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Finite graphs have unique symmetric products

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Abstract

Let X be a metric continuum and let $F_n(X)$ be the nth symmetric product of X ($F_n(X)$ is the hyperspace of nonempty subsets of X with at most n points). In this paper we prove that if $F_n(X)$ is homeomorphic to $F_n(Y)$, where X is a finite graph and Y is a continuum, then X is homeomorphic to Y.

MSC: 54B20; 54F15

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1. Introduction

A *continuum* is a nondegenerate, compact, connected metric space. We consider the following hyperspaces of a continuum X:

$$2^X = \{A \subset X\colon\ A \text{ is closed and nonempty}\},$$

$$C(X) = \{A \in 2^X\colon\ A \text{ is connected}\}, \quad \text{and if } n \text{ is a positive integer,}$$

$$C_n(X) = \{A \in 2^X\colon\ A \text{ has at most } n \text{ components}\},$$

$$F_n(X) = \{A \in 2^X\colon\ A \text{ has at most } n \text{ points}\}.$$

All the hyperspaces are endowed with the Hausdorff metric H. The hyperspace $F_n(X)$ is also known as the *nth symmetric product* of X.

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It is easy to show that if two continua X and Y are homeomorphic, then each one of the hyperspaces of X is homeomorphic to the respective hyperspace of Y. A very natural problem in this area is to determine when this implication can be reversed. This problem can be rephrased in the following way.

Problem 1.1. Suppose that $\mathcal{H}(Z)$ represents one of the hyperspaces 2^Z, $C(Z)$, $C_n(Z)$ or $F_n(Z)$. Then, under what conditions we can assert that the following implication holds: "$\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are homeomorphic implies that X and Y are homeomorphic".

The implication contained in the above paragraph is not always true. For example, if S represents a simple closed curve, then $C([0,1])$ and $C(S)$ are 2-cells (see [19, Example 5.1]), and $2^{[0,1]}$ and 2^S are homeomorphic to the Hilbert cube (see [19, Theorem 11.3]). However, $F_2([0,1])$ is a 2-cell and $F_2(S)$ is homeomorphic to the Möbius strip (see [6, Theorem 6 and pp. 876–875]). It is also known that $F_3([0,1])$ is a 3-cell [6, Theorem 6] while $F_3(S)$ is homeomorphic to the 3-sphere [7]. So it is natural to ask if, for each positive integer n, $F_n([0,1])$ is not homeomorphic to $F_n(S)$. The main result of this paper implies that if X and Y are non-homeomorphic finite graphs, then for each positive integer n, $F_n(X)$ and $F_n(Y)$ are not homeomorphic.

Next we give some definitions and a brief description of what has been done on Problem 1.1.

The continuum X is said to have *unique hyperspace* $C(X)$ (respectively, 2^X, $C_n(X)$ and $F_n(X)$) provided that if Y is a continuum and $C(X)$ (respectively, 2^X, $C_n(X)$ and $F_n(X)$) is homeomorphic to $C(Y)$ (respectively, 2^Y, $C_n(Y)$ and $F_n(Y)$), then X is homeomorphic to Y. A class $\mathcal{C}$ of continua is known to be *C-determined* provided that if $X, Y \in \mathcal{C}$ and $C(X)$ is homeomorphic to $C(Y)$, then X is homeomorphic to Y.

The continua belonging to one of the following classes are known to have unique hyperspace $C(X)$:

(a) Finite graphs different from an arc or a simple closed curve (Duda [9, 9.1], see [1, Theorem 3.3]);
(b) Hereditarily indecomposable continua (Nadler [23, 0.60], see [1, Theorem 3.4]);
(c) Indecomposable continua such that all their proper nondegenerate subcontinua are arcs (Macías [20]);
(d) Metric compactifications of the ray $[0,\infty)$ with nondegenerate remainder (Acosta [1, Theorem 5.3]).

It is known (see [10, Corollary 3.3(c)]) that the class of smooth fans is C-determined.

Hereditarily indecomposable continua have unique hyperspace 2^X (Macías [21]).

Finite graphs have unique hyperspace $C_n(X)$, for each $n > 1$ (Illanes [16,17]).

A. Illanes has shown that the classes of chainable continua and fans are not C-determined [14,15]. In [18] some results about uniqueness of the hyperspace $F_2(X)$ when X is a dendrite are presented. In [12] Herrera, proved that if a dendrite X has a closed set of end-points, then X has unique hyperspace $C(X)$.

Some other results related to unique hyperspaces can be found in [1–5].

The following questions remain open.

Question *([23, Question 0.62]).* Is the class of circle-like continua C-determined?

Question *([18]).* Do hereditarily indecomposable continua have unique hyperspace $F_2(X)$?

In this paper we prove that finite graphs have unique hyperspace $F_n(X)$.

Next, we describe the general strategy followed in the proof of our main result. We start with a positive integer $n \geqslant 2$, a finite graph X and a continuum Y such that $F_n(X)$ and $F_n(Y)$ are homeomorphic, with a homeomorphism h, we need to show that X and Y are homeomorphic.

(a) It is know that a continuum Z is locally connected if and only if $F_n(Z)$ is locally connected (see [6, p. 877]). Thus we conclude that Y is locally connected.
(b) Now, for a locally connected continuum Z, we need to describe a topological property $\mathcal{P}$ defined on $F_n(Z)$ such that Z is a finite graph if and only if $F_n(Z)$ has the property $\mathcal{P}$. This property $\mathcal{P}$ is given by Theorem 3.4. This property $\mathcal{P}$ is described in terms of the set of elements in $F_n(Z)$ having a neighborhood in $F_n(Z)$ which is an n-cell. Since X is a finite graph, $F_n(X)$ has the property $\mathcal{P}$. Thus $F_n(Y)$ has also property $\mathcal{P}$ and Y is a finite graph.
(c) Now, for a finite graph Z, we need to describe a topological property $\mathcal{Q}$ defined for the elements of $F_n(Z)$ which is satisfied exactly by the singletons. That is, if $A \in F_n(Z)$, then $F_n(Z)$ has property $\mathcal{Q}$ at A if and only if $A = \{z\}$ for some $z \in Z$. Once we get the property $\mathcal{Q}$, we can say that $h(F_1(X)) = F_1(Y)$. Thus $F_1(X)$ is homeomorphic to $F_1(Y)$. Since X is homeomorphic to $F_1(X)$ and Y is homeomorphic to $F_1(Y)$, we conclude that X is homeomorphic to Y and we are done. So if it is possible to describe such a property $\mathcal{Q}$, then we have finished. We were able to find an appropriate property $\mathcal{Q}$ only for the case $n \geqslant 4$ (Lemma 4.5).
(d) For the cases $n = 2$ and $n = 3$, we used another approach. For these cases we were able to find a topological property $\mathcal{R}$, defined for elements of $F_n(Z)$ (where Z is a finite graph) such that $F_n(Z)$ has property $\mathcal{R}$ at an element $A \in F_n(Z)$ if and only if $A = \{v\}$ for some ramification point v of Z (Lemma 5.5). In this way, we can give a bijection between the ramification points of X and the ramification points of Y. After that, counting the arcwise components of small neighborhoods of the element $\{v\}$ in $F_n(X)$, where v is a ramification point of X, we are able to show an appropriate bijection between the edges of X and the edges of Y to conclude that X and Y are homeomorphic finite graphs.

Next, we use an example to illustrate why the topological structure of $F_n(X)$ for $n \geqslant 4$ is different from that of $F_n(X)$ for $n \in \{2, 3\}$. It is the same type of behavior that was used to show that $F_3([0, 1])$ is a 3-cell and $F_4([0, 1])$ cannot be embedded in the Euclidean space $\mathbb{R}^4$ [6, Theorem 7]. This difference is used in the proof of the main result of this paper (see Lemma 4.3).

