The degree of approximation by polynomials on some disjoint intervals in the complex plane

Maurice Hasson

Program in Applied Mathematics, The University of Arizona, Tucson, AZ 85721-0089, USA

Received 11 May 2005; received in revised form 10 April 2006; accepted 5 May 2006

Communicated by Manfred v Golitschek

Available online 30 June 2006

Abstract

Let \( f(z) \) be a continuous function defined on the compact set \( K \subset \mathbb{C} \) and let \( E_n(f) = E_n(f, K) \) be the degree of approximation to \( f \), for the supremum norm on \( K \), by polynomials of degree (at most) \( n \). Thus

\[
E_n(f, K) = \inf_{P \in \mathbb{P}_n} \| f - P \|.
\]

Here \( \mathbb{P}_n \) denotes the space of polynomials of degree at most \( n \) and \( \| \cdot \| \) is the supremum norm on \( K \).

For a positive integer \( s \) and for \( 0 < a < b \), let

\[
K = K_{a,b}^s = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b].
\]

We show that for a large class of piecewise analytic functions \( f \) defined on \( K \)

\[
\limsup_{n \to \infty} \left( E_n(f, K) \right)^\frac{1}{n} = \sqrt{\frac{b^2 - a^2}{b^2 + a^2}}
\]

thus recovering several classical results.

The proof of this error estimate is then translated into an algorithm that finds the polynomial of near best approximation.

© 2006 Elsevier Inc. All rights reserved.

MSC: 31E10; 41A10; 41A29; 41A40

Keywords: Complex approximation; Degree of approximation; Conformal mapping

---

\( E-mail \) address: mh047@bucknell.edu, hasson@math.arizona.edu

\( \dagger \) Supported by a VIGRE Postdoctoral Fellowship at the University of Arizona.

\( ^1 \) Current address: Department of Statistics, Texas A&M University, College Station, TX 77843-3143, USA.
1. Introduction

Let \( f(z) \) be a continuous function defined on the compact set \( K \subset \mathbb{C} \) and let \( E_n(f) = E_n(f, K) \) be the degree of approximation to \( f \), for the supremum norm on \( K \), by polynomials of degree (at most) \( n \). Thus

\[
E_n(f, K) = \inf_{P \in \mathbb{P}_n} \| f - P \|.
\]

Here \( \mathbb{P}_n \) denotes the space of polynomials of degree at most \( n \) and \( \| \cdot \| \) is the supremum norm on \( K \). Of special interest is the case where \( K \) consists of the union of several disjoint intervals in the complex plane. Our work originates from problems in the field of digital signal processing where approximation by polynomials (or rational functions) on disjoint intervals is used for the design of digital filters. See [7–9,11,19]. Approximation on disjoint intervals occurs also in acceleration of convergence techniques for iterative methods in linear algebra. See, among others, [2,3,21].

In the case where \( K \) consists of the union of two disjoint intervals results have been obtained by Fuchs and Hasson [9,11]. See also [5,12,13,22]. The results in these papers give estimates on the error \( E_n(f) \) but do not provide information on how to build the polynomial of (near) best approximation. This is a situation which occurs typically when overconvergence techniques (in the sense of Walsh) are used to prove error estimates: construction of a specific Green’s function with logarithmic pole at infinity provides error estimate but is otherwise not constructive.

For a positive integer \( s \) and for \( 0 < a < b \), let

\[
K = K_{a,b}^s = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b].
\]

Approximation on this set plays an important role in potential theoretic techniques for acceleration of convergence of iterative techniques to solve \( Ax = b \) when it is known that the eigenvalues of the matrix \( A \) are located on \( K \).

The set \( K = K_{a,b}^s \) where \( a \to 0 \) and \( b = \sqrt[4]{\frac{s}{4}} \), that is to say

\[
K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} \left[ 0, \sqrt[4]{\frac{s}{4}} \right],
\]

has been analyzed in detail by Henrici in [15,16] and Bartolomeo and He in [4] in their study of the Faber polynomials associated with certain regions of the complex plane. See also [6].

We show that

\[
\limsup_{n \to \infty} \left( E_n(f, K) \right)^{\frac{1}{n}} = \sqrt[2]{\frac{b^2 - a^2}{b^2 + a^2}} \tag{1.1}
\]

for a large class of piecewise analytic functions \( f(z) \). More precisely

**Theorem.** Let \( f_k(z), k = 0, 1, \ldots, s - 1 \) be entire functions with the property that there exist \( 0 \leq i < j \leq s - 1 \) such that \( f_i(z) \neq f_j(z) \) for some \( z \in \mathbb{C} \). Let \( f(z) \) be defined on

\[
\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b], \quad 0 < a < b,
\]
by imposing

\[ f(z) = f_k(z) \quad \text{if} \quad z \in e^{2\pi i \frac{k}{s}}[a, b], \quad k = 0, 1, \ldots, s - 1. \]

Then relation (1.1) holds.

