The Integral Equation Method in Electromagnetic Scattering*

G. A. GRAY

U.S. Naval Surface Weapons Center, Silver Spring, Maryland 20910

AND

R. E. KLEINMAN

Department of Mathematical Sciences,
University of Delaware, Newark, Delaware 19716

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1. Introduction

The problem of scattering of time harmonic electromagnetic waves by a perfectly conducting object may be formulated as an integral equation for the magnetic field on the surface of the scatterer (Maué [16, 10]). The present paper shows that this equation may be solved iteratively for low frequencies when the scattering surface is Lyapunov (e.g., Günter [8, p. 41]).

For slightly more restrictive boundaries ($C^3$), existence and uniqueness proofs for the boundary value problem of perfect reflection of stationary electromagnetic waves in homogeneous media were given by C. Müller [18], H. Weyl [29], W. K. Saunders [22] and A. P. Calderon [5]. In these treatments, the existence of eigenvalues of the adjoint, interior problem complicated the analysis. Werner [27] provided a uniform treatment for all wavenumbers by introducing volume integrals. Knauff and Kress [14] have extended to electromagnetics the idea of Brakhage and Werner [3] to remove the complication of interior eigenvalues using only boundary integrals. The exterior boundary value problem of perfect reflection becomes the scattering problem if the boundary values specified are the values of the known incident field at the surface of the scattering body. Thus, the work of Müller and others establishes the existence and uniqueness of a solution to the scattering problem for $C^3$ boundaries.

The present paper does establish existence and uniqueness for low frequencies and Lyapunov boundaries; however, the main purpose is to prove

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that the boundary integral equation may be solved by iteration. This is
done by showing that the spectral radius of the resulting integral operator
is less than one for small perturbations of the corresponding potential
operator. In our discussion, we shall point out how the proof would
proceed in the more restrictive boundary case. The solution by successive
approximation was suggested by Ormsby [20], who did not provide a
proof of convergence.

This work is an application to electromagnetics of an extended
Neumann's method described by Kleinman [12] which has already found
successful application in acoustics (Ahner and Kleinman [2], Kleinman
and Wendland [13]) and elasticity (Ahner and Hsiao [1]).

In Section 2 we describe the surface of the scattering body and define the
function spaces we will use in our solution of the problem. Section 3 con-
tains a precise statement of the boundary value problem we wish to solve.
Properties of the potential operator, $R_0$, for our problem, including com-
 pactness, are established in Section 4. A version of Plemelj's theorem
(which gives the location and nature of eigenvalues) for $R_0$ is proved in
Section 5. Section 6 contains a proof of the convergence of the sequence of
iterates. We show in Section 7 that with the solution of the integral
equation we can define functions which satisfy the boundary value problem
stated in Section 3, thus establishing an equivalence between the solution of
the boundary value problem and the solution of the integral equation.

Also included is an appendix in which conditions are established under
which the divergence of a simple layer potential with a vector density can
be represented by a simple layer potential with a summable density, a
result which is essential to our solution of the scattering problem.

2. Preliminaries

Let $S$ be a closed, bounded surface in $\mathbb{R}^3$ with interior $D_1$, exterior $D_2$, and
unit normal $\hat{n}$ directed into $D_2$. Denote points in $\mathbb{R}^3$ by $P = (x, y, z)$,
and the distance between two points by $R_{ij} = |P_i - P_j|$. Finally, let $\hat{e}_i$ denote
a unit vector in the direction of $P_i$ and $r_i$ be the magnitude ($r_i := |P_i|,
\hat{n}_i = P_i/|P_i|$).

We shall discuss the solution of the scattering problem for two types of
surfaces: $S$ three times continuously differentiable (Werner [26]) in which
case we say $S$ is smooth, and $S$ Lyapunov, which means, among other
things, that the normal is Hölder continuous with index denoted here by $\alpha$
(Günter [8, p. 1]). For $S$ Lyapunov, the surface belongs in the more
general class characterized by Burago et al. [4]. We assume along with
Burago et al. that there exists a sequence of closed smooth surfaces $S_m^c$
lying exterior to $S$ which reduce to $S$ in the following sense: each $S_m^c$ is
given by a continuous mapping \( \phi_m \) of \( S \) into \( \mathbb{R}^3 \) such that \( \phi_m(P) \to P \in S \) uniformly with respect to \( P \in S \) as \( m \to \infty \) and that \( \int dS_m \to \int dS \). We assume further that \( \hat{n}(\phi_m(P)) \to \hat{n}(P) \) uniformly. A similar assumption is made for closed smooth surfaces \( S_m \) which reduce to \( S \) from the interior. For the case when the Lyapunov index \( \alpha = 1 \), Smirnov [23, Vol. IV, p. 600] actually constructs a sequence of such "parallel surfaces."

If \( \Omega \) is a closed domain in \( \mathbb{R}^3 \), denote by \( C^n(\Omega) \) the space of \( n \) times continuously differentiable complex valued functions on \( \Omega \) with norm \( \| \cdot \|_n \) and by \( C^\alpha(\Omega) \) the space of Hölder continuous complex valued functions on \( \Omega \) with norm

\[
\| u \|_\alpha = \| u \|_0 + h_\alpha[u]
\]

(2.1)

where

\[
\| u \|_0 = \sup_{P \in \Omega} |u(P)|
\]

(2.2)

\[
h_\alpha[u] = \sup_{P_1, P_2 \in \Omega, P_1 \neq P_2} \frac{|u(P_1) - u(P_2)|}{|P_1 - P_2|^\alpha}
\]

(2.3)

(Vekua [25]). These definitions remain valid for vector valued functions if \( \| \cdot \| \) is interpreted as Euclidean length. Furthermore, we denote by \( C^\alpha_\mathbb{R}(S) \), \( C^\alpha(S) \) those Banach spaces of vector valued functions on \( S \) which are tangential to \( S \) (no normal components).

Let \( Va \) denote the simple layer potential

\[
(Va)(P_0) = \frac{1}{2\pi} \int_S \frac{a(P_1)}{R_{01}} dS_1, \quad P_0 \in \mathbb{R}^3.
\]

(2.4)

If \( a \) is bounded and integrable on \( S \), then \( Va \) is Hölder continuous throughout all of \( \mathbb{R}^3 \) (Günter, [8, p. 44]). If \( a \in C^\alpha(S) \) and \( S \) is Lyapunov, the limiting values of \( \nabla \cdot Va \) and \( \nabla \times Va \) as \( P_0 \) approaches \( S \) while remaining in \( D_1 \) or \( D_\epsilon \) can be derived from Lyapunov's theorem on the limiting values of the derivative of a simple layer (Günter [8, p. 286]). These are

\[
(\nabla \times Va)_\ell = (\nabla \times Va)_S \pm \hat{n} \times a
\]

(2.5)

\[
(\nabla \cdot Va)_\ell = (\nabla \cdot Va)_S \pm \hat{n} \cdot a.
\]

(2.6)

The subscript \( i(e) \) denotes the limit from \( D_i \) (\( D_\epsilon \)) and the subscript \( S \) denotes the principal value of the function on \( S \). Moreover, the limiting values belong to \( C^\beta(S) \), \( 0 < \beta < \alpha \leq 1 \).
3. THE BOUNDARY VALUE PROBLEM

The calculation of the electromagnetic field \( (E^\text{scat}, H^\text{scat}) \) produced when a given incoming time-harmonic electromagnetic wave \( (E^\text{inc}, H^\text{inc}) \) is scattered by a closed, bounded perfectly conducting surface \( S \) leads to the following boundary value problem:

\( H^\text{scat} \) and \( E^\text{scat} \) satisfy the time-independent Maxwell's equations

\[
\nabla \times E^\text{scat} = i\omega \mu H^\text{scat}, \quad \nabla \times H^\text{scat} = -i\omega E^\text{scat} \quad \text{in } D_c, \quad (3.1)
\]

\[
\hat{n} \times E^\text{scat} = -\hat{n} \times E^\text{inc}, \quad \hat{n} \cdot H^\text{scat} = -\hat{n} \cdot H^\text{inc} \quad \text{on } S, \quad (3.2)
\]

and the radiation conditions

\[
\lim_{r \to \infty} r \left\{ \hat{r} \times (\nabla \times E^\text{scat}) + i k E^\text{scat} \right\} = 0 \quad (3.3)
\]

\[
\lim_{r \to \infty} r \left\{ \hat{r} \times (\nabla \times H^\text{scat}) + i k H^\text{scat} \right\} = 0 \quad (3.4)
\]

uniformly in all directions. The constant \( k \) in Eqs. (3.3) and (3.4) is related to the constitutive parameters \( \varepsilon \) and \( \mu \) by \( k^2 = \omega^2 \varepsilon / \mu \), \( \text{Im } k \geq 0 \). In the case of smooth \( S \), we shall require

\[
H^\text{scat}, \ E^\text{scat} \in C^1(D_c) \cap C(\bar{D}_c), \quad (3.5)
\]

and in the case of \( S \) Lyapunov, \( 0 < \alpha \leq 1 \), we shall require

\[
H^\text{scat} \in C^1(D^\alpha) \cap C(\bar{D}_c), \quad (3.6)
\]

\[
\hat{n} \times E^\text{inc} = 0 \quad \text{a.e. on } S, \quad (3.7)
\]

and

\[
E \text{ summable on } S
\]

where \( E = E^\text{scat} + E^\text{inc} \) is the total electric field.

The integral equation for the total magnetic surface current \( a = n \times H \),

\[
H = H^\text{inc} + H^\text{scat}, \text{ generally attributed to Maue (Hönl et al. [10])},
\]

is

\[
a(P_0) - \hat{n}(P_0) \times \left[ \nabla \times \frac{1}{2\pi} \int_S \frac{e^{ik R_2}}{R_2} a(P_2) \, dS \right] = 2\hat{n}(P_0) \times H^\text{inc}(P_0). \quad (3.8)
\]

We adopt the convention that the subscript on the \( \nabla \)-operator indicates the differentiation variable.
Define the operator $R_k$ to be
\[ R_k a := \hat{n}(P_0) \times \left[ \nabla_0 \times \frac{1}{2\pi} \int_S \frac{e^{ik R_{20}}}{R_{20}} a(P_2) \, dS_2 \right]. \] (3.9)

In which case, Eq. (3.8) can be written
\[ a - R_k a = c \] (3.10)

where $c = 2\hat{n} \times H^{inc}$. $R_k$ is an operator valued function of $k$ which becomes, for $k = 0$,
\[ R_0 a = \hat{n}(P_0) \times \left[ \nabla_0 \times \frac{1}{2\pi} \int_S \frac{1}{R_{20}} a(P_2) \, dS_2 \right]. \] (3.11)

We propose to solve Eq. (3.10) using the iteration
\[ a^{(n+1)} = R_k a^{(n)} + c, \quad a^{(0)} = c. \] (3.12)

Since $R_k$ is linear, this leads to the sequence
\[ a^{(n+1)} = \sum_{j=0}^{n+1} R_j^k c. \] (3.13)

In a Banach space setting we prove that if $c$ belongs to an appropriate space $B$ with norm $\| \cdot \|_B$, if $R_k$ maps $B$ into $B$ (hence $R^j_k$ does also) and if the spectral radius $r_s(R_k)$ of $R_k$ is less than one, then for small values of $k$ the sequence (3.13) converges in norm to a function in $B$. This limit will satisfy the integral equation (3.10) (Taylor [24, p. 262]). Our proof involves establishing these properties for $R_0$ before considering $R_k$.

4. Basic Properties of $R_0$

In this section, we establish the existence, Hölder continuity, and compactness of $R_0$.

**Theorem 4.1.** If $a \in C_r(S)$, then
\[ |R_0 a| \leq A \| a \|_0 \] (4.1)

where $A$ is a constant which depends only on the surface $S$; i.e., $\| R_0 \|_0 \leq A$.

Werner [26, Lemma 16] has proved Theorem 4.1 in the case when $S$ is smooth and Kress [15, Satz 4.2] has proved it for $S$ Lyapunov, $\varepsilon = 1$. Kress' proof can be modified for $0 < \varepsilon < 1$ using the fact that weakly singular integrals on Lyapunov surfaces exist (Mikhlin [17, p. 158]).
Theorem 4.2. Let $S$ be Lyapunov, $0 < \alpha < 1$. If $\mathbf{a} \in C_{r}(S)$ then $R_{0}\mathbf{a}$ is Hölder continuous with exponent $\beta$, $0 < \beta < \alpha < 1$; i.e.,

$$|R_{0}\mathbf{a}(P_{1}) - R_{0}\mathbf{a}(P_{2})| < A \|\mathbf{a}\|_{0} |P_{1} - P_{2}|^{\beta}. \tag{4.2}$$

The exponent $\beta$ is arbitrary, but must satisfy the strict inequality. The constant $A$ depends only on the surface $S$.

The Hölder continuity of $R_{0}$ has been established by Werner [26, Lemma 6] in the case of smooth $S$ and by Kress [15, Satz 4.3] in the case $S$ Lyapunov, $\alpha = 1$. To show $R_{0}$ is Hölder continuous in the case $S$ Lyapunov, $0 < \alpha < 1$, we write $R_{0}$ as the difference of two integrals:

$$R_{0}\mathbf{a} = R_{0}^{(1)}\mathbf{a} - R_{0}^{(2)}\mathbf{a} \tag{4.3}$$

where

$$R_{0}^{(1)}\mathbf{a} = \frac{1}{2\pi} \int_{S} \left[ \hat{n}(P_{0}) \cdot \mathbf{a}(P_{2}) \right] \nabla_{0} \frac{1}{R_{20}} \, dS_{2} \tag{4.4}$$

and

$$R_{0}^{(2)}\mathbf{a} = \frac{1}{2\pi} \int_{S} \mathbf{a}(P_{2}) \left[ \hat{n}(P_{0}) \cdot \nabla_{0} \frac{1}{R_{20}} \right] \, dS_{2}. \tag{4.5}$$

Observe that $R_{0}^{(2)}\mathbf{a}$ is the normal derivative of a simple layer. Günter [8, p. 61] has proved that if $\mathbf{a}$ is bounded and integrable, then $R_{0}^{(2)}\mathbf{a}$ exists, is bounded and is Hölder continuous. Although Günter did not specifically relate the Hölder index of $R_{0}^{(2)}\mathbf{a}$ to the Hölder index of the surface $S$, such an estimate is contained in his proof. Using techniques similar to those of Günter, it is possible to complete the proof by showing that $R_{0}^{(1)}\mathbf{a}$ also exists and is Hölder continuous (Gray [7, Theorems 4.2 and 4.3]).

In proving compactness, the following result is useful.