In the unit interval $[0,1]$, consider three pairwise disjoint subarcs J_1, J_2 and J_3 of $[0,1]$. Consider the set $\mathcal{S} = \{B \in F_4([0,1])\colon B \subset J_1 \cup J_2 \cup J_3$ and $B \cap J_i \neq \emptyset$ for each $i \in \{1,2,3\}\}$. Let $\mathcal{S}_1 = \{B \in F_4([0,1])\colon B \subset J_1 \cup J_2 \cup J_3$ and, $B \cap J_2$ and $B \cap J_3$ are one-point sets$\}$. In a similar way define $\mathcal{S}_2$ and $\mathcal{S}_3$. A typical element of $\mathcal{S}_1$ contains one point in J_2, one point in J_3 and one or two points in J_1, so it is easy to check that each $\mathcal{S}_i$ is homeomorphic to $F_2([0,1]) \times [0,1] \times [0,1]$. Since $F_2([0,1])$ is a 2-cell, each $\mathcal{S}_i$ is a 4-cell. Let $\mathcal{T} = \{B \in F_4([0,1])\colon B \cap J_1$, $B \cap J_2$ and $B \cap J_3$ are one-point sets$\}$. Notice that $\mathcal{T} = \mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S}_2 \cap \mathcal{S}_3 = \mathcal{S}_1 \cap \mathcal{S}_3$ which is a 3-cell. Since $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. We have that $\mathcal{S}$ is the union of three 4-cells such that each two of them meet at the 3-cell $\mathcal{T}$. Thus $\mathcal{S}$ cannot be embedded in $\mathbb{R}^4$ (the formal proof of this fact is contained in Lemma 4.3). Therefore, $F_4([0,1])$ cannot be embedded in $\mathbb{R}^4$. However, if we consider the respective situation in $F_3([0,1])$, we obtain two 3-cells that meet at one 2-cell and such a situation is permissible in $\mathbb{R}^3$.

2. Conventions

A *finite* (*connected*) *graph* is a continuum which is a finite union of arcs such that every two of them meet at a subset of their end points. If X is a finite graph, in X are defined *edges* and *vertices*. The vertices of X are the end points of the edges of X. A finite graph which is different from a simple closed curve is called an *acircular graph*. We are interested in distinguishing the ramification points of the graph X from the rest of the points, so we assume that each vertex of an acircular graph X is either an end point of X or a ramification point of X. With this restriction the two end points of an edge of X may coincide and such an edge is a simple closed curve. This kind of edges will be called *loops*. Thus the edges of X are arcs or simple closed curves and in X there are only three kind of edges, namely: loops, edges that contain some end point and edges joining ramification points. We assume that the metric d in X is the metric of arc length and each edge of X has length equal to one. The set of ramification points of X is denoted by $R(X)$. Two different vertices p and q of X are said to be *adjacent* provided that there is an edge J of X such that p and q are the end points of J. A *simple n-od Y* is a finite graph which is the union of n arcs $J_1, \ldots, J_n$ such that there exists a point $p \in Y$ with the property $J_i \cap J_j = \{p\}$, if $i \neq j$, and p is an end point of each one of the arcs J_i. The point p is called the *core of Y*. A simple 3-od is called a *simple triod*.

If A is a set, $|A|$ denotes the cardinality of A. A *Peano* continuum is a locally connected continuum. The set of positive integers is denoted by $\mathbb{N}$.

Given a continuum Z and a subset A of Z, $\mathrm{bd}_Z(A)$, $\mathrm{cl}_Z(A)$ and $\mathrm{int}_Z(A)$ denote the respective boundary, closure and interior of A in Z. Let Z be a continuum and $p \in Z$. Let β be a cardinal number. We say that *p is of order less than or equal to β in Z*, written $\mathrm{ord}(p,Z) \leqslant \beta$ provided that for each open subset U of Z such that $p \in U$, there exists an open subset V of Z such that $p \in V \subset U$ and $|\mathrm{bd}_Z(V)| \leqslant \beta$. We say the *$p$ is of order β*, written $\mathrm{ord}(p,Z) = \beta$, provided that $\mathrm{ord}(p,Z) \leqslant \beta$ and $\mathrm{ord}(p,Z) \nleqslant \alpha$ for any cardinal number $\alpha < \beta$. A point $p \in Z$ is called an *end point of* Z provided that $\mathrm{ord}(p,Z) = 1$. A point $p \in Z$ is called a *ramification point of* Z provided that $\mathrm{ord}(p,Z) \geqslant 3$. If A is a subset of Z, $p \in Z$ and $\varepsilon > 0$, let $B_Z(\varepsilon, p) = \{q \in Z\colon d_Z(p,q) < \varepsilon\}$ and $N_Z(\varepsilon, A) = \{q \in Z\colon$ there exists $p \in A$ such that $d_Z(p,q) < \varepsilon\}$.

Given a continuum Z, let $\mathcal{E}_n(Z) = \{A \in F_n(Z)\colon A$ has a neighborhood in $F_n(Z)$ which is an n-cell$\}$. If C is a subset of Z, let $F_n(C) = \{A \in F_n(Z)\colon A \subset C\}$. Given subsets $U_1, \dots, U_m$ in Z, let $\langle U_1, \dots, U_m\rangle_n = \{A \in F_n(Z)\colon A \subset U_1 \cup \dots \cup U_m$ and $A \cap U_i \neq \emptyset$ for each $i \in \{1, \dots, m\}\}$. It is known (see [23, Theorem 0.13]) that the sets of the form $\langle U_1, \dots, U_m\rangle_n$, where the sets $U_1, \dots, U_m$ are open, form a basis of the topology of $F_n(Z)$.

3. Results on Peano continua

Lemma 3.1. *If Z is a Peano continuum and $A \in \mathcal{E}_n(Z)$, then no point of A is the core of a simple triod of Z.*

Proof. Suppose, to the contrary, that $A \in \mathcal{E}_n(Z)$ and A contains a point p such that p is the core of a simple triod T_0 of Z. We will show that each neighborhood $\mathcal{U}$ of A in $F_n(Z)$ contains a topological copy of the product $T \times [0,1]^{n-1}$, where T is a simple triod. It is easy to show that the Theorem on the Invariance of Domain [13, Theorem VI 9, §6, Chapter 6, p. 95] imply that the space $T \times [0,1]^{n-1}$ is not embeddable in $\mathbb{R}^n$, thus we will have a contradiction.

Let $\varepsilon > 0$ be such that $B_{F_n(X)}(\varepsilon, A) \subset \mathcal{U}$.

Suppose that $A = \{p, x_2, \dots, x_m\}$, where $m \leqslant n$ and $p, x_2, \dots, x_m$ are all different. Choosing appropriate points close to x_m, there exists $B \in B_{F_n(Z)}(\varepsilon, A)$ such that $B = \{p, x_2, \dots, x_n\}$, and the points $p, x_2, \dots, x_n$ are all different.