The situation of interest, both in digital signal processing and in acceleration of convergence techniques, consists precisely of the case where \( f(z) \) is made up of \( s \) “different” entire functions each defined on one of the intervals \( e^{2\pi i \frac{k}{s}}[a, b] \). Of particular interest for the purpose of designing band pass filters is the case where \( f(z) \) consists of \( s \) different constants.

The proof of our main result provides, in addition, an algorithm to build the polynomial of near best approximation. We illustrate this algorithm in the special case where the function \( f(z) \) consists of two (different) constants. A more thorough implementation of the algorithm will be presented in a forthcoming paper.

It has to be noted that the theory of overconvergence plays also a fundamental role in the present paper. This technique allows us to prove lower bounds on the degree of approximation \( E_n(f, K) \). These bounds, together with the upper bounds provided by the construction of the polynomials of near best approximation, will show that our error estimates are sharp in an \( n^{th} \) root sense to be described later.

This paper is organized as follows: in the next section we review those fundamental facts of approximation theory in the complex plane which are needed for the investigation of our problem. In Section 3 we prove our main results. In Section 4 we translate the proof of the error estimate into an algorithm that finds the polynomial of near best approximation. This algorithm is illustrated in the last section in the simple case of two intervals.

We end this introduction with the description of another technique which may come to mind for approximation on disjoint intervals. Consider, to fix the ideas, the case \( s = 2 \) and the function \( h(x) \) considered in Section 5 and defined by

\[ h(x) = \begin{cases} 1, & x \in [a, b], \\ -1, & x \in [-b, -a]. \end{cases} \]

Consider a smooth extension \( \tilde{h}(x) \) to \([−b, b]\) of \( h(x) \), and approximate \( \tilde{h}(x) \) on \([−b, b]\). This will give an approximation of \( h(x) \) on \([−b, −a] \cup [a, b] \). Obviously this extension cannot be analytic and is at best \( C^\infty \). It follows from the Walsh–Bernstein Theorem (Theorem 2.1) that the speed of decrease to 0 of \( E_n(\tilde{h}; [-b, b]) \) cannot be exponential. Consider in fact such a \( C^\infty \) extension \( \tilde{h}(x) \) built as usual with combinations of functions of the form \( e^{\frac{-1}{x}} \). Then it can be shown that

\[ E_n(\tilde{h}; [-b, b]) \geq Ke^{-c\sqrt{n}}, \]

whereas our technique gives, for this particular function \( h(x) \),

\[ E_n(h; [-b, −a] \cup [a, b]) \leq K \left( \sqrt{\frac{b-a}{b+a}} \right)^n, \]

as shown in Proposition 5.1.
2. The fundamental approximation theorem of Bernstein

In this section we state, mostly without proof, results of complex approximation theory which will be needed in the next section. We follow essentially three sources: [16,18,20]. In [14,10] complex approximation is used in different contexts.

Let $E$ be a compact simply connected set of the complex plane containing more than one point and let $\omega = \phi(z)$ map conformally $\text{Ext}(E)$ into $|\omega| > 1$ and with $\phi(\infty) = \infty$. The map $\phi(z)$ has the form

$$\phi(z) = \frac{z}{c} + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots.$$ 

The number $c > 0$ is the capacity $\text{Cap}(E)$ of the set $E$. The following theorem of Walsh and Bernstein [23] will play a fundamental role in the sequel.

**Theorem 2.1 (Walsh–Bernstein).** Let $E$ be as above and let $P_n(z)$ be a polynomial of degree (at most) $n$ with

$$\|P_n(z)\|_E \leq 1.$$ 

Then, for $\rho > 1$, one has

$$\|P_n(z)\|_{\Gamma_{\rho}} \leq \rho^n.$$ 

As a consequence we have the following famous:

**Theorem 2.2 (Bernstein).** Let $E_{\rho}, \rho > 1$, be the ellipse with foci $-1$ and $1$ and large axis $\rho + \frac{1}{\rho}$. Let $f(z)$ be analytic in the interior of $E_{\rho}$. Then

$$\liminf_{n \to \infty} (E_n(f, [-1, 1])) \frac{1}{n} \leq \frac{1}{\rho}.$$ 

Moreover if $f(z)$ has a singular point on $E_{\rho}$. Then

$$\liminf_{n \to \infty} (E_n(f, [-1, 1])) \frac{1}{n} = \frac{1}{\rho}.$$ 

2.1. Extension of Bernstein’s Theorem to the case of several intervals

**Lemma 2.1.** Let $0 < a < b$. Then the mapping

$$\gamma(t) := \left( a^t + \frac{(t+1)(b^t - a^t)}{2} \right)^{\frac{1}{t}}, \quad -1 \leq t \leq 1,$$ 

transforms the interval $[-1, 1]$ into $K$.

