JOURNAL OF NUMBER THEORY 26, 274–285 (1987)

Hilbert Irreducibility Sequences and Nonstandard Arithmetic

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Communicated by P. Roquette

Received July 6, 1985; revised July 30, 1986

We give sufficient conditions for a sequence of integers to be a Hilbert irreducibility sequence and give such a sequence explicitly. © 1987 Academic Press, Inc.

Let R be an integral domain. A sequence $a_1, a_2, ..., a_n, ...$ $(n \in \mathbb{N})$ of elements of R is called an m-irreducibility sequence if for any irreducible polynomial $f(X, Y) \in R[X, Y]$ with $\deg_X(f) \leq m$, there are only finitely many a_n such that $f(X, a_n)$ is reducible. A sequence is called a Hilbert irreducibility sequence (H.i.seq.) if it is an m-irreducibility sequence for all natural numbers m. It is easily proved that if R is countable, then the existence of a H.i.seq. is equivalent to Hilbert's irreducibility theorem. Since it is well known that Hilbert's irreducibility theorem holds for $\mathbf{Z}$, we know that there is a H.i.seq. of integers. In his papers [11, 12], Sprindzuk showed that $[\exp \sqrt{\log\log n}] + n!2^{n^2}$ is a Hilbert irreducibility sequence. Moreover he proved that there is an effectively computable function $n_0(f)$ such that $f(X, a_n)$ is irreducible for all $n > n_0(f)$. (See also [2]). Our purpose in this paper is to give a sufficient condition for a sequence of integers to be an m-irreducibility sequence (Theorem 1). Our condition gives H.i.seq. explictly which are simpler and whose growth orders are smaller than Sprindzuk's. Unfortunately, we cannot effectively find finite exceptions a_n that $f(X, a_n)$ is reducible. The lack of effectivity comes from the use of the nonstandard version of Siegel's theorem. No proof of Theorem 1 without nonstandard method is known and it seems difficult to prove an effective version of Theorem 1.[1]

* This paper was written during the author's stay at University of Illinois at Urbana–Champaign.

[1] By contrast, the main result of our earlier paper [13], also obtained by nonstandard methods, is of effective nature. Namely, the effective computability of the constant N in [13,

0022-314X/87 $3.00

Let K be an algebraic number field of finite degree. By an arithmetical prime divisor of K, we mean a class of nontrivial valuations of K with respect to the equivalence relation for valuations. For each non-archimedean arithmetical prime p, let $\|x\|_p$ denote the normalized absolute value, i.e.,

$$\|x\|_p = Np^{-v_p(x)},$$

where Np is the norm of p and v_p is the normalized valuation which belongs to p. For each archimedean arithmetical prime p, we define

$$\|x\|_p = \begin{cases} |x|_p & \text{if } p \text{ is real} \\ |x|_p^2 & \text{if } p \text{ is complex,} \end{cases}$$

where p is called real (resp. complex) if the p-adic completion of K is isomorphic to the field of real (resp. complex) numbers. Let S be a finite set of arithmetical primes. We define

$$H_S(x) = \prod_{p \in S} \max(1, \|x\|_p),$$

$$H(x) = \prod_{p} \max(1, \|x\|_p).$$

THEOREM 1. *Let $a_n (n \in \mathbb{N})$ be a sequence of elements of K. Assume*

(i) *there is a finite set S of arithmetical prime divisors such that*

$$\liminf_{n \to \infty} \frac{\log(H_S(a_n)\, H_S(a_n^{-1}))}{\log H(a_n)} > 2 - \frac{1}{m!},$$

(ii) *for any nonzero $r \in K$ and any natural number j with $2 \leqq j \leqq m!$, there are only finitely many $y \in K$ and a_n such that $ra_n = y^j$.*

Then a_n is an m-irreducibility sequence.

Let us consider some examples. Let $K = \mathbb{Q}$ and $a_n = 2^n p_n$, where p_n is the nth prime number. Let $S = \{2, p_\infty\}$, where p_∞ denotes the archimedean prime divisor. Then

$$\liminf_{n \to \infty} \frac{\log(H_S(a_n)\, H_S(a_n^{-1}))}{\log H(a_n)} \geqq \lim_{n \to \infty} \frac{\log(2^{2n} p_n)}{\log(2^n p_n)} = 2.$$

Theorem 1] comes from the fact that the proof of that theorem is valid in all nonstandard models of a recursive set of axioms of the rational number field. Whereas any nonstandard model in which the proof of Theorem 1 in the present paper is valid, has to satisfy Roth's inequalities (see [5]) which are not known to be recursively presentable. By the way, our Theorem 1 in [13] contains Fried's Proposition 4.5 in [2] as a special case.