Choose $\delta > 0$ such that the sets $B_Z(\delta, p), B_Z(\delta, x_2), \dots, B_Z(\delta, x_n)$ are pairwise disjoint and $B_{F_n(Z)}(\delta, B) \subset \mathcal{U}$. Choose arcs $I_2, \dots, I_n$ of Z such that $x_i \in I_i$ and diameter$(I_i) < \delta$, for each $i \in \{2, \dots, n\}$. Finally, choose a simple subtriod T of T_0 such that p is the core of T and diameter$(T) < \delta$. Then $\langle T, I_2, \dots, I_n\rangle_n \subset B_{F_n(Z)}(\delta, B) \subset \mathcal{U}$. Notice that $T, I_2, \dots, I_n$ are pairwise disjoint. Thus $T \times I_2 \times \dots \times I_n$ is homeomorphic to $\langle T, I_2, \dots, I_n\rangle_n$ (using the homeomorphism that sends $(t_1, t_2, \dots, t_n)$ into $\{t_1, t_2, \dots, t_n\}$). Therefore, the space $T \times I_2 \times \dots \times I_n$ can be embedded in $\mathcal{U}$. Hence $\mathcal{U}$ cannot be embedded in $\mathbb{R}^n$. This contradiction completes the proof of the lemma. □

Lemma 3.2. *If Z is a Peano continuum which is not a finite graph, then for each $k \in \mathbb{N}$, Z contains a finite graph with at least k edges.*

Proof. In the case that there exist arcs α and β in Z such that $\alpha \cap \beta$ has infinitely many components, we have that $\alpha - (\alpha \cap \beta)$ has infinitely many components. If we choose k components $J_1, \dots, J_k$ of $\alpha - (\alpha \cap \beta)$, then $\beta \cup (J_1 \cup \dots \cup J_k)$ is a finite graph with at least k edges. So, in this case, we are done. Hence, we are going to assume that $\alpha \cap \beta$ has finitely many components for all arcs α and β in Z. Under this assumption, if α and β are arcs in Z and $\alpha \cap \beta \neq \emptyset$, we have that $\alpha \cup \beta$ is a finite graph.

By [22, Theorem 9.10], Z has one of the following two properties:

(a) there exist infinitely many points $p \in Z$ such that $\mathrm{ord}(p, Z) > 2$,

(b) there exists a point $q \in Z$ such that $\mathrm{ord}(q, Z) \geqslant \aleph_0$.

We first assume that (a) holds.

In this case there exists a sequence of points $\{p_m\}_{m=1}^{\infty}$ such that $\mathrm{ord}(p_m, Z) > 2$ for each $m \in \mathbb{N}$ and the points p_m are all different. By [11, Example 8 of §51 p. 277], for each $m \in \mathbb{N}$, there exists a simple triod T_m such that p_m is the core of T_m.

Fix a point $p \in Z$. For each $i \in \{1, \ldots, k+1\}$, let α_i be an arc joining p and p_i. By the assumption in the first paragraph of this proof, the continuum $Y = \alpha_1 \cup \cdots \cup \alpha_{k+1} \cup T_1 \cup \cdots \cup T_{k+1}$ is a finite graph. Since each one of the points p_i is a ramification point of Y, then Y contains at least $k+1$ ramification points. Thus, Y contains at least k edges.

Now, suppose that (b) holds.

Then there exists a point $q \in Z$ such that $\mathrm{ord}(q, Z) \geqslant \aleph_0$. From [11, Example 8 of §51 p. 277], q is the vertex of a simple k-od Y. Therefore, Y contains k edges. This finishes the proof of the lemma. □

The next result easily follows from [8, Lemma 2.2].

Lemma 3.3. *If α is an arc in $F_n(Z)$ and α joins the elements A and B, then $\bigcup \alpha$ has a finite number of components, each one of them is locally connected and intersects both sets A and B.*

Theorem 3.4. *A Peano continuum Z is a finite graph if and only if, for some (each) $n \in \mathbb{N}$, $\mathcal{E}_n(Z)$ is an open dense subset of $F_n(Z)$ with a finite number of components.*

Proof. First suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{E}_n(Z)$ is an open dense subset of $F_n(Z)$, with r components ($r \in \mathbb{N}$) and Z is not a finite graph. Since $F_n(Z)$ is a Peano continuum, the components of $\mathcal{E}_n(Z)$ are arcwise connected. By Lemma 3.2 there exists a finite graph $Y \subset Z$ such that Y contains at least $k = 2r+1$ edges. Choose different edges $J_1, \ldots, J_k$ of Y and points $p_i \in \mathrm{int}_Y(J_i)$, for each $i \in \{1, \ldots, k\}$. Choose open connected and pairwise disjoint subsets $V_1, \ldots, V_k$ of Z such that $p_i \in V_i$ and $V_i \cap Y \subset \mathrm{int}_Y(J_i)$ for each $i \in \{1, \ldots, k\}$. Given $i \in \{1, \ldots, k\}$, since $\{p_i\} \in \langle V_i \rangle_n$ and $\mathcal{E}_n(Z)$ is dense, we can choose an element $A_i \in \langle V_i \rangle_n \cap \mathcal{E}_n(Z)$.

Since $\mathcal{E}_n(Z)$ has r components and we have $2r+1$ sets $A_1, \ldots, A_{2r+1}$, by the box principle, there exists a component $\mathcal{C}$ of $\mathcal{E}_n(Z)$ having three of the sets A_i. We may assume that A_1, A_2 and A_3 belong to $\mathcal{C}$. Since $\mathcal{C}$ is arcwise connected, there exist arcs α_1 and α_2 in $\mathcal{C}$ such that α_1 joins A_3 and A_1, and α_2 joins A_3 and A_2. Choose a point $x \in A_3$. Let C_1 and C_2 be the components of $\bigcup \alpha_1$ and $\bigcup \alpha_2$, respectively, such that $x \in C_1 \cap C_2$. By Lemma 3.3, the set $C = C_1 \cup C_2$ is a locally connected subcontinuum of $(\bigcup \alpha_1) \cup (\bigcup \alpha_2)$ that intersects A_1, A_2 and A_3.

Each point $p \in C$ belongs to an element of $\mathcal{E}_n(Z)$. By Lemma 3.1, p is not the core of any simple triod of Z. In particular, C is a Peano continuum without simple triods, therefore C is an arc or a simple closed curve. In any case, we conclude that there exists an arc in Z which intersects the three sets A_1, A_2 and A_3. For the rest of the proof, we may assume, without loss of generality that there exist an arc $\beta \subset C$ and points $a_1 \in A_1$, $a_2 \in A_2$ and $a_3 \in A_3 \cap \beta - \{a_1, a_2\}$ such that β joins a_1 and a_2. Since $a_3 \in V_3$ and V_3 is arcwise connected, there exists an arc α in V_3 that joins a_3 and p_3. Since the end points of β are not in V_3, the end points of β are not in α. Since the points of β are not cores of simple triods of Z, we have that $\alpha \subset \beta$. Thus, β intersects the edge J_3 which is an arc or a simple closed curve. Notice that J_3 contains a vertex v of Y which is the core of a simple triod in Z, thus $v \notin \beta$. Thus J_3 is not contained in β. Since β does not contain the core of a simple triod, one of the end points of β belongs to J_3. We may assume that $a_1 \in J_3$. Then $a_1 \in V_1 \cap J_3$. But V_1 was chosen in such a way that $V_1 \cap Y \subset \mathrm{int}_Y(J_1)$. Thus $a_1 \in J_3 \cap \mathrm{int}_Y(J_1)$. This is impossible since J_1 and J_3 are edges of the finite graph Y. This contradiction proves that Z is a finite graph.

Now suppose that Z is a finite graph and let n be an arbitrary positive integer. We are going to prove that $\mathcal{E}_n(Z)$ is an open dense subset of $F_n(Z)$ with a finite number of components. Let $\mathcal{G} = \{A \in F_n(Z)\colon A$ does not contain ramification points of X and $|A| = n\}$. Given $A = \{p_1, \ldots, p_n\} \in \mathcal{G}$, let $J_1, \ldots, J_n$ be pairwise disjoint arcs of Z such that $J_1 \cup \cdots \cup J_n$ does not contain ramification points of Z and $p_i \in \mathrm{int}_Z(J_i)$ for each $i \in \{1, \ldots, n\}$. It is easy to show the map from $J_1 \times \cdots \times J_n$ to $\langle J_1, \ldots, J_n\rangle_n$ which sends $(x_1, \ldots, x_n)$ to $\{x_1, \ldots, x_n\}$ is a homeomorphism. Thus $\langle J_1, \ldots, J_n\rangle_n$ is an n-cell which is a neighborhood of A is $F_n(Z)$. Thus $A \in \mathcal{E}_n(Z)$. We have shown that $\mathcal{G} \subset \mathcal{E}_n(Z)$. Clearly, $\mathcal{G}$ is dense in $F_n(Z)$. Therefore $\mathcal{E}_n(Z)$ is an open dense subset of $F_n(Z)$.