**Proof.** The mapping $\gamma_1(t) := a^t + \frac{(t+1)(b^t - a^t)}{2}$ transforms $[-1, 1]$ into $[a^t, b^t]$. Hence, because $\gamma(t) = (\gamma_1(t))^{\frac{1}{t}}$, we obtain the result. \qed

**Corollary 2.1.** Let $f(z)$ be an entire function and let $g(z) := f(\gamma(z))$. Then

$$\liminf_{n \to \infty} (E_n(g, [-1, 1])) \frac{1}{n} = \frac{b^\frac{1}{t} - a^\frac{1}{t}}{b^\frac{1}{t} + a^\frac{1}{t}}.$$
Proof. \( g(z) \) is analytic in \( \mathbb{C} \setminus (−∞, t_0] \) where \( t_0 = −\frac{b^r + a^r}{b^r - a^r} \). Hence

\[
\lim_{n \to \infty} (E_n(g, [−1, 1]))^{\frac{1}{n}} \leq \frac{1}{\rho},
\]

where \( \rho + \frac{1}{\rho} = 2\frac{b^r + a^r}{b^r - a^r} \). The corollary follows by solving this second degree equation and from Bernstein’s Theorem 2.2. \( \square \)

3. Construction of the fundamental conformal mapping

Recall that the set \( K \) was defined by

\[
K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b],
\]

with \( 0 < a < b \).

Lemma 3.1. The \( s \)-to-one mapping function

\[
\omega = g(z) = \frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} + \sqrt{\left( \frac{2z^s}{b^s - a^s} - \frac{b^s + a^s}{b^s - a^s} \right)^2 - 1}
\]

transforms \( \text{Ext}(K) \) into \(|\omega| > 1 \). Its (continuous) extension to \( \mathbb{C} \) maps each of the intervals \( e^{2\pi i \frac{k}{s}} [a, b] \), traversed twice, into \(|\omega| = 1 \).

Proof. The function

\[
\omega = z + \sqrt{z^2 - 1}
\]

transforms conformally \( \text{Ext}[−1, 1] \) into \(|\omega| > 1 \). Hence the function

\[
\omega = f(z) = \frac{2z}{b - a} - \frac{b + a}{b - a} + \sqrt{\left( \frac{2z}{b - a} - \frac{b + a}{b - a} \right)^2 - 1}
\]

transforms conformally \( \text{Ext}[a, b] \) into \(|\omega| > 1 \). Here the square root in the function \( \omega = z + \sqrt{z^2 - 1} \) is uniquely chosen in such a way that its branch cut is the interval \([−1, 1] \). Its (continuous) extension to \( \mathbb{C} \) maps the interval \([a, b] \), traversed twice, into \(|\omega| = 1 \). Hence the continuous extension of \( g(z) \) to \( \mathbb{C} \) maps each of the intervals \( e^{2\pi i \frac{k}{s}} [a, b] \), traversed twice, into \(|\omega| = 1 \). The lemma follows now by direct computation. \( \square \)

Corollary 3.1. The harmonic function

\[
G(z) = \frac{1}{s} \log |g(z)|
\]

is the Green function with logarithmic pole at \( \infty \) of \( \text{Ext}(K) \) with boundary values

\[
\lim_{z \to z_0 \in K} G(z) = 0.
\]

Here \( g(z) \) is given by (3.1).
3.1. Study of the level curve $|h(z)| = \sqrt{|g(z)|} = C$.

In the remaining of this paper $h(z)$ denotes the $s$ root of $g(z)$ as given is (3.1):

$$h(z) = \sqrt[4]{g(z)}.$$

**Theorem 3.1.** With $0 < a < b$, the level curve

$$|h(z)| = \sqrt[4]{\frac{b^s + a^s}{b^s - a^s}}$$

passes through the point $0$.

