - \left( \exp \int_0^t \frac{g(s)(q + 1)}{r} \, ds \right) \int_0^t \frac{qf(s)}{r} \, ds \right\}^{-1/q}, \]

\[ 0 \leq t < \bar{v}, \tag{31} \]
where $\bar{v}$ is a positive number satisfying

$$\frac{1}{e^{q}} > \left( \int_{0}^{e} \frac{qf(s)}{r} \, ds \right) \left( \exp \int_{0}^{e} \frac{g(s)(q + 1)}{r} \, ds \right).$$

Obviously, in many situations, the bound in (18) is not only better than that given in (31), but also the validity of (31) when $0 < r < 1$ cannot be established by using Corollary 1.

4. APPLICATION

Consider the delay integral equation

$$x^p(t) = F \left( t, x(\sigma(t)), \int_{0}^{t} K[t, s, x(\sigma(s))] \, ds \right), \quad t \in \mathbb{R}_+$$

with the initial condition (D$^*$),

$$x(t) = \psi(t), \quad t \in [a, 0] \text{ with } \psi(\sigma(t)) \leq n^{1/p}(t), \quad \text{for every } t \geq 0 \text{ with } \sigma(t) \leq 0,$$

where $F \in C(\mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R}), K \in (\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}), p \geq 1$, is a constant, and $x, \sigma, a$ are as defined in Theorem 1.

Assume that

$$|F[t, u, v]| \leq n(t) + m|v|, \quad t \in \mathbb{R}, \; u, v \in \mathbb{R}, \quad t \in \mathbb{R}_+ \text{ with } \sigma(t) \leq 0,$$

$$|K[t, u, v]| \leq |v|^{p-1} f(s) W(|v|) + g(s)|v| + h(s), \quad t, s \in \mathbb{R}_+, \; v \in \mathbb{R},$$

where $m > 0$ is a constant, and the functions $n, f, g, W$ are defined as in Theorem 1.

For every continuous solution $x(t)$ of (32) satisfying the condition (D$^*$), from (32), (33), and (34) we obtain

$$|x(t)|^p = \left| F \left( t, x(\sigma(t)), \int_{0}^{t} K[t, s, x(\sigma(s))] \, ds \right) \right|$$

$$\leq n(t) + m \int_{0}^{t} |x(\sigma(s))|^{p-1} \left| f(s) W(|x(\sigma(s))|) + g(s)|x(\sigma(s))| + h(s) \right| \, ds, \quad t \in J(x),$$

where $J(x)$ denotes the maximal existent interval of $x(t)$. 
An application of Corollary 1 to the last inequality yields

\[
|x(t)| \leq G^{-1}\left\{ G\left[ \left( \exp \int_0^t \frac{mg(s)}{p} \, ds \right) \left( n^{1/p}(t) + \int_0^t \frac{mh(s)}{p} \, ds \right) \right] \\
+ \left( \exp \int_0^t \frac{mg(s)}{p} \, ds \right) \int_0^t \frac{mf(s)}{p} \, ds \right\},
\]

where \( G \) and \( G^{-1} \) are as defined in Theorem B, and \( \gamma > 0 \) is chosen so that the quantity in the curly brackets is in the range of \( G \).

\section*{REFERENCES}