Let $E_1, \ldots, E_m$ be all different edges of Z, and $\mathcal{K}(i_1, \ldots, i_m)$ be the subset of $F_n(Z)$ such that each member of $\mathcal{K}(i_1, \ldots, i_m)$ has exactly i_j elements in the interior of edge E_j for each $j \in \{1, \ldots, m\}$. It is obvious that each $\mathcal{K}(i_1, \ldots, i_m)$ is connected. In the case that $i_1 + \cdots + i_m = n$, $\mathcal{K}(i_1, \ldots, i_m) \subset \mathcal{G} \subset \mathcal{E}_n(Z)$, and the union of all sets $\mathcal{K}(i_1, \ldots, i_m)$, with $i_1 + \cdots + i_m = n$, is dense in $\mathcal{E}_n(Z)$. Since $\mathcal{E}_n(Z)$ is an open subset of the Peano continuum $F_n(Z)$, $\mathcal{E}_n(Z)$ is locally arcwise connected. Thus each component of $\mathcal{E}_n(Z)$ intersects one set of the form $\mathcal{K}(i_1, \ldots, i_m)$. Since there is only finitely many sets $\mathcal{K}(i_1, \ldots, i_m)$, $\mathcal{E}_n(Z)$ has only finitely many components. The proof of the theorem is complete. □

Corollary 3.5. *If continua X and Y have homeomorphic symmetric products $F_n(X)$ and $F_n(Y)$ for some $n \in \mathbb{N}$, then X is a finite graph if and only if Y is.*

4. The case $n \geqslant 4$

If S is a simple closed curve, let $R(S) = \emptyset$. Given a finite graph X and $n \in \mathbb{N}$, let $R_n(X) = \{A \in F_n(X)\colon A \cap R(X) \neq \emptyset\}$. Notice that $R_1(X) = F_1(R(X))$.

Lemma 4.1. *Let X be a finite graph and $n \in \mathbb{N}$, then the components of $F_n(X) - R_n(X)$ are exactly the sets of the form $\langle \mathrm{int}_X(I_1), \ldots, \mathrm{int}_X(I_r)\rangle_n$, where $I_1, \ldots, I_r$ are pairwise different edges of X and $r \in \{1, \ldots, n\}$.*

Proof. Take pairwise different edges $I_1, \ldots, I_r$. Then $\mathrm{int}_X(I_1), \ldots, \mathrm{int}_X(I_r)$ are open connected and pairwise disjoint. In order to see that $\langle \mathrm{int}_X(I_1), \ldots, \mathrm{int}_X(I_r)\rangle_n$ is connected, consider the map $f : X^n \to F_n(X)$ given by $f(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$. It is easy to show that, if $\langle \mathrm{int}_X(I_1), \ldots, \mathrm{int}_X(I_r)\rangle_n \neq \emptyset$, then $r \leqslant n$ and $\langle \mathrm{int}_X(I_1), \ldots, \mathrm{int}_X(I_r)\rangle_n = \bigcup\{f(\mathrm{int}_X(I_{i_1}) \times \cdots \times \mathrm{int}_X(I_{i_n}))\colon \{I_1, \ldots, I_r\} = \{I_{i_1}, \ldots, I_{i_n}\}\}$. Fix elements $p_1 \in \mathrm{int}_X(I_1)$,

$\dots, p_n \in \operatorname{int}_X(I_r)$. Then $\{p_1, \dots, p_r\}$ is in the image of each set of the form $f(\operatorname{int}_X(I_{i_1}) \times \dots \times \operatorname{int}_X(I_{i_n}))$, where $\{I_1, \dots, I_r\} = \{I_{i_1}, \dots, I_{i_n}\}$. Since $f(\operatorname{int}_X(I_{i_1}) \times \dots \times \operatorname{int}_X(I_{i_n}))$ is connected, we conclude that $\langle \operatorname{int}_X(I_1), \dots, \operatorname{int}_X(I_r)\rangle_n$ is connected. Therefore, each set of the form $\langle \operatorname{int}_X(I_1), \dots, \operatorname{int}_X(I_r)\rangle_n$ is connected (and open).

It is easy to see that if $\{I_1, \dots, I_r\} \neq \{J_1, \dots, J_s\}$, then $\langle \operatorname{int}_X(I_1), \dots, \operatorname{int}_X(I_r)\rangle_n \cap \langle \operatorname{int}_X(J_1), \dots, \operatorname{int}_X(J_s)\rangle_n = \emptyset$. Finally, since $X - R(X) = \bigcup\{\operatorname{int}_X(J)\colon J \text{ is an edge } X\}$, it follows that the union of all sets of the form $\langle \operatorname{int}_X(I_1), \dots, \operatorname{int}_X(I_r)\rangle_n$ is equal to $F_n(X) - R_n(X)$. This completes the proof of the lemma. □

Proceeding as in the proof that $\mathcal{G} \subset \mathcal{E}_n(X)$ in Theorem 3.4, the following lemma can be proved.

Lemma 4.2. *Let X be a finite graph and $A \in F_n(X) - (F_{n-1}(X) \cup R_n(X))$. Then A has a neighborhood in $F_n(X)$ which is an n-cell (i.e. $A \in \mathcal{E}_n(X)$).*

Lemma 4.3. *Let X be a finite graph, $A \in F_{n-1}(X)$ and $n \geqslant 4$. Then no neighborhood of A in $F_n(X)$ can be embedded in $\mathbb{R}^n$.*

Proof. Let $\mathcal{U}$ be a neighborhood of A in $F_n(X)$. Since $A \in F_{n-1}(X)$, it is possible to find different points $p_1, \dots, p_{n-1}$ of X and pairwise disjoint subarcs $I_1, \dots, I_{n-1}$ of X such that $p_i \in I_i$, p_i is not an end point of I_i, for each $i \in \{1, \dots, n-1\}$ and $\langle I_1, I_2, \dots, I_{n-1}\rangle_n \subset \mathcal{U}$.

Given $i \in \{1, \dots, n-1\}$, there exists a homeomorphism $f_i : [0,1]^2 \to F_2(I_i)$ such that $f_i([0,1] \times \{0\}) = F_1(I_i)$ and $f_i(\frac{1}{2}, 0) = \{p_i\}$. Let $\alpha_i : [0,1] \to F_1(I_i)$ be given by $\alpha_i(t) = f_i(t, 0)$.

Let $\varphi : [0,1]^{n-1} \times [-1,1] \to \mathcal{U}$ be given by

$$\varphi(t_1, t_2, \dots, t_n) = \begin{cases} f_1(t_1, t_n) \cup \alpha_2(t_2) \cup \dots \cup \alpha_{n-1}(t_{n-1}), & \text{if } t_n \geqslant 0, \\ \alpha_1(t_1) \cup f_2(t_2, -t_n) \cup \alpha_3(t_3) \cup \dots \cup \alpha_{n-1}(t_{n-1}), & \text{if } t_n \leqslant 0. \end{cases}$$

Clearly, φ is a well defined map and, for each $z \in [0,1]^{n-1} \times [-1,1]$, $\varphi(z) \in \langle I_1, I_2, \dots, I_{n-1}\rangle_n \subset \mathcal{U}$.

Now, we see that φ is one-to-one. Suppose that $\varphi(t_1, \dots, t_n) = \varphi(s_1, \dots, s_n)$.

In the case that $t_n, s_n \geqslant 0$, we have $f_1(t_1, t_n) \cup \alpha_2(t_2) \cup \dots \cup \alpha_{n-1}(t_{n-1}) = f_1(s_1, s_n) \cup \alpha_2(s_2) \cup \dots \cup \alpha_{n-1}(s_{n-1})$. Since $I_1, \dots, I_{n-1}$ are pairwise disjoint, $f_1(t_1, t_n) = f_1(s_1, s_n)$, $f_2(t_2, 0) = f_2(s_2, 0)$, $\dots$, $f_{n-1}(t_{n-1}, 0) = f_{n-1}(s_{n-1}, 0)$. Since each one of the maps $f_1, f_2, \dots, f_{n-1}$ is one-to-one, $(t_1, \dots, t_n) = (s_1, \dots, s_n)$.