**Proof.** This is a direct consequence of the version Corollary 2.1. □

**Corollary 3.2.** The one-to-one mapping function $\omega = h(z)$, maps conformally

$$z; |h(z)| > \sqrt[4]{\frac{b^s + a^s}{b^s - a^s}}$$

into

$$|\omega| > \sqrt[4]{\frac{b^s + a^s}{b^s - a^s}}.$$

Our mapping function $\omega = h(z)$ allows us to recover, as a special case, the following result.

**Corollary 3.3** ([Henrici 15]). The mapping function

$$z = \psi(\omega) = \left(\omega^\frac{1}{4} + \frac{1}{\omega^\frac{1}{4}}\right)^{\frac{1}{4}}$$

maps conformally

$$|\omega| > 1$$

into

$$\text{Ext} \left( \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{4}} \left[ 0, \sqrt[4]{4} \right] \right).$$

**Proof.** A rather lengthy but, at the same time, elementary computation shows that the inverse map $z = \psi(\omega)$, $|\omega| > \sqrt[4]{\frac{b^s + a^s}{b^s - a^s}}$, of the conformal map $\omega = h(z)$ is

$$z = \psi(\omega) = \sqrt[4]{\frac{1}{4} \left( b^s - a^s \right) \left( \omega^s + \frac{1}{\omega^s} \right) + \frac{b^s + a^s}{2}}.$$
Letting $a = 0$ and $b = \sqrt{4}$ in $z = \psi(\omega)$ gives $z = \psi(\omega) = (\omega^{\frac{k}{2}} + \frac{1}{\omega^{\frac{k}{2}}})^{\frac{2}{s}}, |\omega| > 1$. The image by $z = \psi(\omega)$ of $|\omega| > 1$ is $\text{Ext}(K)$ with $a = 0$ and $b = \sqrt{4}$. □

We summarize at this point those consequences of Theorem 3.1 and Corollary 3.2 that, in conjunction with the Walsh–Bernstein theorem, will be the principal tools in the proofs of our main results.

**Theorem 3.2.** Consider the level curve

$$|h(z)| = C.$$

If $C = 1$, then the level curve $|h(z)| = C$ consists of the set $K$.

If $1 < C < \frac{b^2 + a^2}{b^2 - a^2}$ then the level curve $|h(z)| = C$ consists of $s$ pairwise disjoint closed curves each of which contains in its interior exactly one of the intervals $e^{2\pi i \frac{k}{s}} [a, b]$ making up $K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b]$.

If $C = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}}$ then the level curve $|h(z)| = C$ is a lemniscate-like curve which consists of $s$ closed branches, whose only common point is the point 0 and all of which coalesce at the point 0. Moreover each of these branches contains in its interior exactly one of the intervals $e^{2\pi i \frac{k}{s}} [a, b]$.

If $C > \sqrt{\frac{b^2 + a^2}{b^2 - a^2}}$ then the level curve $|h(z)| = C$ consists of a single closed curve and contains 0 (as well as $\bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b]$) in its interior.

### 4. Statements and proofs of the main results

In what follows we will consider functions analytic in the interior of the level curve

$$|h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}}$$

that passes through the point 0. We now give an example of such a function, built out of $s$ entire functions, that, in addition, are not analytic in the interior of the level curve

$$|h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}} + \varepsilon$$

no matter how small $\varepsilon > 0$ is chosen. These are precisely those functions that occur in digital signal processing as well as in acceleration of convergence techniques.

Recall that in view of Theorem 3.2 the level curve $|h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}}$ is a lemniscate-like curve which consists of $s$ closed branches, whose only common point is the point 0 and all of which coalesce at the point 0. Moreover each of these branches contains exactly one of the intervals $e^{2\pi i \frac{k}{s}} [a, b]$ making up $K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b]$. Let now $f_k(z), k = 0, 1, \ldots, s - 1$ be entire functions with the property that there exist $0 \leq i < j \leq s - 1$ such that $f_i(z) \neq f_j(z)$ for some
Let \( z \in \mathbb{C} \). Let \( f(z) \) be defined on the interior of the level curve \( |h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}} \) by the following rule:

\[
f(z) = f_k(z) \quad \text{if} \quad z \in e^{2\pi i k}[a, b].
\]

Then it is clear that \( f(z) \) has an analytic extension in the interior of \( |h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}} \) by imposing

\[
f(z) = f_k(z) \quad \text{if} \quad z \in \text{the interior of the } k\text{th branch of the level curve}
\]
and (this extension of) \( f(z) \) cannot be analytically continued at the point 0. Because of Theorem 3.2 the level curve \( |h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}} + \varepsilon, \quad \varepsilon > 0 \), consists of a single closed curve that contains 0 in its interior, \( f(z) \) cannot be analytically continued to the interior of this level curve.