Lemma 4.1. If $\mathbf{a} \in C_{r}^{\beta}(S)$ then $R_{0} : C_{r}^{\beta}(S) \rightarrow C_{r}^{\beta}(S)$ and $\|R_{0}\|_{\beta} < M, 0 < \beta < \alpha < 1$.

Proof. Since $C_{r}^{\beta}(S) \subset C_{r}(S)$, the fact that $R_{0}$ maps $C_{r}^{\beta}(S)$ into itself follows from Theorem 4.2. To estimate the operator norm we have from (4.2) an estimate of the Hölder constant:

$$h_{\alpha}[R_{0}\mathbf{a}] = \sup_{\substack{P_{1}, P_{2} \in S \\ P_{1} \neq P_{2}}} \frac{|R_{0}\mathbf{a}(P_{1}) - R_{0}\mathbf{a}(P_{2})|}{|P_{1} - P_{2}|^{\beta}} < B \|\mathbf{a}\|_{0}. \tag{4.6}$$
With (4.1) and (4.6), we estimate the $\beta$-norm of $R_0a$ as follows:

$$\|R_0a\|_\beta = \|R_0a\|_0 + h_\beta [R_0a] < M \|a\|_0. \quad (4.7)$$

Since $C^\alpha(S) \subset C_\gamma(S)$, we can replace $\|a\|_0$ by $\|a\|_\mu$ in inequality (4.7). Thus, for $\|a\|_\beta \neq 0$,

$$\|R_0\|_\beta = \sup_{a \in C_\gamma(S)} \frac{\|R_0a\|_\beta}{\|a\|_\beta} < M \quad (4.8)$$

which completes the proof.

For $S$ smooth, Werner [27, Lemma 5] has proved that $R_0$ is compact, mapping $C_\gamma(S)$ into $C_\gamma(S)$. Kress [15, Satz 4.5] has proved this result for $S$ Lyapunov, $\alpha = 1$. This also holds for $S$ Lyapunov, $0 < \alpha < 1$. We summarize these properties in

**Theorem 4.3.** The operator $R_0$ is compact in the following cases:

(a) $C_\gamma(S) \to C_\gamma(S)$,

(b) $C^\alpha_\gamma(S) \to C_\gamma(S)$, $0 < \alpha \leq 1$,

(c) $C^\beta_\gamma(S) \to C^\beta_\gamma(S)$, $0 < \beta < \alpha < 1$,

(d) $C^\beta_\gamma(S) \to C^\beta_\gamma(S)$, $0 < \beta < \alpha < 1$.

**Proof.** The proof of (a) is given by Werner [27] and is omitted here. The validity of (b) follows immediately since $C^\alpha_\gamma(S) \subset C_\gamma(S)$. To prove (c), we note that $a \in C^\beta_\gamma(S)$ implies $a \in C_\gamma(S)$ and Theorem 4.2 implies that $R_0a \in C^\beta_\gamma(S)$.

Let $\{a_n\}$ be a uniformly bounded sequence of functions from $C^\gamma(S)$; i.e., there exists a constant $M$ independent of $n$ such that

$$\|a_n\|_\chi < M. \quad (4.9)$$

It follows from (4.9) that

$$h_\gamma [a_n] \leq \|a_n\|_\chi < M. \quad (4.10)$$

Hence each $a_n$ satisfies

$$|a_n(P_1) - a_n(P_2)| \leq h_\gamma [a_n] |P_1 - P_2|^\alpha < M |P_1 - P_0|^\gamma. \quad (4.11)$$

Inequality (4.11) implies that the sequence $\{a_n\}$ is equicontinuous. Inequalities (4.9) and (4.11) imply via the Ascoli–Arzelà theorem that there is a subsequence $\{a_{n_k}\}$ which converges in the supremum norm.
To show compactness in case (c), we must show that the sequence \( \{ R_0 a_n \} \) converges in the \( \beta \)-norm. We have with inequality (4.2)

\[
\| R_0 (a_{n_1} - a_{m_1}) \|_\beta \leq M \| a_{n_1} - a_{m_1} \|_0. 
\]

Since the sequence \( \{ a_{n_1} \} \) is a Cauchy sequence in the supremum norm, inequality (4.12) implies that \( \{ R_0 a_{n_1} \} \) is a Cauchy sequence in the \( \beta \)-norm. The space \( C^\beta(S) \), however, is complete in the \( \beta \)-norm, which proves our assertion.

To establish compactness in case (d), observe that \( C^\beta(S) \subset C^\gamma(S) \). The result follows directly from (c).

5. The Eigenvalues of \( R_0 \)

Plemelj's theorem (Haack and Wendland [9]) gives the nature and location of the eigenvalues of the integral operator of classical potential theory and ensures the convergence of Neumann's series in that context as well as for the Helmholtz problem considered by Kleinman and Wendland [13]. Kleinman and Wendland have remarked that this extended Neumann method might be used to solve integral equations which arise in other problems provided an analogue of Plemelj's Theorem were valid. We show that such an analogue is valid for the operator \( R_0 \).

The proof of Plemelj's theorem for \( R_0 \) is built on properties of the simple layer potential \( V a \) (2.4), together with vector forms of the Divergence Theorem (Günter, [8, p. 26]) and the eigenvalue equation

\[
a - \lambda R_0 a = 0. \tag{5.1}
\]

A first step in the proof of Plemelj's Theorem for \( R_0 \) is

**Theorem 5.1.** If \( a \) satisfies the eigenvalue equation (5.1) then the eigenvalues \( \lambda \) are real and \(| \lambda | \geq 1 \).

Müller and Niemeyer [19, Lemma 5] have proved Theorem 5.1 for \( S \) smooth using a representation of \( \nabla \times \nabla \times V a \) as the gradient of a simple layer; i.e.,

\[
\nabla \times \nabla \times V a = -\nabla \frac{1}{2\pi} \int_S (V_S \cdot a) \frac{1}{R} dS, \quad P \notin S, \tag{5.2}
\]

where \( V_S \cdot a \) is the surface divergence of \( a \) (see also Werner [27, p. 363]). Kress [15, Satz 5.1] has proved Theorem 5.1 for \( S \) Lyapunov, \( \alpha = 1 \). Kress
proved that if \( a \) satisfies (5.1), then \( \nabla \cdot Va \) can be represented as a double layer; i.e.,

\[
-\frac{\lambda}{2\pi} \int_S (\nabla \cdot Va) \frac{\partial}{\partial n_i} \frac{1}{R_{\alpha}} \, dS = (1 + \lambda) \nabla \cdot Va, \quad P \in D_i,
\]

\[
= \nabla \cdot Va, \quad P \in S,
\]

\[
= (1 - \lambda) \nabla \cdot Va, \quad P \in D_\epsilon. \tag{5.3}
\]

In this case, the eigenvalue equation (5.1) becomes

\[
(1 - \lambda) \frac{\partial}{\partial n_e} (\nabla \cdot Va) - (1 + \lambda) \frac{\partial}{\partial n_i} (\nabla \cdot Va) = 0. \tag{5.4}
\]

Kress' proof can be modified for S Lyapunov, \( 0 < \alpha < 1 \), provided we give an interpretation to the normal derivative of \( \nabla \cdot Va \). An example by Günter [8, p. 72] shows that for Lyapunov surfaces, \( 0 < \alpha < 1 \), the Hölder continuity of the density is not sufficient for the existence of the normal derivative of a double layer. Thus, it is evident by Günter's example and Kress' representation of \( \nabla \cdot Va \) as a double layer that \((\partial/\partial n)(\nabla \cdot Va)\) may not exist in the usual sense; i.e., as a point function.