Hence the first condition (i) of Theorem 1 is satisfied for all m. It is easily seen that the sequence $2^n p_n$ satisfies (ii) for any m. Therefore $2^n p_n$ is an m-irreducibility sequence for all m, hence it is a H.i.seq. By the same way, we can show that the sequence $2^n(n^3+1)$ is also a H.i.seq. On the other hand, $2^n(n^2+1)$ is not a H.i.seq. In fact, let (u, v) be an integer solution of

$$U^2 - 2V^2 = -1. \tag{1}$$

Let $f(X, Y) = X^2 - 2Y$ and $g(X, Y) = X^2 - Y$. If u is even,

$$f(X, 2^u(u^2+1)) = X^2 - 2^{u+2}v^2 = (X + 2^{(u+2)/2}v)(X - 2^{(u+2)/2}v),$$

if u is odd,

$$g(X, 2^u(u^2+1)) = (X + 2^{(u+1)/2}v)(X - 2^{(u+1)/2}v).$$

Since it is well known that there are infinitely many integer solutions of (1), there are infinitely many integers n such that $f(X, 2^n(n^2+1))$ or $g(X, 2^n(n^2+1))$ is reducible. Therefore $2^n(n^2+1)$ is not a 2-irreducibility sequence. The next example is $a^n(b^n+1)$, where a and b are integers. Let S be a finite set of prime divisors which contains the archimedean prime and all primes which appear in the prime factorization of a. Then

$$\liminf_{n\to\infty} \frac{\log(H_S(a_n)\, H_S(a_n^{-1}))}{\log H(a_n)} \geqq \lim_{n\to\infty} \frac{\log|a^{2n}(b^n+1)|}{\log|a^n(b^n+1)|}$$
$$= 2 - \lim_{n\to\infty} \frac{\log|b^n+1|}{\log|a^n(b^n+1)|} = 2 - \frac{\log|b|}{\log|ab|}.$$

It is easily proved by Siegel's theorem that the condition (ii) is satisfied for all m. Hence for any m with $m! < \log ab/\log b$, the sequence $a^n(b^n+1)$ is an m-irreducibility sequence. It does not seem to be known whether p_n and $2^n + n$ are Hilbert irreducibility sequences.

The author expresses his thanks to Professor A. Macintyre and Professor G. Takeuti for their valuable suggestions and encouragements.

1. Let R be an integral domain and K its quotient field. $^*R$ and $^*\mathbb{N}$ denote enlargements of R and $\mathbb{N}$, respectively. For the definition of enlargement, refer to [5], in this paper we assume the reader is familiar with nonstandard arithmetic. We use the terminology and the notations as introduced in [5].

In nonstandard arithmetic, we have a beautiful characterization of a H.i.seq., due to Gilmore and Robinson [3].

PROPOSITION 1. *Let a_n ($n \in \mathbb{N}$) be a sequence of elements of R. Then a_n is a H.i.seq. in R if and only if for ay nonstandard natural number $\omega \in {}^*\mathbb{N} - \mathbb{N}$, $K(a_\omega)$ is relatively algebraically closed in ${}^*K$.*

As for m-irreducibility we have the following suffcient condition for a sequence to be an m-irreducibility sequence.

PROPOSITION 2. *If for any nonstandard natural number $\omega \in {}^*\mathbb{N} - \mathbb{N}$, $K(a_\omega)$ has no proper algebraic extension of degree not more than $m!$ within ${}^*K$, then a_n is an m-irreducibility sequence in R.*

Unfortunately, the converse of Proposition 2 is not true, but if $m!$ is replaced by m, then its converse holds.

PROPOSITION 3. *If a_n is an m-irreducibility sequence in R, then for any nonstandard natural number $\omega \in {}^*\mathbb{N} - \mathbb{N}$, $K(a_\omega)$ has no proper algebraic extension of degree not more than m within ${}^*K$.*

It is easily shown that Proposition 1 is a consequence of Propositions 2 and 3. First, we prove Proposition 2.