We now have built the necessary tools for the proofs of our main results.

**Theorem 4.1.** Let \( 0 < a < b \) and let \( f(z) \) be analytic in the interior of the level curve

\[
|h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - a^2}}
\]

and let

\[
K = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{z}{2}}[a, b].
\]

Then

\[
\limsup_{n \to \infty} (E_n(f, K))^{\frac{1}{n}} \leq \sqrt{\frac{b^2 + a^2}{b^2 - a^2}}.
\]

**Proof.** Let \( P_{n}^0(z) \) be the polynomial of degree at most \( n \) of best approximation of

\[
f \left( \sqrt[n]{z} \right) \quad \text{on} \quad [a^* , b^*].
\]

Here \( \sqrt[n]{z} \) is that determination of the \( n \)th root whose branch cut is the negative real axis.

The hypothesis on the domain of analyticity of \( f(z) \), together with a standard computation, shows that \( f \left( \sqrt[n]{z} \right) \) is analytic in the interior of the ellipse with foci \( a^* \) and \( b^* \) and vertex zero.

Recall now the classical result [1]:

Let \( g(z) \) be analytic in the interior of the ellipse with foci \(-1\) and \(1\) and whose big axis has length \( \rho + \frac{1}{\rho} \). Then

\[
\limsup_{n \to \infty} (E_n(f, [-1, 1]))^{\frac{1}{n}} \leq \rho.
\]

A slight modification of the proof in [1] shows that:

Let \( f(z) \) be analytic in the interior of the ellipse with foci \( a \) and \( b \) and vertex 0. Then

\[
\limsup_{n \to \infty} (E_n(f, [a, b]))^{\frac{1}{n}} \leq \frac{\sqrt{b} + \sqrt{a}}{\sqrt{b} - \sqrt{a}}.
\]
Hence
\[
\limsup_{n \to \infty} \left( E_n \left( \frac{f \left( \sqrt[\frac{s}{b}]z \right)}{z^\frac{k}{s}} \right), \left[ a^s, b^s \right] \right)^\frac{1}{n} \leqslant \frac{b^s + a^s}{b^s - a^s}.
\]

For \(0 \leqslant k \leqslant s - 1\), let now \(P^k_n(z)\) be the polynomial of degree at most \(n\) of best approximation of
\[
\frac{f \left( \sqrt[\frac{s}{b}]z \right)}{z^\frac{k}{s}} \text{ on } [a^s, b^s].
\]

Remark that, as \(f \left( \sqrt[\frac{s}{b}]z \right)\), the function \(f \left( \frac{s}{b}\sqrt[\frac{s}{b}]z \right)\) is analytic in the interior of the ellipse with foci \(a^s\) and \(b^s\) and vertex zero. Then an argument similar to the one used above shows that
\[
\limsup_{n \to \infty} \left( E_n \left( \frac{f \left( \sqrt[\frac{s}{b}]z \right)}{z^\frac{k}{s}} \right), \left[ a^s, b^s \right] \right)^\frac{1}{n} \leqslant \frac{b^s + a^s}{b^s - a^s}.
\]
(4.1)

Consider now the polynomial of degree \(ns + s - 1\) \(Q_{ns+s-1}(z)\) defined by
\[
Q_{ns+s-1}(z) := \frac{1}{s} \left( P^0_n(z^s) + z P^1_n(z^s) + \cdots + z^{s-1} P^{s-1}_n(z^s) \right).
\]
(4.2)

We remark that, by construction, and in view of (4.1),
\[
\limsup_{n \to \infty} \left( \left\| P^k_n(z) - \frac{f \left( \sqrt[\frac{s}{b}]z \right)}{z^\frac{k}{s}} \right\|_{[a^s, b^s]} \right)^\frac{1}{n} \leqslant \frac{b^s + a^s}{b^s - a^s}, \quad k = 0, 1, \ldots, s - 1.
\]
(4.3)

Hence (4.3) yields
\[
\limsup_{n \to \infty} \left( \left\| z^k P^k_n(z^s) - f(z) \right\|_{[a, b]} \right)^\frac{1}{n} \leqslant \frac{b^s + a^s}{b^s - a^s}, \quad k = 0, 1, \ldots, s - 1.
\]
(4.4)

Hence relations (4.2) and (4.4) give
\[
\limsup_{n \to \infty} \left( \left\| Q_{ns+s-1}(z) - f(z) \right\|_{[a, b]} \right)^\frac{1}{s} \leqslant \frac{b^s + a^s}{b^s - a^s}.
\]