Radon [21] introduced the idea of boundary flow to generalize the normal derivative for irregular domains. A slightly different version, which we also employ, was used by Burago et al. [4, p. 15]. A function \( u \) has interior boundary flow if for any \( \phi \in C^\infty_c(\mathbb{R}^3) \) and any sequence of smooth surfaces \( S_m \subset D, \) which reduce to \( S \) in the sense of Section 2, the limit

\[
Lu(\phi) = \lim_{m \to \infty} \int_S \phi \frac{d}{dn_m} u \, dS \tag{5.5}
\]

exists, where \( \hat{n}_m \) is the unit normal exterior to \( S_m \), and, moreover, the functional \( Lu(\phi) \) defined by (5.5) can be extended to \( C(S) \), the space of all continuous functions on \( S \). If the functional \( Lu(\phi) \) of (5.5) defined on \( C^\infty_c(\mathbb{R}^3) \) is bounded in \( C(S) \), then the extension of \( Lu(\phi) \) to \( C(S) \) can be represented by the Lebesgue–Stieltjes integral

\[
Lu(\phi) = \int_S \phi \sum^{(1)} (dS) \tag{5.6}
\]

where \( \Sigma^{(1)} \) is a set function of bounded total variation (Yosida [30, p. 119]). The set function \( \Sigma^{(1)} \) is called the interior boundary flow of \( u \). The exterior boundary flow \( \Sigma^{(e)} \) of \( u \) is defined analogously.

With the idea of boundary flow, it is possible to prove the following two theorems.
THEOREM 5.2. If $a \in C^2_+(S)$, then $\nabla \cdot V a$ has interior and exterior boundary flow.

THEOREM 5.3. If $a \in C^2_+(S)$, then $\nabla \cdot V a$ has a representation as a simple layer potential with a summable density. The proofs of Theorems 5.2 and 5.3 are deferred to the Appendix.

Since $\nabla \cdot V a$ has a representation as a simple layer with a summable density, it follows that the limiting values of the normal derivatives $\partial(\nabla \cdot V a)/\partial n_i$ and $\partial(\nabla \cdot V a)/\partial n_e$ exist a.e. on $S$ and are summable (Günter [8, p. 114]). We remark that Günter [8, p. 114] has also shown that the jump conditions for the normal derivative of a simple layer with a summable density remain valid a.e. We can use the approximating surface argument of Theorems 5.2 and 5.3 to establish that if $a$ satisfies (5.1), then Eq. (5.4) is valid a.e. With this established, the rest of the proof of Theorem 5.1 for $S$ Lyapunov, $0 < \alpha < 1$, follows that of Kress [15, Satz 5.1].

The next step in the proof of Plemelj's theorem for $R_o$ is to show that the eigenvalues $\lambda$ satisfy the strict inequality $|\lambda| > 1$; i.e., that $\lambda = \pm 1$ are not eigenvalues. For $\lambda = -1$, we have

THEOREM 5.4. If $a$ satisfies

$$a + R_0 a = 0$$

then $a \equiv 0$; i.e., $\lambda = -1$ is not an eigenvalue of $R_o$.

Proof. Werner [27, Lemma 13] has established this theorem for smooth surfaces $S$. It is easy to show using the approximating argument of Theorems 5.2 and 5.3 that the representation (5.3) holds for $S$ Lyapunov, $0 < \alpha < 1$. Hence, with $\lambda = -1$, (5.3) becomes

$$(\nabla \cdot V a) - \frac{1}{2\pi} \int_S (\nabla \cdot V a) \frac{d}{dn_2} R_2 dS_2 = 0. \quad (5.8)$$

It follows that $\nabla \cdot V \equiv 0$ on $S$ (Mikhlin, [17, p. 375]). Since $a$ is tangential

$$\langle \nabla \cdot V a \rangle_i = \langle \nabla \cdot V a \rangle_S = \langle \nabla \cdot V a \rangle_e = 0. \quad (5.9)$$

But, $\nabla \cdot V a$ is harmonic in $D_i$ and $D_e$. Thus, with the boundary conditions (5.9), $\nabla \cdot V a \equiv 0$ in all of $\mathbb{R}^3$. Hence

$$\nabla \times \nabla \times V a = A V a = 0 \quad \text{in } D_i \text{ and } D_e. \quad (5.10)$$

Equation (5.7) written in terms of the jump condition (2.5) is

$$\hat{n} \times (\nabla \times V a)_e = 0. \quad (5.11)$$
Application of the Divergence Theorem to $Va \cdot \hat{n} \times (\nabla \times Va)$ with (5.11) yields

$$- \int_{D_\epsilon} |\nabla \times Va|^2 \, d\tau = \int_S V a \cdot \hat{n} \times (\nabla \times Va)_e \, dS = 0. \tag{5.12}$$

Thus,

$$\nabla \times Va = 0 \quad \text{in } D_\epsilon. \tag{5.13}$$

The jump conditions (2.5) and Eq. (5.13) imply

$$\hat{n} \cdot (\nabla \times Va)_i = \hat{n} \cdot (\nabla \times a)_c = 0. \tag{5.14}$$

With (5.10), there exists a potential function $\phi$ such that

$$\nabla \times Va = \nabla \phi \quad \text{in } D_\epsilon. \tag{5.15}$$

Application of the Divergence Theorem to $\phi(\nabla \times Va)$ with (5.14) yields

$$\int_{D_\epsilon} |\nabla \times Va|^2 \, d\tau = \int_{D_\epsilon} (\nabla \times Va)_e \nabla \phi \, d\tau \\
= \int_{D_\epsilon} \nabla \cdot (\phi \nabla \times Va) \, d\tau = \int_S \hat{n} \cdot (\nabla \times Va)_e \phi \, dS = 0. \tag{5.16}$$

Thus,

$$\nabla \times Va = 0 \quad \text{in } D_\epsilon. \tag{5.17}$$

Equations (5.13) and (5.17) with the jump condition (2.5) imply

$$\hat{n} \times a = 0.$$

But $a$ is tangential; i.e., $\hat{n} \cdot a = 0$. Thus $a \equiv 0$.

To show that $\lambda = +1$ is not an eigenvalue of $R_0$ we first discuss the adjoint $R_0^*$. Here we need two preliminary theorems.

**Theorem 5.5.** For $a, b \in C_\tau(S)$, the operator $R_0^*$ defined by

$$R_0^* a = \hat{n}(P_0) \times \left[ \hat{n}(P_0) \times \frac{1}{2\pi} \int_S \frac{1}{R_{20}} \times (a(P_2) \times \hat{n}(P_2)) \, dS_2 \right] \tag{5.18}$$

is adjoint to the operator $R_0$ in the sense

$$\int_S a \cdot R_0 b \, dS = \int_S b \cdot R_0^* a \, dS. \tag{5.19}$$
THEOREM 5.6. If \( a \in C_r(S) \) and \( b = a \times \hat{n} \), then \( b \) is a solution of
\[
b + R_0^* b = 0
\]
if and only if \( a \) is a solution of
\[
a - R_0 a = 0.
\]

Werner [28] proved Theorems 5.5 and 5.6 for \( S \) smooth but the same proof suffices for \( S \) Lyapunov. The applicability of Fubini's theorem, needed in Theorem 5.5, is assured by Theorem 4.1.