The case $t_n, s_n \leqslant 0$ is similar.

Finally, suppose that $s_n \leqslant 0 \leqslant t_n$. Then $f_1(t_1, t_n) \cup \alpha_2(t_2) \cup \dots \cup \alpha_{n-1}(t_{n-1}) = \alpha_1(s_1) \cup f_2(s_2, -s_n) \cup \alpha_3(s_3) \cup \dots \cup \alpha_{n-1}(s_{n-1})$. Thus $f_1(t_1, t_n) = f_1(s_1, 0)$, $f_2(t_2, 0) = f_2(s_2, -s_n)$, $f_3(t_3, 0) = f_3(s_3, 0)$, $\dots$, $f_{n-1}(t_{n-1}, 0) = f_{n-1}(s_{n-1}, 0)$. Hence $(t_1, \dots, t_{n-1}) = (s_1, \dots, s_{n-1})$ and $t_n = 0 = s_n$.

Therefore, φ is one-to-one.

Let $\mathcal{C} = \operatorname{Im} \varphi$, so $\mathcal{C}$ is an n-cell in $\mathcal{U}$. Consider the arc $\mathcal{A} = \{p_1, p_2, p_4, \dots, p_{n-1}\} \cup f_3(\{\frac{1}{2}\} \times [0,1])$. Clearly, $\mathcal{A} \subset \mathcal{U}$. Notice that the element $\varphi(\frac{1}{2}, \dots, \frac{1}{2}, 0) = f_1(\frac{1}{2}, 0) \cup f_2(\frac{1}{2}, 0) \cup \dots \cup f_{n-1}(\frac{1}{2}, 0) = \{p_1, \dots, p_{n-1}\}$ belongs to $\mathcal{A}$. On the other hand, if $t > 0$, the set $\{p_1, p_2, p_4, \dots, p_{n-1}\} \cup f_3(\frac{1}{2}, t)$ has two different points in the arc I_3 and, for each

$z = (t_1, \dots, t_n) \in [0,1]^{n-1} \times [-1,1]$, $\varphi(z) \cap I_3 = \alpha_3(t_3)$ is a one-point set. We have shown that $\mathcal{A} \cap \mathcal{C} = \{\varphi(\frac{1}{2}, \dots, \frac{1}{2}, 0)\}$. Therefore, $\mathcal{C} \cup \mathcal{A}$ is the union of the n-cell $\mathcal{C}$ and the arc $\mathcal{A}$ which intersects $\mathcal{C}$ only in one point which is an end point of $\mathcal{A}$ and it is in the manifold interior of $\mathcal{C}$ (such spaces are called n-dimensional umbrellas). Theorem of the Invariance of Domain imply that the space $\mathcal{C} \cup \mathcal{A}$ is not embeddable in $\mathbb{R}^n$. We conclude that $\mathcal{U}$ cannot be embeddable in $\mathbb{R}^n$. □

Corollary 4.4. *Let X be a finite graph and $n \geqslant 4$. Then $\mathcal{E}_n(X) = F_n(X) - (R_n(X) \cup F_{n-1}(X))\}$.*

The next lemma uses the fact that for every nondegenerate continuum X the hyperspace $F_n(X)$ is locally separated by $F_{n-1}(X)$. Indeed, for example if $A = \{p, q\}$ with $n > 2$ and $p \neq q$, then A has in its small neighborhoods both sets $B \in F_n(X) - F_{n-1}(X)$ having the point p and the rest near q, and sets $C \in F_n(X) - F_{n-1}(X)$ having the point q and the rest near p. However, there is no connected collection of sets in $F_n(X) - F_{n-1}(X)$, each near A, that contains B and C.

Lemma 4.5. *Let X be a finite graph and $n \geqslant 4$. For every $A \in F_n(X)$ the following conditions are equivalent*:

(a) $A \in F_1(X) - R_n(X)$;
(b) *$A \notin \mathcal{E}_n(X)$ and A has a basis $\mathcal{B}$ of neighborhoods in $F_n(X)$ such that $\mathcal{U} \cap \mathcal{E}_n(X)$ is arcwise connected for each $\mathcal{U} \in \mathcal{B}$.*

Proof. ((b) ⇒ (a)) Suppose that A has a basis of neighborhoods $\mathcal{B}$ in $F_n(X)$ such that, for each $\mathcal{U} \in \mathcal{B}$, $\mathcal{U} \cap \mathcal{E}_n(X)$ is arcwise connected and $A \notin \mathcal{E}_n(X)$. Since $A \notin \mathcal{E}_n(X)$, then $A \in R_n(X) \cup F_{n-1}(X)$. Let $A = \{p_1, p_2, \dots, p_r\}$, where $1 \leqslant r \leqslant n$ and all the points $p_1, p_2, \dots, p_r$ are different. Let $\delta_1 > 0$ be such that $B(\delta_1, p_1), \dots, B(\delta_1, p_r)$ are pairwise disjoint $A \cap R(X) = (B(\delta_1, p_1) \cup \dots \cup B(\delta_1, p_r)) \cap A$ and $\delta_1 < \frac{1}{3}$. Choose $\mathcal{U} \in \mathcal{B}$ such that $\mathcal{U} \subset B(\delta_1, A)$. Let $\delta > 0$ be such that $B(\delta, A) \subset \mathcal{U}$.

First, we consider the case that $A \in R_n(X)$. In this case, we may assume that $p_r \in R(X)$. Let J and L be edges of X such that $J \neq L$ and $p_r \in J \cap L$. Choose two subsets with n different points $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ of $X - R(X)$ such that $d(x_i, p_i) < \delta$, $d(y_i, p_i) < \delta$ for each $i \in \{1, \dots, r-1\}$, $\{x_r, x_{r+1}, \dots, x_n\} \subset B_X(\delta, p_r) \cap J$ and $\{y_r, y_{r+1}, \dots, y_n\} \subset B_X(\delta, p_r) \cap L$. Then the sets $B = \{x_1, \dots, x_n\}$ and $C = \{y_1, \dots, y_n\}$ belong to $\mathcal{U} \cap \mathcal{E}_n(X)$. From the choice of $\mathcal{B}$, there exists a map $\alpha : [0,1] \to \mathcal{U} \cap \mathcal{E}_n(X) \subset B_{F_n(X)}(\delta_1, A)$ such that $\alpha(0) = B$ and $\alpha(1) = C$.

Notice that p_r is a point that separates the set $B_X(\delta_1, p_r)$ in two open subsets U and V such that $\{x_r, x_{r+1}, \dots, x_n\} \subset U$ and $\{y_r, y_{r+1}, \dots, y_n\} \subset V$. Given $t \in [0,1]$, $\alpha(t) \in \mathcal{E}_n(X)$, so $\alpha(t)$ does not contain ramification points and $\alpha(t)$ contains n different points. In particular, $p_r \notin \alpha(t)$. Moreover, $\alpha(t) \in \mathcal{U}$, so $\alpha(t) \in B(\delta_1, A)$. Hence, $\alpha(t) \subset B_X(\delta_1, p_1) \cup \dots \cup B_X(\delta_1, p_{r-1}) \cup U \cup V$.