Fix now one of the intervals \(e^{2\pi i \frac{k}{s}}[a, b]\) making up \(K = K_{a,b}^s = \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}}[a, b], k \neq 0\), say \(e^{2\pi i \frac{1}{s}}[a, b]\). Because
\[
P^k_n(z) \to \frac{f \left( \sqrt[\frac{s}{b}]z \right)}{z^\frac{k}{s}} \text{ on } [a^s, b^s], \quad 0 \leqslant k \leqslant s - 1,
\]
we have
\[
P^0_n(z^s) \to f \left( z e^{2\pi i \frac{1}{s}} \right), \quad z \in e^{2\pi i \frac{1}{s}}[a, b],
\]
(4.6)
\[
z P^1_n(z^s) \to e^{2\pi i \frac{1}{s}} f \left( z e^{2\pi i \frac{1}{s}} \right), \quad z \in e^{2\pi i \frac{1}{s}}[a, b],
\]
(4.7)
\[ z^2 P_n^2(z^s) \to e^{2\pi i \frac{2}{s}} f \left( z e^{2\pi i \frac{1}{s}} \right), \quad z \in e^{2\pi i \frac{1}{s}} [a, b], \]  

(4.8)

\[ \vdots \]

\[ z^{s-1} P_n^{s-1}(z^s) \to e^{2\pi i \frac{s-1}{s}} f \left( z e^{2\pi i \frac{1}{s}} \right), \quad z \in e^{2\pi i \frac{1}{s}} [a, b]. \]  

(4.9)

Now

\[ 1 + z + \cdots + z^{s-1} = 0, \quad z \in e^{2\pi i \frac{k}{s}} [a, b], \quad 1 \leq k \leq s - 1. \]  

(4.10)

Hence relations (4.6)–(4.10) show that

\[ P_0^0(z^s) + z P_1^1(z^s) + \cdots + z^{s-1} P_{s-1}^{s-1}(z^s) \to 0, \quad z \in e^{2\pi i \frac{1}{s}} [a, b]. \]

A similar argument shows that

\[ P_0^0(z^s) + z P_1^1(z^s) + \cdots + z^{s-1} P_{s-1}^{s-1}(z^s) \to 0, \quad z \in e^{2\pi i \frac{k}{s}} [a, b], \quad 1 \leq k \leq s - 1. \]

Moreover in view of relation (4.4)

\[
\limsup_{n \to \infty} \left( \| P_n^0(z^s) + z P_n^1(z^s) + \cdots + z^{s-1} P_n^{s-1}(z^s) - 0 \|_{e^{2\pi i \frac{1}{s}} [a, b]} \right) \leq \frac{b^s + a^s}{b^s - a^s},
\]

\[ 1 \leq k \leq s - 1. \]  

Hence

\[
\limsup_{n \to \infty} \left( \| Q_{ns+s-1}(z) - 0 \|_{e^{2\pi i \frac{k}{s}} [a, b]} \right) \leq \frac{b^s + a^s}{b^s - a^s}, \quad 1 \leq k \leq s - 1.
\]

To summarize we have for the polynomials \( Q_{ns+s-1}(z) \), that we rename \( Q_{ns+s-1}^0(z) \),

\[ Q_{ns+s-1}^0(z) \to f(z), \quad z \in [a, b] \]

and

\[ Q_{ns+s-1}^0(z) \to 0, \quad z \in e^{2\pi i \frac{k}{s}} [a, b], \quad 1 \leq k \leq s - 1. \]

Moreover the speed of convergence is, in an nth root sense the exponential speed, \( \left( \frac{b^s + a^s}{b^s - a^s} \right)^n \).

In a similar manner we build, for \( 0 \leq k \leq s - 1 \), polynomials \( Q_{ns+s-1}^k(z) \) with the following property:

\[ Q_{ns+s-1}^k(z) \to f(z), \quad z \in e^{2\pi i \frac{k}{s}} [a, b] \]  

(4.11)

and

\[ Q_{ns+s-1}^k(z) \to 0, \quad z \in e^{2\pi i \frac{l}{s}} [a, b], \quad l \neq k. \]  

(4.12)

In addition the speed of convergence is the one alluded above.