Finally, since \( R_0 \) is compact, we show via the Fredholm alternative that \( \lambda = +1 \) is a regular value of \( R_0 \).

THEOREM 5.7. If \( a \) satisfies (5.21), then \( a \equiv 0 \), i.e., \( \lambda = +1 \) is not an eigenvalue of \( R_0 \).

Proof. By hypothesis \( a \) is a solution of (5.1) for \( \lambda = +1 \). Define \( b = \hat{n} \times a \). By Theorem 5.6, \( b \) satisfies the adjoint equation \( b - \lambda R_0^* b = 0 \) for \( \lambda = -1 \). We have shown that \( \lambda = -1 \) is a regular value of \( R_0 \) (Theorem 5.4) and that \( R_0 \) is compact (Theorem 4.3). By Fredholm's alternative, \( \lambda = -1 \) is also a regular value of \( R_0^* \). Since \( b \) satisfies the adjoint equation (5.20), \( b \equiv 0 \). By definition of \( b \), \( \hat{n} \times a = 0 \). But \( a \) is tangential; i.e., \( \hat{n} \cdot a = 0 \). Thus \( a \equiv 0 \).

We have shown that the eigenvalues \( \lambda \) of \( R_0 \) are real and \( |\lambda| \geq 1 \) (Theorem 5.1) and that neither \( \lambda = \pm 1 \) are eigenvalues (Theorems 5.4 and 5.7). Collectively, these results constitute the analogue of Plemelj's theorem which we set out to prove; namely,

THEOREM 5.8. The eigenvalues \( \lambda \) such that \( a - \lambda R_0 a = 0 \) where \( R_0 : C_r(S) \to C_r(S) \) are real and strictly greater than one in absolute value (\( |\lambda| > 1 \)).

This result, together with the fact \( R_0 \) is a compact mapping of \( C_r(S) \) into \( C_r(S) \) (Theorem 4.3), enables us to prove the convergence of the sequence of iterates \( \{ R_0 a \} \). This will be done in the next section, where we also establish convergence of the sequence of iterates \( \{ R_k a \} \).

6. CONVERGENCE OF THE SUCCESSIVE APPROXIMATIONS

The kernel of the operator \( R_k \) can be written
\[
\nabla \frac{e^{ik R}}{R} = \nabla \frac{1}{R} + \nabla \frac{e^{ik R} - 1}{R}.
\]
Using elementary calculations, it is possible to show, for points $P_1, P_2$ and $P_0$ on $S$, that the function $\nabla(e^{ikR} - 1/R)$ is bounded, i.e.,

$$\left| \nabla \frac{e^{ikR} - 1}{R} \right| \leq A |k|^2 \max\{1, e^{-|kD|}\}.$$  \hspace{1cm} (6.2)

and Hölder continuous, i.e.,

$$\left| \nabla_1 \frac{e^{ikR_{10}} - 1}{R_{10}} - \nabla_2 \frac{e^{ikR_{20}} - 1}{R_{20}} \right| \leq |B| |k|^{3} \max\{1, e^{-|kD|}\} |P_1 - P_2|$$ \hspace{1cm} (6.3)

where $D = \sup_{P_2, P_1 \in S} |P_2 - P_1|$ and $A$ and $B$ are constants which depend only on the surface $S$.

Since

$$(R_k - R_0) a = \hat{n}(P_0) \times \frac{1}{2\pi} \int_S a(P_2) \times \nabla_0 \frac{e^{ikR} - 1}{R} dS_2,$$ \hspace{1cm} (6.4)

inequalities (6.2) and (6.3) can be used to show that $R_k$ has the properties

$$R_k : C_\gamma(S) \rightarrow C_\gamma(S)$$ \hspace{1cm} (6.5)

$$\|R_k - R_0\|_0 \leq |k|^2 C_1 \max\{1, e^{-|kD|}\}$$ \hspace{1cm} (6.6)

$$\|R_k - R_0\|_\beta \leq [A \|k\|^2 + B_1 \|k\|^3] \max\{1, e^{-|kD|}\}$$ \hspace{1cm} (6.7)

where $A_1, B_1,$ and $C_1$ are constants which depend only on the surface $S$. It is clear that (6.6) can be made arbitrarily small provided $|k|$ is sufficiently small.

Recall that $R_0$ is completely continuous, mapping $C_\gamma(S)$ into $C_\gamma(S)$ (Theorem 4.3), and that the eigenvalues $\lambda$ of $R_0$ are real and strictly greater than one in absolute value (Theorem 5.8). Hence, if $\lambda_1$ denotes the smallest eigenvalue of $R_0$, then $|\lambda_1| > 1$. Thus, the points $A = \{\lambda : |\lambda| \leq 1\}$ are not eigenvalues and hence belong to the resolvent set. Consequently, the spectrum $\sigma(R_0)$ is in the complement of the set $A$. Furthermore, $R_0$ is compact. Hence, the spectrum is discrete and the only limit point is $\lambda = \infty$. Therefore,

$$\inf_{\lambda \in \sigma(R_0)} |\lambda| = |\lambda_1| > 1.$$  \hspace{1cm} (6.8)

If in inequality (6.8) we replace $\lambda$ by its reciprocal $1/\lambda$, then the left-hand side of (6.8) is the reciprocal of the spectral radius of $R_0$ (Taylor [24, p. 262]). Thus, the spectral radius of $R_0$ is strictly less than one. Hence we have established
**Theorem 6.1.** The Neumann series for \((I - \lambda R_0)^{-1}\) converges in the operator norm \(\| \cdot \|_0\) for \(|\lambda| < 1\).

An immediate consequence of Theorem 6.1 is that for \(k = 0\) the iteration scheme (3.12) converges to the actual solution of the integral equation (3.10). Furthermore, since \(R_0\) maps \(C_\tau(S)\) into \(C_\beta(S)\), \(\beta < \alpha\), the solution \(a = \sum_{n=0}^{\infty} (\lambda R_0)^n c\) of \(a - R_0 a = c\) is not only in \(C_\tau(S)\), provided \(c \in C_\beta(S)\), but is actually in \(C_\beta(S)\), \(\beta < \alpha\).

That the successive approximation method of solution remains valid for small values of \(k\) is proved in

**Theorem 6.2.** For any \(\rho, 1 < \rho < |\lambda|\), there exists a \(k(\rho) > 0\) such that for all \(k\) with \(|k| < k(\rho)\) the Neumann series for \((I - R_k)^{-1}\) converges in the operator norm \(\| \cdot \|_0\); i.e., the approximation scheme (3.12) converges. Moreover, for all \(k\) with \(|k| < k(\rho)\),

\[
\| R_0 - R_k \| \leq (\| (I - R_0)^{-1} \|_0)^{-1}. \tag{6.9}
\]

The proof of Theorem 6.2 is essentially the same as that used by Kleinman and Wendland \([13]\) in the scalar Neumann problem. As before, even though convergence is established initially in the norm \(\| \cdot \|_0\), the fact that \(R_k\) actually maps \(C_\tau(S) \rightarrow C_\beta(S)\), \(\beta < \alpha\) means that the iterative solution of (3.10) is in \(C_\beta(S)\), provided of course that the inhomogeneous term \(c\) also lies in this space.