Proof of Proposition 2. Assume otherwise, then there is an irreducible polynomial $f(X, Y) \in R[X, Y]$ with $\deg_X(f) \leqq m$ such that $f(X, a_n)$ is reducible in $R[X]$ for infinitely many $n \in \mathbb{N}$. By nonstandard principle, there is a nonstandard natural number $\omega \in {}^*\mathbb{N} - \mathbb{N}$ such that $f(X, a_\omega)$ is reducible in ${}^*(R[X])$. Let

$$f(X, a_\omega) = g(X)\, h(X),$$

where $g(X)$, $h(X) \in {}^*(R[X])$ are not constants. Since $\deg(f(X, a_\omega)) \leqq m$, $\deg(g(X))$, $\deg(h(X)) \leqq m$. This means $g(X)$, $h(X) \in {}^*R[X]$ because ${}^*R[X]$ is the set of all polynomials in ${}^*(R[X])$ whose degrees are finite. We may assume that $g(X)$ is monic.

Let F be the extension of $K(a_\omega)$ generated by all coefficients of $g(X)$. Then $[F: K(a_\omega)] \leqq m!$ because F is included in the splitting field of $f(X, a_\omega)$ over $K(a_\omega)$. Since F is included in ${}^*K$, $F = K(a_\omega)$ by the assumption of Proposition 2. Hence $g(X) \in K(a_\omega)[X]$, and therefore $h(X) = f(X, a_\omega)\, g(X)^{-1} \in K(a_\omega)[X]$. Since a_ω is transcendental over K, we have

$$f(X, Y) = g(X, Y)\, h(X, Y),$$

where $g(X, Y)$, $h(X, Y) \in K(Y)[X]$. This contradicts the fact that $f(X, Y)$ is an irreducible polynomial.

Proof of Proposition 3. Assume that there is a proper algebraic extension $K(a_\omega, b)$ of degree at most m over $K(a_\omega)$ within ${}^*K$. Let $f(X, a_\omega)$ be the irreducible polynomial of b over $K(a_\omega)$. We may assume that $f(X, Y) \in K[X, Y]$ and $f(X, Y)$ is irreducible. Since b is a root of $f(X, a_\omega) = 0$, there are infinitely many a_n such that $f(X, a_n) = 0$ has a root in K, hence $f(X, a_n)$ is reducible. This means that the sequence a_n is not an m-irreducibility sequence.

2. In this section, we always assume that the sequence a_n satisfies the conditions (i) and (ii) in Theorem 1. To prove Theorem 1, it suffices to show by Proposition 2 that for any nonstandard natural number $\omega \in {}^*\mathbb{N} - \mathbb{N}$, $K(a_\omega)$ has no proper algebraic extension of degree not more than $m!$ within ${}^*K$. The original idea of our proof is essentially the same as that of the nonstandard proof of Hilbert's irreducibility theorem by P. Roquette [6] and [7], so we will show how to modify his proof to prove Theorem 1.

First we consider an algebraic function field F of one variable over K which is included in ${}^*K$, i.e.,

$$K \subset F \subset {}^*K.$$

In this situation we have two kinds of prime divisors, i.e., the functional prime divisors of F and the arithmetical prime divisors of ${}^*K$. Between these prime divisors we have the following fundamental relation.

THEOREM [5, Lemma 4.1.]. *Every functional prime divisor of F is induced by some arithmetical prime divisor of ${}^*K$.*

A functional prime divisor is called exceptional if it is induced by standard arithmetical prime divisors only. A functional divisor A is called exceptional if $A = P_1 + P_2 + \cdots + P_r$, where P_i are distinct exceptional prime divisors. In their paper [5], A. Robinson and P. Roquette gave a bound for degrees of exceptional divisors.