Let now the polynomials \( R_{ns+s-1}(z) \) be defined by

\[ R_{ns+s-1}(z) := Q_{ns+s-1}^0(z) + Q_{ns+s-1}^1(z) + \cdots + Q_{ns+s-1}^{s-1}(z). \]
We then have in view of relations (4.11) and (4.12)
\[ R_{ns+s−1}(z) \to f(z), \quad z \in \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b], \]
with the above exponential speed of convergence \( \left( \frac{b^\frac{1}{s} + a^\frac{1}{s}}{b^\frac{1}{s} - a^\frac{1}{s}} \right)^n \), again in an nth root sense. Hence
\[ \forall \varepsilon > 0 \ \exists N > 0 \ \text{s.t.} \ n \geq N \ \text{implies} \ E_{ns+s−1}(f; K) \leq \left( \frac{b^\frac{1}{s} + a^\frac{1}{s}}{b^\frac{1}{s} - a^\frac{1}{s}} + \varepsilon \right)^n. \]

It follows easily, using the fact that \( E_n(f; K) \) tends to zero decreasingly, that
\[ \limsup_{n \to \infty} \left( E_n(f; K) \right)^\frac{1}{n} \leq \sqrt[1]{\frac{b^\frac{1}{s} - a^\frac{1}{s}}{b^\frac{1}{s} + a^\frac{1}{s}}}, \]
as was to be proved. □

We now show that if the function \( f(z) \) is made up of different entire function \( f_k(z) \), as described before the statement of Theorem 4.1, then the upper bound is sharp, again in an nth root sense.

The lower bound is now easily established because most of the work for its proof has been carried on in Section 3 where the properties of the level curves of the conformal mapping \( \omega = h(z) \) are analyzed in Theorem 3.1. As expected it is the theorem of Walsh–Bernstein that is the major tool.

**Theorem 4.2.** Let \( f_k(z), \ k = 0, 1, \ldots, s − 1 \) be entire functions with the property that there exist \( 0 < i < j < s − 1 \) such that \( f_i(z) \neq f_j(z) \) for some \( z \in \mathbb{C} \). Let \( f(z) \) be defined on
\[ \bigcup_{k=0}^{s-1} e^{2\pi i \frac{k}{s}} [a, b] \]
by imposing
\[ f(z) = f_k(z) \quad \text{if} \quad z \in e^{2\pi i \frac{k}{s}} [a, b], \quad k = 0, 1, \ldots, s − 1. \]

Then
\[ \limsup_{n \to \infty} \left( E_n(f, K) \right)^\frac{1}{n} = \sqrt[1]{\frac{b^\frac{1}{s} - a^\frac{1}{s}}{b^\frac{1}{s} + a^\frac{1}{s}}}. \]

**Proof.** Assume to the contrary that there exists \( \varepsilon > 0 \) and a sequence of polynomials \( P_n(z) \), each of degree at most \( n \), such that
\[ \| P_n(z) - f(z) \|_K \leq \left( \frac{b^\frac{1}{s} + a^\frac{1}{s}}{b^\frac{1}{s} - a^\frac{1}{s}} \right)^\frac{n}{s} \cdot \left( \frac{b^\frac{1}{s} + a^\frac{1}{s}}{b^\frac{1}{s} - a^\frac{1}{s}} - \varepsilon \right)^\frac{n}{s}. \]

Then in view of Walsh–Bernstein theorem \( f(z) \) is analytic in the interior of the level curve
\[ |h(z)| = \sqrt[1]{\frac{b^\frac{1}{s} - a^\frac{1}{s}}{b^\frac{1}{s} + a^\frac{1}{s}} + \delta}. \]
for some $\delta > 0$. Theorem 3.1 guarantees that this set is simply connected and contains $K$ (and consequently 0) so that in view of the principle of analytic continuation

$$f_0(z) = f_1(z) = \cdots = f_{s-1}(z) \quad \forall z \in \mathbb{C}.$$  

Hence the condition

$$\exists \ 0 \leq i < j \leq s-1 \quad \text{such that} \quad f_i(z) \neq f_j(z) \quad \text{for some} \ z \in \mathbb{C}$$

is not possible.

The conclusion now follows from Theorem 4.1 if we remark that the function $f(z)$ has an analytic extension in the interior of the level curve $|h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - c^2}}$. \hfill $\Box$

Note that in the case $s = 1$ the level curve $|h(z)| = \sqrt{\frac{b^2 + a^2}{b^2 - c^2}}$ is the ellipse with foci $a$ and $b$ and vertex 0. Moreover in that case $K = [a, b]$. So in the case $s = 1$ we recover from Theorem 4.1 (an equivalent form of) the classical result [1]:

**Corollary 4.1.** Let $f(z)$ be analytic in the interior of the ellipse with foci $a$ and $b$ and vertex 0. Then

$$\limsup_{n \to \infty} \left( E_n(f, [a, b]) \right)^{1/n} \leq \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}.$$  