### 7. The Equivalence Problem

The vector fields

\[
W_a = \nabla \times \left( \frac{1}{2\pi} \int_S a e^{ik\cdot r} dS \right), \quad U_a = -\frac{1}{i\omega\epsilon} \nabla \times W_a \tag{7.1}
\]

satisfy the time-independent Maxwell's equations in \(D_e\)

\[
\nabla \times W_a = i\omega\epsilon U_a, \quad \nabla \times U_a = i\omega\mu W_a \tag{7.2}
\]

and the radiation conditions

\[
\lim_{r \to \infty} \{ \hat{r} \times (\nabla \times W_a) + ik W_a \} = 0 \tag{7.3}
\]

\[
\lim_{r \to \infty} \{ \hat{r} \times (\nabla \times U_a) + ik U_a \} = 0 \tag{7.4}
\]
uniformly in $\theta$ and $\phi$. Thus, it is apparent that $W_a$ and $U_a$ act like the scattered fields $H^\text{scat}$ and $E^\text{scat}$, respectively. We define the total fields

$$H = H^\text{inc} + \frac{1}{2} W_a, \quad P \in D_e, \quad (7.5)$$

$$E = -\frac{1}{io\varepsilon} \nabla \times H, \quad P \in D_e. \quad (7.6)$$

We will show that if $a$ is a solution of

$$a - R_k a = c \quad (7.7)$$

then for small values of $k$, $H^\text{scat} = 1/2 W_a$ and $E^\text{scat} = 1/2 U_a$ satisfy (3.2) and (3.5) in the case $S$ smooth and (3.6) and (3.7) in the case $S$ Lyapunov.

For smooth $S$, if $a$ satisfies (7.7), then the tangential derivatives of $a$ exist ([Werner [27, Lemmas 1 and 2]]. Thus, $W_a$ and $\nabla \times W_a$ are continuous in $D_a \cup S$ and $D_a \cup S$ ([Werner [27, Lemma 3 and Eqs. (2.21)-(2.22)]]. Hence, for smooth $S$, $W_a$ and $U_a$ satisfy (3.5). The function $W_a$ is the sum of the curl of a simple layer potential and an integral which is Hölder continuous; i.e.,

$$W_a = \nabla \times \left( \frac{1}{2\pi} \int_S a \frac{1}{R} dS + \frac{1}{2\pi} \int_S a \times \nabla \frac{e^{ikR}}{R} dS \right). \quad (7.8)$$

Hence, if $a \in C^2(S)$ and $S$ is Lyapunov, the jump conditions (2.5) hold; moreover, $W_a \in C^0(S)$. Thus for $S$ Lyapunov, $W_a$ satisfies (3.6).

To establish the boundary conditions (3.2) and (3.7) we first show that $H$ has no tangential components on $S$. The limit from the exterior of (7.5) is, with (2.5),

$$H_e = H^\text{inc} + \frac{1}{2} (W_a)_e = H^\text{inc} + \frac{1}{2} (W_a)_S - \frac{1}{2} \hat{n} \times a; \quad (7.9)$$

hence the scalar product of $\hat{n}$ with Eq. (7.9) is

$$\hat{n} \cdot H_e = \hat{n} \cdot H^\text{inc} + \frac{1}{2} \hat{n} \cdot (W_a)_S. \quad (7.10)$$

We claim that if $a$ satisfies Eq. (7.7) and $H$ is defined by (7.5), then for $k$ sufficiently small

$$\hat{n} \cdot H_e = 0. \quad (7.11)$$

To validate this assertion, we proceed as follows.

Werner has shown that for smooth $S$ the operator $R_k$ is an entire function of $k$ and the same proof holds for $S$ Lyapunov. Werner has also shown that for smooth $S$ the inverse operator $(I - R_k)^{-1}$ exists and
depends analytically on \( k \) in a neighborhood of \( k = 0 \). To extend this to the Lyapunov case, we argue as follows. Since \( (I - R_k)^{-1} \) exists and is bounded (Theorem 6.1) and inequality (6.9) holds for all \( |k| < k(\rho) \), the inverse \( (I - R_k)^{-1} \) exists, is bounded, and is given by

\[
(I - R_k)^{-1} = \sum_{j=0}^{\infty} \left[ -(I - R_0)^{-1}(R_k - R_0) \right]^j (I - R_0)^{-1}
\]  

(7.12)

(Taylor [24, p. 164]). Moreover, the series converges uniformly with respect to the operator norm \( \| \|_0 \) for \( |k| < k(\rho) \). Since each term of (7.12) is analytic in \( k \) and since convergence is uniform, the inverse operator \( (I - R_k)^{-1} \) is analytic in \( k \) for \( |k| < k(\rho) \) and similarly for \( (I + R_k)^{-1} \).

Next, we prove a preliminary lemma.

**Lemma 7.1.** The interior boundary value problem

\[
\nabla \cdot \psi = 0 \quad \nabla \times \nabla \times \psi - k^2 \psi = 0 \quad \text{in} \ D_i.
\]

(7.13)

\[
\hat{n} \times \psi = 0 \quad \text{on} \ S_i.
\]

(7.14)

has, for small values of \( k \), only the solution \( \psi \equiv 0 \) in \( D_i \).

**Proof.** Let \( \psi = \hat{n} \times \nabla \times \psi_S \). It may be shown that in \( D_i \)

\[
\nabla_0 \times \psi = -\nabla_0 \times \frac{1}{4\pi} \int_S \psi (P_2) \frac{e^{ikR}}{R} \ dS.
\]

(7.15)

The jump condition (2.5) implies

\[
\nabla \times \psi_S = -\frac{1}{2} (\nabla \times \psi)_S \quad \text{in} \ \nabla \times \psi_S.
\]

(7.16)

Equation (7.16) written in terms of the operator \( R_k \) is

\[
\psi + R_k \psi = 0.
\]

(7.17)

We have shown that for \( k = 0 \), Eq. (7.17) has only the solution \( \psi \equiv 0 \) (Theorem 5.4). Since \( R \) is analytic in \( k \) and \( (I + R_k)^{-1} \) exists and is analytic in \( k \) in a neighborhood of \( k = 0 \), Eq. (7.17) has only the trivial solution \( \psi \equiv 0 \), for small values of \( k \). Hence \( \nabla \times \psi \), defined by (7.16), is identically zero for small values of \( k \) and, with (7.13), \( \psi \) also vanishes. A separate potential theoretic argument is required to show that \( \psi = 0 \) when \( k = 0 \).

Now we are able to prove

**Theorem 7.1.** If \( \mathbf{a} \) satisfies (7.7) and \( \mathbf{H} \) is defined by (7.5), then for \( k \) sufficiently small \( \hat{n} \cdot \mathbf{H}_a = 0 \).
Proof. Consider, for $P_0 \in D_i$, the function
\[ \psi = H^{\text{inc}} + \frac{1}{2} W a. \] (7.18)

In $D_i$, the function $\psi$ satisfies
\[ \nabla \times \nabla \times \psi - k^2 \psi = 0, \quad \nabla \cdot \psi = 0 \] (7.19)
since both $W a$ and $H^{\text{inc}}$ satisfy these equations in the interior. By hypothesis, $a$ satisfies (7.7). Hence, with the jump condition (2.5)
\[ n \times [H^{\text{inc}} + \frac{1}{2} (W a)] = 0, \]
i.e.,
\[ n \times \psi_i = 0 \quad \text{on } S. \] (7.20)

For small values of $k$, the only solution of the interior boundary value problem (7.18b)-(7.19) is $\psi \equiv 0$ (Lemma 7.1). The limiting values of $\psi$ exist and are continuous on $S$; i.e., $n \cdot \psi_i = 0$. Hence, with the jump conditions (2.6) and the fact that $a$ is tangential, we obtain
\[ n \cdot H_e = n \cdot H^{\text{inc}} + \frac{1}{2} (n \cdot W a) = n \cdot H^{\text{inc}} + \frac{1}{2} (n \cdot W a) = n \cdot \psi_i = 0. \]

Thus far we have shown that the solutions $W a$ and $U a$ satisfy the time-harmonic Maxwell equations and the radiation conditions. In addition, it has been shown that $W a$ satisfies the appropriate boundary conditions and smoothness properties on $S$. It remains to demonstrate the corresponding properties for $U a$.