THEOREM [5, Theorem 5.4]. *Let A be an exceptional divisor. Then*

$$\deg(A) \leq \min_{x \in F - K} [F : K(x)]$$

Let S be a finite set of (standard or nonstandard) arithmetical prime divisors. We define the size, S-size and S-degree of a functional divisor as follows. Let P be a functional prime, then we define

$$\sigma(P) \doteq \sum_p v_p(P) \log(Np)$$

$$\sigma^S(P) \doteq \sum_{p \in S} v_p(P) \log(Np)$$

$$\deg^S(P) \simeq \deg(P)\, \sigma^S(P)/\sigma(P).$$

Let $A = \sum_i d_i P_i$ be a functional divisor

$$\sigma(A) \doteq \sum_i d_i \sigma(P_i) \doteq \sum_p v_p(A) \log(Np)$$

$$\sigma^S(A) \doteq \sum_i d_i \sigma^S(P_i)$$

$$\deg^S(A) \simeq \sum_i d_i \deg^S(P_i).$$

If A is exceptional, then there is a finite set S of standard arithmetical primes such that $\deg^S(A) = \deg(A)$. We will prove that if S is a finite set of standard arithmetical primes and A has no multiple component, i.e., $A = P_1 + P_2 + \cdots + P_r$, where $P_1, P_2, \ldots, P_r$ are distinct primes, then

$$\deg^S(A) \gtrsim 2 \min_{x \in F - K} [F: K(x)].$$

Its proof is almost the same as that of Theorem 5.4 in [5], so let us recall the basic relations between size and degree. First we show that the size and the S-size are not changed by a function field extension $F \subset E \subset {}^*K$, in other words, $\sigma_F(A) \doteq \sigma_E(A)$ and $\sigma_F^S(A) \doteq \sigma_E^S(A)$, where σ_F and σ_F^S (resp. σ_E and σ_E^S) are the size and S-size defined in F (resp. in E). It suffices to prove this for any positive functional divisor A. Let x and y be elements of F such that

$$A = \max(0, \min([x], [y])), \tag{2}$$

where $[x]$ and $[y]$ denote the functional principal divisors of x and y, respectively. Then

$$\begin{aligned} \sigma_F(A) &\doteq \sum_p \max(0, \min(v_p(x), v_p(y))) \log(Np) \\ \sigma_F^S(A) &\doteq \sum_{p \in S} \max(0, \min(v_p(x), v_p(y))) \log(Np). \end{aligned} \tag{3}$$

On the other hand, Eq. (2) also holds in E, hence the right-hand sides of Eq. (3) are also equal to $\sigma_E(A)$ and $\sigma_E^S(A)$, so we conclude that $\sigma_F(A) \doteq \sigma_E(A)$ and $\sigma_F^S(A) \doteq \sigma_E^S(A)$ as contended. Hence we omit F in σ_F and σ_F^S if it is clear in the context.

In [5], the following Theorem is proved.

THEOREM [5, Theorem 4.4]. *There exists an infinitely large number $\rho \in {}^*\mathbb{R}$ such that $\sigma(A)/\rho \simeq \deg(A)$ for all functional divisor A. The number ρ is uniquely determined up to infinitesimals in the following multiplicative sense: If $\lambda \in {}^*\mathbb{R}$ is another such number then $\lambda/\rho \simeq 1$.*

Precisely speaking, $\rho = \rho_F$ is determined by F. Let E be a finite extension of F included in $^*K$, then

$$\frac{\sigma_E(A)}{\rho_E} \simeq \deg_E(A) = [E\colon F]\deg_F(A) \simeq \frac{[E\colon F]\,\sigma_F(A)}{\rho_F}.$$

Since $\sigma_E(A) \doteq \sigma_F(A)$,

$$[E\colon F] \simeq \rho_F/\rho_E. \tag{4}$$

By the definition of S-degree,

$$\deg^S(P) \simeq \frac{\deg(P)\,\sigma^S(P)}{\sigma(P)} \simeq \frac{\sigma^S(P)}{\rho}$$

hence by linearity for every functional divisor A,

$$\deg^S(A) \simeq \frac{\sigma^S(A)}{\rho}$$

we conclude:

LEMMA 1 [c.f. [5, Corollary 4.6.]]. *Let A and B be functional divisors and* $\deg(A) > 0$. *Then*

$$\frac{\deg^S(B)}{\deg(A)} \simeq \frac{\sigma^S(B)}{\sigma(A)}.$$

By the same way as in p. 155 [5], we get

LEMMA 2 [c.f. [5, Corollary 4.7.]]. *Let $x \in F$ be a nonconstant. For every functional divisor B,*

$$\frac{\sigma^S(B)}{\log H(x)} \simeq \frac{\deg^S(B)}{[F\colon K(x)]}.$$

Let A be a functional divisor. An element $x \in F$ is called A-integral if $v_p(x) \geqq 0$ for every component P of A.