Let $K_1 = \{t \in [0,1]\colon\ \alpha(t) \subset B_X(\delta_1, p_1) \cup \dots \cup B_X(\delta_1, p_{r-1}) \cup U\}$ and $K_2 = \{t \in [0,1]\colon\ \alpha(t) \cap V \neq \emptyset\}$. Hence, $[0,1] = K_1 \cup K_2$. Since V does not intersect $B_X(\delta_1, p_1) \cup \dots \cup B_X(\delta_1, p_{r-1}) \cup U$, we have $K_1 \cap K_2 = \emptyset$. Clearly, K_1 and K_2 are open in $[0,1]$. Since

$\alpha(0) = B$ and $\alpha(1) = C$, $0 \in K_1$ and $1 \in K_2$. Thus K_1 and K_2 is a separation of $[0, 1]$. This contradiction proves that $A \notin R_n(X)$.

Therefore, $A \in F_{n-1}(X) - R_n(X)$.

Now, suppose that A is not degenerate. That is $r > 1$. Choose subsets $\{x_{r+1}, \ldots, x_n\} \in B_X(\delta, p_1) - \{p_1\}$ and $\{y_{r+1}, \ldots, y_n\} \in B_X(\delta, p_r) - \{p_r\}$, where the points $x_{r+1}, \ldots, x_n$ are all different and the same happens with $y_{r+1}, \ldots, y_n$. Let $B = \{p_1, \ldots, p_r, x_{r+1}, \ldots, x_n\}$ and $C = \{p_1, \ldots, p_r, y_{r+1}, \ldots, y_n\}$. Then the sets B and C belong to $\mathcal{U} \cap \mathcal{E}_n(X)$. By the choice of $\mathcal{B}$, there exists a map $\alpha : [0, 1] \to \mathcal{U} \cap \mathcal{E}_n(X) \subset B_{F_n(X)}(\delta_1, A)$ such that $\alpha(0) = B$ and $\alpha(1) = C$.

Let $K_1 = \{t \in [0, 1]$: $\alpha(t)$ contains exactly one point in $B_X(\delta_1, p_1)\}$ and $K_2 = \{t \in [0, 1]$: $\alpha(t)$ contains more than one point in $B_X(\delta_1, p_1)\}$. Clearly, $[0, 1] = K_1 \cup K_2$, $1 \in K_1$, $0 \in K_2$ and $K_1 \cap K_2 = \emptyset$.

Next, we show that K_1 and K_2 are open in $[0, 1]$.

Given $t \in K_2$, let $\alpha(t) = \{w_1, \ldots, w_n\}$, where all the points $w_1, \ldots, w_n$ are different. We know that $\alpha(t)$ contains at least two elements in $B_X(\delta_1, p_1)$. Suppose that w_1 and w_2 belong to $B_X(\delta_1, p_1)$. Let $\delta_0 > 0$ be such that $B_X(\delta_0, w_1), \ldots, B_X(\delta_0, w_n)$ are pairwise disjoint and $B_X(\delta_0, w_1) \cup B_X(\delta_0, w_2) \subset B_X(\delta_1, p_1)$. If s is close to t, $\alpha(s)$ has an element in $B_X(\delta_0, w_1)$ and another one in $B_X(\delta_0, w_2)$, both points are in $B_X(\delta_1, p_1)$. Hence $s \in K_2$. We have shown that K_2 is open.

Now take $t \in K_1$. Suppose that $\alpha(t) = \{w_1, \ldots, w_n\}$, where all the points $w_1, \ldots, w_n$ are different. Suppose that w_1 is the only element of $\alpha(t)$ that belongs to $B_X(\delta_1, p_1)$. Since $\alpha(t) \in \mathcal{U}$, $\alpha(t) \subset B_X(\delta_1, p_1) \cup \cdots \cup B_X(\delta_1, p_r)$. Thus we can take $\delta_0 > 0$ such that the sets $B_X(\delta_0, w_1), \ldots, B_X(\delta_0, w_n)$ are pairwise disjoint and each one of them is contained in one set of the form $B_X(\delta_1, p_j)$. Since w_1 is the only w_i that belongs to $B_X(\delta_1, p_1)$, $B_X(\delta_1, p_1) \cap (B_X(\delta_0, w_2) \cup \cdots \cup B_X(\delta_0, w_n)) = \emptyset$. Let $s \in [0, 1]$ be such that $H(\alpha(s), \alpha(t)) < \delta_0$. Then $\alpha(s)$ intersects each one of the sets of the form $B_X(\delta_0, w_i)$ and it is contained in their union. Thus $\alpha(s)$ has $n - 1$ elements in $B_X(\delta_0, w_2) \cup \cdots \cup B_X(\delta_0, w_n)$ and one in $B_X(\delta_0, w_1)$. Hence, $\alpha(s)$ has exactly one element in $B_X(\delta_1, p_1)$. Hence $s \in K_1$. This completes the proof that K_1 is open.

We have found a disconnection of $[0, 1]$. This contradiction proves that A is degenerate.

Therefore, $A \in F_1(X) - R_n(X)$.

((a) $\Rightarrow$ (b)) Suppose that $A \in F_1(X) - R_n(X)$. Thus $A = \{p\}$ for some $p \in A - R(X)$. Hence, there exists $\delta > 0$ such that $B_X(\delta, p)$ is contained in some edge J of X and $B_X(\delta, p) \cap R(X) = \emptyset$. Thus $B_X(\delta, p)$ is homeomorphic to a subinterval L of $[0, 1]$. We identify $B_X(\delta, p)$ with L. Let $\mathcal{B} = \{B_{F_n(X)}(\eta, \{p\})$: $0 < \eta < \delta\}$. We are going to prove that if $\eta > 0$ and $B, C \in B_{F_n(X)}(\eta, \{p\}) \cap \mathcal{E}_n(X)$. Then there exists an arc contained in $B_{F_n(X)}(\eta, \{p\}) \cap \mathcal{E}_n(X)$ which joins B and C. Since $B, C \subset N(\delta, p) = L$, we may assume that $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$, where $b_1 < \cdots < b_n$ and $c_1 < \cdots < c_n$. Thus define $\alpha : [0, 1] \to B_{F_n(X)}(\eta, \{p\}) \cap \mathcal{E}_n(X)$ by $\alpha(t) = \{tb_1 + (1-t)c_1, \ldots, tb_n + (1-t)c_n\}$. Clearly, α is a map, $\alpha(0) = C$ and $\alpha(1) = B$. Thus $B_{F_n(X)}(\eta, \{p\}) \cap \mathcal{E}_n(X)$ is arcwise connected. Since A is degenerate, $A \notin \mathcal{E}_n(X)$. This completes the proof of the theorem. □

Theorem 4.6. *Let X and Y be finite graphs. Suppose that $F_n(X)$ is homeomorphic to $F_n(Y)$ and $n \geqslant 4$. Then X is homeomorphic to Y.*

Proof. Let $h : F_n(X) \to F_n(Y)$ be a homeomorphism.

Notice that $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$. By Lemma 4.5, $h(F_1(X) - R_n(X)) = F_1(Y) - R_n(Y) \subset F_1(Y)$. Since $F_1(X) - R_n(X)$ is dense in $F_1(X)$ and $F_1(Y)$ is compact, we have that $h(F_1(X)) \subset F_1(Y)$. Similarly, $h^{-1}(F_1(Y)) \subset F_1(X)$. Thus, $h(F_1(X)) = F_1(Y)$. Hence $F_1(X)$ is homeomorphic to $F_1(Y)$. Therefore, X is homeomorphic to Y. □

5. The case $n \leqslant 3$

Lemma 5.1. *Let X be a finite graph, $n \in \{2, 3\}$ and $A \in F_n(X) - R_n(X)$. Then $A \in \mathcal{E}_n(X)$.*

Proof. We only analyze the case that $n = 3$, the case $n = 2$ is simpler.

First suppose that $A = \{x\}$ for some $x \in X$. Since x is not a ramification point of X, there exists a neighborhood J of x such that J is an arc. Thus $A \in \langle J\rangle_3$ and $\langle J\rangle_3 = F_3(J)$ is a neighborhood of A in $F_3(X)$ which is homeomorphic to $[0, 1]^3$ (see [6, Theorem 6]).