Werner [27, Lemma 4] has shown in the smooth case that if $a$ satisfies (7.7) then
\[ n \times (\nabla \times W a)_i = n \times (\nabla \times W a)_e. \] (7.21)

Since $H^{\text{inc}}$ is continuously differentiable,
\[ n \times [\nabla \times (H^{\text{inc}} + \frac{1}{2} W a)] = n \times [\nabla \times (H^{\text{inc}} + \frac{1}{2} W a)]. \] (7.22)

But for small $k$, $H^{\text{inc}} + \frac{1}{2} W a \equiv 0$ in $D_i$ (Lemma 7.1). Thus, we have established
\[ n \times E_e = n \times [\nabla \times (H^{\text{inc}} + \frac{1}{2} W a)] = 0 \] (7.23)
in the case $S$ smooth. For $S$ Lyapunov, we prove

**Theorem 7.2.** If $a$ satisfies (7.7) and $E$ is defined by (7.6) then for $k$ sufficiently small
\[ n \times E_e = 0 \] (7.24)
almost everywhere on $S$ and, moreover, the $E$-field is defined almost everywhere on $S$ and is summable.
Proof. We have shown that the limit $H_c$ exists (Eq. (7.9)). Hence, the limit
\[
(\nabla \times E)_c = i\omega \mu H_c
\] (7.25)
exists. Since $\hat{n} \cdot H_c = 0$ for small values of $k$ (Theorem 7.1), it follows by Stokes' theorem for a closed surface that
\[
\int_S (\hat{n} \times E)_c \cdot \nabla \phi dS = \int_S \hat{n} \cdot (\nabla \times \phi E) dS - \int_S \hat{n} \cdot \nabla \phi \times E dS = \int_S \hat{n} \cdot H_c \phi dS = 0
\] (7.26)
for every $\phi \in C_0^\infty(S)$. Thus,
\[
\hat{n} \times E_c = 0
\] (7.27)
almost everywhere on $S$. This shows that the boundary condition (3.7) for the $E$-field is satisfied. If we compute $\hat{n} \cdot E_c$, we obtain
\[
\hat{n} \cdot E_c = -\frac{1}{i\omega \epsilon} \hat{n} \cdot \nabla \times H^{inc} + \frac{\epsilon}{\epsilon_n \epsilon} \nabla \cdot \frac{1}{4\pi} \int_S a(P_2) e^{ikR_2} dS_2
\]
\[
+ k^2 \frac{1}{4\pi} \int_S \hat{n}(P_0) \cdot a(P_2) e^{ikR_2} dS_2.
\] (7.28)
Referring to Eq. (7.28), the first and third terms in the braces are defined and Hölder continuous and hence summable on $S$. The second term, the limiting value of the normal derivative of the divergence of a simple layer, exists almost everywhere on $S$ and is summable (Theorem 5.3). Thus $\hat{n} \cdot E_c$ is defined almost everywhere and is summable. Since we have already shown that $\hat{n} \times E_c = 0$ almost everywhere on $S$, we have that $E_c$ exists almost everywhere on $S$.

Appendix

We provide here the proofs of Theorems 5.2 and 5.3.

Proof of Theorem 5.2. We will prove that $Va$ has interior boundary flow. The proof of the existence of the exterior boundary flow is similar. Let $S_m$ be a sequence of smooth surfaces lying in $D_i$ which reduce to $S$ in the sense of Section 2. First we show that for all $\phi \in C_0^\infty(R^3)$ the limit
\[
F(\phi) = \lim_{m \to \cdot} \int_{S_m} \phi \frac{\partial}{\partial n_m} \nabla \cdot Va dS_m
\] (A.1)
exists and defines a bounded linear functional on $C_0^\infty(R^3)$. The function $\nabla \cdot V\mathbf{a}$ is harmonic in $D_j$. Hence,

$$\nabla(\nabla \cdot V\mathbf{a}) = \nabla \times \nabla \times V\mathbf{a}. \quad (A.2)$$

Applying first (A.2), the vector identity $\nabla \times \mathbf{f} = \nabla \phi \times \mathbf{f} + \phi \nabla \times \mathbf{f}$, and Stokes' theorem for the closed surface $S_m$, we obtain for all $\phi \in C_0^\infty(R^3)$

$$\int_{S_m} \phi \frac{\partial}{\partial n_m} \nabla \cdot V\mathbf{a} \, dS_m$$

$$= \int_{S_m} \phi \hat{n} \cdot \nabla \times \nabla \times V\mathbf{a} \, dS_m$$

$$= \int_{S_m} \phi \hat{n} \cdot \nabla \times \nabla \times V\mathbf{a} \, dS_m - \int_{S_m} \hat{n} \cdot \nabla \times [\phi \nabla \times V\mathbf{a}] \, dS_m$$

$$= \int_{S_m} \hat{n} \cdot [(\nabla \times V\mathbf{a}) \times \nabla \phi] \, dS_m. \quad (A.3)$$

Convergence of the right-hand side of (A.3) is shown as follows. Denote $(\nabla \times V\mathbf{a}) \times \nabla \phi$ by $\mathbf{A}$ and partition $S = \bigcup_{j=1}^N S_j$ such that for a given $\varepsilon$

$$p, q \in S_j \rightarrow |\hat{n}(p) \cdot \mathbf{A}(p) - \hat{n}(q) \cdot \mathbf{A}(q)| < \varepsilon \quad (A.4)$$

($N$ may depend on $\varepsilon$). This can be done by choosing $S_j$ to lie in a ball of sufficiently small radius. The intersection of $S$ with a finite number of such balls will provide a finite cover from which the $S_j$ may be chosen. Partition $S_m$ by the image of the partition of $S$ under $\phi_m$, i.e., $S_m = \phi_m(S_j)$. Then, with the Mean Value Theorem and some standard manipulation, it follows that

$$\left| \int_{S_m} \hat{n} \cdot \mathbf{A} \, dS_m - \int_S \hat{n} \cdot \mathbf{A} \, dS \right|$$

$$\leq \sup_j |\hat{n}(\phi_m(p_j)) \cdot \mathbf{A}(\phi_m(p_j)) - \hat{n}(p_j) \cdot \mathbf{A}(p_j)| \left| \int_{S_j} \, dS_m \right|$$

$$+ \sup_j |\hat{n}(p_j) \cdot \mathbf{A}(p_j) - \hat{n}(q_j) \cdot \mathbf{A}(q_j)| \left| \int_{S_m} \, dS_m \right|$$