LEMMA 3 [c.f. [5, Corollary 5.2.]]. *Let A be a positive functional divisor which has no multiple component. We assume that every component of A is of degree* 1. *Given any nonconstant element $x \in F$ which is A-integral, there are elements $c_p \in K$ (for $p \in S$) such that*

$$\sigma^S(A) \mathrel{\dot{\leqq}} \sum_{p \in S} v_p(x - c_p)\log(Np).$$

Proof. Let $S' \subset S$ be the set of arithmetical primes which are effective on A, in other words, $p \in S'$ if and only if $p \in S$ and p induces some component of A. For each $p \in S'$, let c_p denote the P-adic residue of x where P is the component of A which p induces. Since $\deg(P) = 1$, we have $c_p \in K$. By construction, $x - c_p$ has a zero at P which p induces. On the other hand, P is a simple component of A; this implies

$$v_p(A) = 1 \leqq v_P(x - c_p),$$

hence

$$v_p(A)\log(Np) \mathrel{\dot{\leqq}} v_p(x - c_p)\log(Np).$$

For $p \in S - S'$, let c_p be any element in K such that $v_p(x - c_p) \geqq 0$. Let $A = P_1 + P_2 + \cdots + P_r$. Since $v_p(P_i)\log(Np)$ is finite for $p \in S - S'$

$$\begin{aligned}
\sigma^S(A) &\doteq \sum_i \sigma^S(P_i) \\
&\doteq \sum_{p \in S} \sum_i v_p(P_i)\log(Np) \\
&\doteq \sum_{p \in S'} \sum_i v_p(P_i)\log(Np) \\
&\doteq \sum_{p \in S'} v_p(A)\log(Np) \\
&\mathrel{\dot{\leqq}} \sum_{p \in S'} v_p(x - c_p)\log(Np) \\
&\mathrel{\dot{\leqq}} \sum_{p \in S} v_p(x - c_p)\log(Np)
\end{aligned}$$

as contended.

By the same way as in p. 158–161 [5] we now obtain:

THEOREM 2. *Let S be a finite set of standard arithmetical primes and A a functional divisor which has no multiple component. Then*

$$\deg^S(A) \gtrsim 2 \min_{x \in F - K} [F\colon K(x)].$$

Next we consider a finite unramified extension E of F within $^*K$.

LEMMA 4. *Let A be a functional divisor of F. Then*

$$\deg_E^S(A) \simeq [E\colon F] \deg_F^S(A).$$

Proof.

$$\begin{aligned}\deg_E^S(A) &\simeq \frac{\sigma^S(A)}{\rho_E} \\ &\simeq \frac{\rho_F}{\rho_E} \cdot \frac{\sigma^S(A)}{\rho_F} \\ &\simeq [E:F] \circ \deg_F^S(A).\end{aligned}$$

As proved in [5] and [7], if F is of genus $g > 0$, then there is an unramified extension E_n of F in $^*K$ such that

$$[E_n : F] = n^{2g}$$

$$\min_{x \in E_n - K} [E_n : K(x)] \leqq gdn^{2g-2},$$

where $d = \min_{x \in F-K} [F: K(x)]$. Therefore, applying Theorem 2 to E_n instead of F,

$$\begin{aligned}\deg_F^S(A) &\simeq \frac{\deg_{E_n}^S(A)}{n^{2g}} \\ &\lesssim \frac{gd}{n^2}.\end{aligned}$$

Taking n to infinity, we have proved

PROPOSITION 4. *Let S be a finite set of standard arithmetical primes and A a positive functional divisor which has no multiple component. If $F \subset {}^*K$ is of genus $g > 0$, then* $\deg_F^S(A) \simeq 0$.

Recall that the S-degree is additive by definition. Therefore if $\deg_F^S(A) \not\simeq 0$, then there is a component P of A such that $\deg_F^S(P) \not\simeq 0$. So we can eliminate the assumption in Proposition 4 that A is a positive functional divisor which has no multiple component.