5. Description of the algorithm in the case of a step function

Examination of Theorem 4.1 shows that its proof contains an algorithm to build the polynomial of near best approximation to the function $f(z)$ on $K$. Indeed the polynomials $Q_{ns+s-1}(z)$, which provide the near best approximation, are made up of polynomials of near best approximation to the functions $f\left(\frac{\sqrt{z}}{z}\right), f\left(\frac{\sqrt{z}}{z}\right), \ldots, f\left(\frac{\sqrt{z}}{z}\right)$ on the single intervals $e^{2\pi i l/z}[a^s, b^s], l = 0, 1, \ldots, s-1$. Several techniques are available to build the polynomial of near best approximation in that setting. Our experiments show that expansion of the function in series of modified Chebyshev polynomials provide a stable and very economical construction of such a polynomial.

In the description of the algorithm below the term “polynomial of near best approximation” refers to the polynomial as described above.

The algorithm proceeds as follows, as essentially contained in the proof of Theorem 4.1.

For $0 \leq k, l \leq s-1$, let $P_{n,l}^k(z)$ be the polynomial of degree at most $n$ of near best approximation to

$$f\left(\frac{\sqrt{z}}{z}\right) \quad \text{on} \quad e^{2\pi i l/z}[a^s, b^s].$$

The desired polynomial of near best approximation of degree at most $ns + s - 1$ to

$$f(z) \quad \text{on} \quad \bigcup_{k=0}^{s-1} e^{2\pi i l/z}[a, b]$$

is
is then
\[ R_{ns + s - 1}(z) := \sum_{l=0}^{s-1} \frac{1}{s} \sum_{k=0}^{s-1} z^k P_n^k(z^s). \] (5.1)

We now consider the problem of approximating the function \( h(x) \) defined on \([-b, -a] \cup [a, b] \), \( 0 < a < b \), by
\[ h(x) = \begin{cases} 1, & x \in [a, b], \\ -1, & x \in [-b, -a]. \end{cases} \]

Although the function \( h(x) \) is very simple, its approximation by polynomials plays an important role in the design of digital filters. See [19]. (To be more precise it is the function \( \frac{1}{2}(1 + h(x)) \) that is used).

Here \( s = 2 \) in the algorithm and, with its notations, the polynomial \( R_{2n+1}(x) \) of near best approximation to \( h(x) \) takes the form
\[ R_{2n+1}(x) = \frac{1}{2} \left( x P_n(x^2) + x Q_n(x^2) \right), \]
where
\[ P_n(x) \rightarrow \frac{1}{\sqrt{x}}, \quad x \in [a^2, b^2] \] (5.2)
and
\[ Q_n(x) \rightarrow \frac{1}{\sqrt{-x}}, \quad x \in [-b^2, -a^2]. \] (5.3)

It follows from relations (5.2), (5.3) and from uniqueness considerations of the polynomial of best approximation [17] that
\[ P_n(x^2) = Q_n(x^2), \]
so that the resultant polynomial \( R_{2n+1}(x) \) reduces here to the determination of one single polynomial \( P_n(x) \) given by relation (5.2):
\[ R_{2n+1}(x) = x P_n(x^2). \]

We built in a stable manner, and at low cost, the polynomials \( P_n(x) \), and so \( R_{2n+1}(x) \), using standard Fourier–Chebyshev expansions. Moreover, in this particular case, using somewhat lengthy but, otherwise, standard asymptotic expansion techniques of integrals we obtain:

**Proposition 5.1.**
\[ \| R_{2n+1}(x) - h(x) \|_{[-b, -a] \cup [a, b]} \leq K \left( \frac{b-a}{b+a} \right)^n \sqrt{n}. \]

Hence
\[ E_n (h; [-b, -a] \cup [a, b]) \leq K \left( \frac{b-a}{b+a} \right)^n \sqrt{n}. \] (5.4)

Here the constant \( K \) varies from occurrence to occurrence.
Hence, in this particular setting, we recover (a more precise form of) Theorem 4.1 which only gave us

\[
\limsup_{n \to \infty} \left( E_n (h; [-b, -a] \cup [a, b]) \right)^{\frac{1}{n}} \leq \sqrt{\frac{b-a}{b+a}}.
\]

Indeed it is clear that relation (5.4) implies relation (5.5) and hence, in this setting, Theorem 4.1.

Acknowledgments

The author wishes to express his appreciation to Professors Dick Gundy and Yanyan Li of Rutgers University, and to Professor Tom Kennedy of the University of Arizona for fruitful discussions.

References