$$+ \sup_j |\hat{n}(q_j) \cdot \mathbf{A}(q_j)| \sum_{j=1}^N \left| \int_{S_j} \, dS_m - \int_{S_j} \, dS \right|. \quad (A.5)$$
Continuity of $A$ in the closed domain $\overline{D}$, together with the assumed uniform convergence of the normal $n_m$ implies that the first term in (A.5) approaches zero with $m$, independently of $p$ and $N$. The second term in (A.5) is small since both $p$ and $q_J \in S^{(p)}$, i.e., Eq. (A.4). The convergence of $\int_{S'} dS_m$ to $\int_{S''} dS$ for any $J$ (Burago et al. [4, Lemma 5]) guarantees that the last term in (A.5) can be made small by taking $m$ sufficiently large; e.g., with $N$ fixed, $m$ may be chosen so that

$$\left| \int_{S'} dS_m - \int_{S''} dS \right| < \frac{\varepsilon}{N}.$$  

Thus,

$$\lim_{m \to \infty} \int_{S_m} n_m \cdot (\nabla \times V \mathbf{a}) \times \nabla \phi dS_m = \int_S n \cdot (\nabla \times V \mathbf{a}) \times \nabla \phi dS \quad \text{(A.6)}$$

and the limit (A.1) exists for all $\phi \in C^\infty_0(R^3)$ and remains valid for $\phi \in C^\infty(S)$. To prove that $V \cdot V \mathbf{a}$ has interior boundary flow, we must show further that the limit (A.6) can be extended to functions $\psi \in C(S)$. Let $\psi \in C^\infty(S)$. Then certainly $\psi \in C^1(S)$. Define $\psi'$, the extension of $\psi$ to the region $D'$, to be the solution of the interior Dirichlet problem: $V \psi' = 0$ in $D'$, $\psi' = \psi$ on $S$. The solution $\psi'$ is unique, depends continuously on the data, and since $\psi \in C^1(S)$, $\psi' \in C^1(D')$. $0 < \beta < \alpha \leq 1$ (Günter [8, p. 213]). For functions $\psi \in C^\infty(S)$, let

$$F_m(\psi) = F_m(\psi') = \int_{S_m} \psi' \cdot \nabla \phi dS_m.$$  

Since $\psi'$ is unique and depends continuously on the data, $F_m$ is well defined and continuous for each $m$.

The surface $S$ is closed and bounded and, hence, a compact subset of $R^3$. Let $\mathcal{D}$ denote the space $C^\infty(S)$ of infinitely differentiable functions defined on $S$ with the topology induced by the countable collection of norms

$$\|\phi\|_a = \sup_{p \in S} \{ |\phi(P)|, |D\phi(P)|, \ldots, |D^a \phi(P)| \} \quad \text{(A.8)}$$

for each multi-index $a$. Then $\mathcal{D}$ is a complete countably normed space (Gel'fand and Shilov [6, Vol. 2, p. 20]). In (A.7), we claim that the limit $m \to x$ $F_m(\psi)$ defines a continuous linear functional on $\mathcal{D}$. From (A.6), we see that $\lim_{m \to x} F_m(\psi)$ exists for all $\psi \in C^\infty(S)$. The function $\partial(\nabla \cdot V \mathbf{a})/\partial n_m$ is locally integrable off $S$. Hence, for each $m$, $F_m$ is a continuous linear functional on $\mathcal{D}$. Thus, for each $m$, $F_m \in \mathcal{D}'$, the dual space of $\mathcal{D}$. Since the dual space of a complete countably normed space is weakly sequentially complete (Gel'fand and Shilov [6, Vol. 2, p. 49]), there exists a function $F' \in \mathcal{D}'$ such that
A continuous linear functional defined on $\mathcal{D}$ generates a continuous linear functional on $C^p(S)$, $p > 1$, a closed subspace of $C(S)$ (Gel'fand and Shilov [6, Vol. 2, p. 37]). With the Hahn-Banach theorem, this functional can be extended to all of $C(S)$. An application of the Riesz theorem (Yosida [30, p. 19]) shows that there exists a function $\Sigma^{(i)}$ of bounded total variation such that

$$F(\psi) = \int_S \psi \Sigma^{(i)}(dS).$$

This proves the theorem.

**Proof of Theorem 5.3.** The function $\nabla \cdot V_\alpha$ is harmonic in $D_i$ and $D_e$ and is regular at infinity (Kellogg [11]). The limits $(\nabla \cdot V_\alpha)$, and $(\nabla \cdot V_\alpha)_e$ exist and are Hölder continuous on $S$ and, moreover, since $\alpha$ is tangential

$$(\nabla \cdot V_\alpha)_e = (\nabla \cdot V_\alpha)_e$$

(Eq. (2.5)). Thus, the existence and uniqueness of $u_i = \nabla \cdot V_\alpha$, $P \in D_i$, and $u_e = \nabla \cdot V_\alpha$, $P \in D_e$, as solutions of an interior and exterior Dirichlet problem, respectively, are guaranteed by a theorem of Günter [8, p. 178]. Let $S'_m$ be a sequence of surfaces in $D_j$ which reduce to $S$ (Section 2). Since $\Delta u_i = 0$ in $D_i$, we obtain

$$\frac{1}{4\pi} \int_{S'_m} u_i \frac{\partial}{\partial n_m} \frac{1}{R} dS - \frac{1}{4\pi} \int_{S'_m} \frac{1}{R} \frac{\partial}{\partial n_m} u_i dS = u_i, \quad P \in D^{m}_i,$$

$$= u_i, \quad P \in S'_m,$$  \hspace{1cm} (A.12)

$$= 0, \quad P \in D_e.$$

When $S'_m \to S$, the limit of the first integral exists (Burago et al. [4, Theorem 2]). By Theorem 5.2, the function $\nabla \cdot V_\alpha$ has interior boundary flow. Hence, the limit of the second integral exists. Therefore, the limiting value of (A.12) as $S'_m \to S$ exists and is given by

$$\frac{1}{4\pi} \int_S u_i \frac{\partial}{\partial n} \frac{1}{R} dS - \frac{1}{4\pi} \int_S \frac{1}{R} d\Sigma^{(i)} = u_i, \quad P \in D_i,$$

$$= u_i, \quad P \in S,$$  \hspace{1cm} (A.13)

$$= 0, \quad P \in D_e.$$
where } \Sigma^{(i)} \text{ is the function of bounded total variation generated by the interior boundary flow of } u_e = \nabla \cdot V a.

On the other hand, let } S_m \text{ be a sequence of surfaces in } D_e \text{ which reduce to } S \text{ (Section 2). Since } u_e = 0 \text{ in } D_e \text{ and } u_e \text{ is regular at infinity, we obtain a representation for } u_e \text{ similar to (A.12). When } S_m \to S, \text{ by reasoning similar to the above, we obtain

\[
\frac{1}{4\pi} \int_{S_m} u_e \frac{c}{\epsilon n} \frac{1}{R} \, dS - \frac{1}{4\pi} \int_{S} \frac{1}{R} \, d\Sigma^{(e)} = 0, \quad P \in D_e,
\]

\[
= \frac{u_e}{2}, \quad P \in S, \quad (A.14)
\]

\[
= -u_e, \quad P \in D_e,
\]

where } \Sigma^{(e)} \text{ is the function of bounded total variation generated by the exterior boundary flow of } u_e = \nabla \cdot V a.

Subtracting (A.14) from (A.13) and using the boundary condition } u_i = u_e \text{ on } S \text{ (A.11), we obtain

\[
\nabla \cdot V a = \frac{1}{4\pi} \int_{S} \frac{1}{R} \, d(\Sigma^{(i)} - \Sigma^{(e)}).
\]

Thus, } \nabla \cdot V a \text{ has a representation as a simple layer with a summable density.

REFERENCES