THEOREM 3. *Let S be a finite set of standard arithmetical primes and A a functional divisor. If $F \subset {}^*K$ is of genus $g > 0$, then*

$$\deg_F^S(A) \simeq 0.$$

Let F be a finite extension of $K(a_\omega)$ included in $^*K$. Let $j = [F: K(a_\omega)]$ and $A = [a_\omega]_0 + [a_\omega]_\infty$, where $[a_\omega] = [a_\omega]_0 - [a_\omega]_\infty$ is the functional principal divisor of a_ω. Then the first condition (i) of Theorem 1 implies,

$$\begin{aligned}
\deg_F^S(A) &\simeq \deg_F(A)\,\sigma^S(A)/\sigma(A)\\
&\simeq j\deg_{K(a_\omega)}(A)\,\sigma^S(A)/\sigma(A)\\
&\simeq 2j\sum_{p\in S} v_p(A)\log(Np)/2\log H(a_\omega)\\
&\simeq j\left(\sum_{p\in S} v_p([a_\omega]_\infty)\log(Np)+\sum_{p\in S} v_p([a_\omega]_0)\log(Np)\right)\Big/\log H(a_\omega)\\
&\simeq j\log(H_S(a_\omega)\,H_S(a_\omega^{-1}))/\log H(a_\omega)\\
&> j\left(2-\frac{1}{m!}\right)+je, \qquad (5)
\end{aligned}$$

where e is a standard positive number. Therefore by Theorem 3, the genus of F is 0. Since F is included in $*K$, then F has infinitely many functional primes of degree 1, hence F is a rational function field.

LEMMA 5. *If $j \leqq m!$, then there are functional primes P and Q of F of degree* 1 *such that*

$$[a_\omega]=j(P-Q).$$

Proof. Assume not. Without loss of generality we may assume

$$[a_\omega]_0=\sum j_i P_i,$$

where $0<j_i\leqq j-1$ and P_i are distinct primes. Let

$$[a_\omega]_\infty=\sum l_i Q_i,$$

where $0<l_i\leqq j$ and Q_i are distinct primes. Let $B=\sum P_i$ and $C=\sum Q_i$, then $B+C$ is a positive functional divisor which has no multiple component. Then

$$\begin{aligned}
[a_\omega]_0 &\leqq (j-1)\,B\\
[a_\omega]_\infty &\leqq jC.
\end{aligned} \qquad (6)$$

Since by (5) for some standard real $e>0$,

$$\deg_F^S([a_\omega]_0+[a_\omega]_\infty) \gtrsim j\left(2-\frac{1}{m!}\right)+je$$

and

$$\deg_F^S([a_\omega]_\infty)\leqq j\deg_{K(a_\omega)}([a_\omega]_\infty)=j,$$

we get

$$\deg_F^S([a_\omega]_0) \gtrsim j\left(1-\frac{1}{m!}\right)+je.$$

Hence by the inequation (6),

$$\deg_F^S(B) \gtrsim \frac{j}{j-1}\left(1-\frac{1}{m!}\right)+e$$

and

$$\begin{aligned} j\deg_F^S(B+C) &= \deg_F^S((j-1)B)+\deg_F^S(jC)+\deg_F^S(B) \\ &\geqq \deg_F^S([a_\omega]_0+[a_\omega]_\infty)+\deg_F^S(B) \\ &\gtrsim j\left(2-\frac{1}{m!}\right)+\frac{j}{j-1}\left(1-\frac{1}{m!}\right)+je. \end{aligned}$$

Since we are assuming $j<m!$,

$$\deg_F^S(B+C) \gtrsim 2+e.$$

This contradicts Theorem 2 because F is a rational function field. Lemma 5 is proved.

Lemma 5 means that if $j \leqq m!$, then there exists $y \in F$ and $r \in K$ such that

$$F=K(y)$$

and

$$ra_\omega = y^j.$$

But if $2 \leqq j \leqq m!$, this contradicts condition (ii) in Theorem 1. Hence we conclude $j=1$ or $j>m!$, in other words, $K(a_\omega)$ has no proper algebraic extension of degree not more than $m!$ within $^*K$. Sincce ω is an arbitrary infinite natural number, a_n is an m-irreducibility sequence by Proposition 2, hence Theorem 1 is proved.

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