Now suppose that $A = \{x, y\}$, where $x \neq y$. Let J_1 and J_2 be disjoint arcs in $X - R(X)$ such that J_1 and J_2 are neighborhoods of x and y, respectively. Thus $\langle J_1, J_2\rangle_3$ is a neighborhood of A in $F_3(X)$. For each $i \in \{1, 2\}$, there exists a homeomorphism $f_i : [0, 1]^2 \to F_2(J_i)$ such that $f_i([0, 1] \times \{0\}) = F_1(J_i)$. Let $\varphi : [0, 1]^2 \times [-1, 1] \to \langle J_1, J_2\rangle_3$ be given by

$$\varphi(t_1, t_2, t_3) = \begin{cases} f_1(t_1, t_3) \cup f_2(t_2, 0), & \text{if } t_3 \geqslant 0, \\ f_1(t_1, 0) \cup f_2(t_2, -t_3), & \text{if } t_3 \leqslant 0. \end{cases}$$

It is easy to show that φ is a homeomorphism. Therefore, $\langle J_1, J_2\rangle_3$ is a 3-cell.

The last case is $A = \{x, y, z\}$, where x, y and z are all different. This case follows from Lemma 4.2. □

Let Z be a continuum and $\mathcal{W}$ an open subset of Z. For each open subset U of Z, let $c(U) =$ (number of components of $U \cap \mathcal{W}$), if this number is finite and $c(U) = \infty$, otherwise. For each $p \in \mathrm{cl}_Z(\mathcal{W})$, define $v(p) = \min(\{m \in \mathbb{N}\colon\ p$ has a basis of neighborhoods $\mathcal{B}$ in Z such that $c(U) = m$ for each $U \in \mathcal{B}\} \cup \{\infty\})$.

Lemma 5.2. *Let Z be a continuum, $p \in Z$, $\mathcal{W}$ an open subset of Z and $m \in \mathbb{N}$. Suppose that p has a basis of neighborhoods $\mathcal{B}$ in Z such that, for each $U \in \mathcal{B}$, $c(U) = m$ and for each component C of $U \cap \mathcal{W}$, $p \in \mathrm{cl}_Z(C)$. Then $v(p) = m$.*

Proof. By the definition of $v(p)$, $v(p) \leqslant m$ and p has a basis of neighborhoods $\mathcal{B}_1$ in Z such that $c(U) = v(p)$ for each $U \in \mathcal{B}_1$. Let $V \in \mathcal{B}$ and $U \in \mathcal{B}_1$ be such that $U \subset V$. By hypothesis $V \cap \mathcal{W}$ has m components $C_1, \ldots, C_m$ and $p \in \mathrm{cl}_Z(C_i)$ for each $i \in \{1, \ldots, m\}$. Thus $U \cap C_i \neq \emptyset$ for each $i \in \{1, \ldots, m\}$. Hence $U \cap \mathcal{W} = U \cap V \cap \mathcal{W} = (U \cap C_1) \cup \cdots \cup (U \cap C_m)$. Since the sets in this union are mutually separated and they are nonempty, $U \cap \mathcal{W}$ has at least m components. Thus, $v(p) = c(U) \geqslant m$. Therefore, $v(p) = m$. □

For an acircular graph X, let v_X be the index defined as before for the set $\mathcal{E}_3(X)$. We simply write v if it is not necessary to mention the space X.

Lemma 5.3. *Let X be a finite graph, let p, q, r, w, x and y be points of X such that* $\mathrm{ord}(p, X) = n \geqslant 3$, $\mathrm{ord}(q, X) = m \geqslant 3$, $\mathrm{ord}(r, X) = k \geqslant 3$ *and x, y and w are not ramification points of X. Given $A \in F_3(X)$, then the possible values for $v(A)$ are*:

(a) *if $A = \{p\}$, then $v(A) = n + \binom{n}{2} + \binom{n}{3}$,*
(b) *if $A = \{p, x\}$, then $v(A) = n + \binom{n}{2}$,*
(c) *if $A = \{p, x, y\}$ and $x \neq y$, then $v(A) = n$,*
(d) *if $A = \{p, q\}$ and $p \neq q$, then $v(A) = n \cdot \binom{m}{2} + m \cdot \binom{n}{2} + n \cdot m$,*
(e) *if $A = \{p, q, w\}$ and $p \neq q$, then $v(A) = n \cdot m$,*
(f) *if $A = \{p, q, r\}$ and p, q and r are all different, then $v(A) = n \cdot m \cdot k$,*
(g) *if $A \in F_3(X) - R_3(X)$, then $v(A) = 1$.*

Proof. We use Lemma 5.2. Let $\delta_0 > 0$ be such that $N_X(\delta_0, A) \cap R(X) = A \cap R(X)$, and $N_X(\delta_0, A)$ has as many components as the number of points of A and $\delta_0 < \frac{1}{3}$. Let $\mathcal{B} = \{B_{F_3(X)}(\delta, A) \subset F_3(X)\colon 0 < \delta < \delta_0\}$. Then $\mathcal{B}$ is a basis of neighborhoods of A in $F_3(X)$.

(a) Suppose that $A = \{p\}$. Let $\delta \in (0, \delta_0)$. Then $\mathrm{cl}_X(N_X(\delta, A))$ is a simple n-od, so $\mathrm{cl}_X(N_X(\delta, A)) = J_1 \cup \cdots \cup J_n$, where $J_i \cap J_j = \{p\}$, if $i \neq j$ and each J_i is an arc with end points p and a point a_i. Notice that an element $B \in F_3(X)$ belongs to $B_{F_3(X)}(\delta, A)$ if and only if $B \subset N_X(\delta, A) = \mathrm{cl}_X(N_X(\delta, A)) - \{a_1, \ldots, a_n\}$. Thus $B \in B_{F_3(X)}(\delta, A) \cap \mathcal{E}_3(X)$ if and only if $B \subset N_X(\delta, A) - \{p\} = (J_1 - \{p\}) \cup \cdots \cup (J_n - \{p\})$. Proceeding as in Lemma 4.1, we have that the components of $B_{F_3(X)}(\delta, A) \cap \mathcal{E}_3(X)$ are the sets of the form $\langle J_{i_1} - \{p\}, \ldots, J_{i_r} - \{p\}\rangle_3$, where $i_1, \ldots, i_r \in \{1, \ldots, n\}$ are all different numbers and $r \in \{1, 2, 3\}$. Thus $c(B_{F_3(X)}(\delta, A)) = n + \binom{n}{2} + \binom{n}{3}$. It is easy to show that $A \in \mathrm{cl}_X(\langle J_{i_1} - \{p\}, \ldots, J_{i_r} - \{p\}\rangle_3)$. Applying Lemma 5.2, we conclude that $v(A) = n + \binom{n}{2} + \binom{n}{3}$.

The proofs of (b)–(g) are similar. □

The proof of the following lemma is similar to the proof of Lemma 5.3.

Lemma 5.4. *Let X be a finite graph, let p, q and x be points of X such that* $\mathrm{ord}(p, X) = n \geqslant 3$, $\mathrm{ord}(q, X) = m \geqslant 3$ *and x is not a ramification point of X. Given $A \in F_2(X)$, then the possible values for $v(A)$ ($v(A)$ is defined as in the previous paragraph to Lemma* 5.2 *for the open set $\mathcal{E}_2(X)$) are*:

(a) *if $A = \{p\}$, then $v(A) = n + \binom{n}{2}$,*
(b) *if $A = \{p, x\}$, then $v(A) = n$,*
(c) *if $A = \{p, q\}$ and $p \neq q$, then $v(A) = n \cdot m$,*
(d) *if $A \in F_2(X) - R_2(X)$, then $v(A) = 1$.*

Lemma 5.5. *Let X and Y be finite graphs and $n \in \{2, 3\}$. Suppose that there exists a homeomorphism $h : F_n(X) \to F_n(Y)$. If p is a ramification point of X, then $h(\{p\}) = \{u\}$ for some ramification point u of Y.*

Proof. We only prove the lemma for $n = 3$, the proof for $n = 2$ is similar and simpler. By Lemmas 3.1 and 5.1, $h(R_3(X)) = R_3(Y)$ and $h(\mathcal{E}_3(X)) = \mathcal{E}_3(Y)$. In particular, $h(\{p\}) \in R_3(Y)$. Notice that for each $A \in F_3(X)$, $v_X(A) = v_Y(h(A))$.

Given $A \in F_3(X)$, if A contains a point $x \in X - R(X)$, then there exists an arc J in X such that $x \in J$ and $J \cap R(X) = \emptyset$. By Lemma 5.3, $v_X(A) = v_X((A - \{x\}) \cup \{u\})$, for each $u \in J$. This shows that, for each neighborhood $\mathcal{U}$ of A in $F_3(X)$, $v_X(A)$ coincides with $v_X(A_1)$ for infinitely many elements A_1 of $\mathcal{U}$.

Given $A \in F_3(X)$, if $A \subset R(X)$, then A is of one of the forms described in (a), (d) or (f) of Lemma 5.3. If A is of the form described in (a), then the elements A_1 of $F_3(X)$ which are close to A and are different from A is of one of the forms described in (b), (c) or (g), and for each one of them $v_X(A_1) < v_X(A)$. Hence v_X attains an absolute local maximum at A. Similarly, if A is of one of the forms (d) or (f), then $v_X(A)$ attains also an absolute local maximum at A.

Therefore, if $A \in F_3(X)$, then $A \subset R(X)$ if and only if $v_X(A)$ attains an absolute local maximum at A.

Therefore, if $A \in F_3(X)$, then $A \subset R(X)$ if and only if $h(A) \subset R(Y)$.

Hence, $h(\{p\})$ is of one of the following forms: $\{u\}$, $\{u, z\}$, $\{u, z, w\}$, where $u, z, w \in R(Y)$ are all different.

First, we analyze the case that $h(\{p\}) = \{u, z\}$, with $u \neq z$, where $\operatorname{ord}(p, X) = k$, $\operatorname{ord}(u, Y) = r$ and $\operatorname{ord}(z, Y) = s$. Since $v_X(\{p\}) = v_Y(\{u, z\})$, we have that

$$k + \binom{k}{2} + \binom{k}{3} = r \cdot \binom{s}{2} + s \cdot \binom{r}{2} + r \cdot s.$$

According to Lemma 5.3, the possible values for v_X in the elements different from $\{p\}$ and in a small neighborhood of $\{p\}$ in $F_3(X)$ are $k + \binom{k}{2}$, k and 1. On the other hand, the possible values for v_Y in the elements different from $\{u, z\}$ and in a small neighborhood of $\{u, z\}$ in $F_3(Y)$ are $r + \binom{r}{2}$, $s + \binom{s}{2}$, $r \cdot s$, r, s and 1. Since these two sets of values must coincide, we obtain that $r + \binom{r}{2}$ and r belong to $\{k + \binom{k}{2}, k\}$. Thus, $r = k$. Similarly, $s = k$. But then $r \cdot s = k^2$ must be equal to either $k + \binom{k}{2}$ or k. Clearly, $k^2 \neq k$, and if $k^2 = k + \frac{k(k-1)}{2}$, then $k = 1$, which is a contradiction. We have shown that it is impossible that $h(\{p\}) = \{u, z\}$.

Now, we analyze the case that $h(\{p\}) = \{u, z, w\}$, where u, z, w are all different, $\operatorname{ord}(p, X) = k$, $\operatorname{ord}(u, Y) = r$, $\operatorname{ord}(z, Y) = s$ and $\operatorname{ord}(w, Y) = t$. Since $v_X(\{p\}) = v_Y(\{u, z, w\})$, we have that

$$k + \binom{k}{2} + \binom{k}{3} = r \cdot s \cdot t.$$

According to Lemma 5.3, the possible values for v_X in the elements different from $\{p\}$ and in a small neighborhood of $\{p\}$ in $F_3(X)$ are $k + \binom{k}{2}$, k and 1. On the other hand, the possible values of v_Y in the elements different from $\{u, z, y\}$ and in a small neighborhood of $\{u, z, y\}$ in $F_3(Y)$ are r, s, t, $r \cdot s$, $r \cdot t$, $s \cdot t$ and 1. Since these two sets of values must coincide, we obtain that r and $r \cdot s$ must belong to the set $\{k + \binom{k}{2}, k\}$. Thus $r = k$. Similarly, $s = k = t$. Hence, $k^2 = k + \frac{k(k-1)}{2}$. Thus $k = 1$ which is a contradiction. We have shown that this case is also impossible.

Therefore, the only possibility is that $h(\{p\})$ is of the form $h(\{p\}) = \{u\}$, for some $u \in R(Y)$. □

Theorem 5.6. *Let X and Y be acircular graphs different from an arc and $n \in \{2,3\}$. Suppose that $F_n(X)$ is homeomorphic to $F_n(Y)$. Then X is homeomorphic to Y.*

Proof. We only prove the theorem for $n = 3$. The proof for $n = 2$ is simpler. Let $h : F_3(X) \to F_3(Y)$ be a homeomorphism. According to Lemma 5.5, for each point $p \in R(X)$, there exists a point $k(p) \in R(Y)$ such that $h(\{p\}) = \{k(p)\}$. Thus $h^{-1}(\{k(p)\}) = \{p\}$. Applying again Lemma 5.5 to h^{-1}, for each $q \in R(Y)$, there exists a point $k'(q) \in R(X)$ such that $h^{-1}(\{q\}) = \{k'(q)\}$. Given $p \in R(X)$, $\{k'(k(p))\} = h^{-1}(\{k(p)\}) = \{p\}$. Thus $k'(k(p)) = p$. Similarly, $k(k'(q)) = q$ for each $q \in R(Y)$. Hence k is a bijection between $R(X)$ and $R(Y)$. As usual, k' will be denoted by k^{-1}. We are going to prove the theorem by showing a series of claims.

Claim 1. *Let $p, x \in R(X)$. Then p and x are adjacent vertices in X if and only if $k(p)$ and $k(x)$ are adjacent vertices in Y.*

Clearly, we only need to show the necessity of Claim 1. Since p and x are adjacent, there exists an edge L of X such that p and x are the end points of L. Let $\mathcal{U} = \langle \mathrm{int}_X(L)\rangle_3$. By Lemma 4.1, $\mathcal{U}$ is open and it is a component of $F_3(X) - R_3(X)$. By Lemmas 3.1 and 5.1, $h(F_3(X) - R_3(X)) = F_3(Y) - R_3(Y)$. Thus $h(\mathcal{U})$ is a component of $F_3(Y) - R_3(Y)$. By Lemma 4.1, $h(\mathcal{U}) = \langle \mathrm{int}_Y(J_1), \ldots, \mathrm{int}_Y(J_r)\rangle_3$ for some edges $J_1, \ldots, J_r$ of Y and some $r \in \{1,2,3\}$. Since $\{p\}, \{x\} \in \mathrm{cl}_{F_3(X)}(\mathcal{U})$, $\{k(p)\}, \{k(q)\} \in \mathrm{cl}_{F_3(Y)}(\langle \mathrm{int}_Y(J_1), \ldots, \mathrm{int}_Y(J_r)\rangle_3)$. Hence there exists a sequence $\{B_n\}_{n=1}^{\infty}$ in $\langle \mathrm{int}_Y(J_1), \ldots, \mathrm{int}_Y(J_r)\rangle_3$ such that $\lim B_n = \{k(p)\}$. Since each B_n intersects J_1 and J_1 is closed, J_1 intersects $\{k(p)\}$. Thus, $k(p) \in J_1$. Similarly, $k(x) \in J_1$. Hence, $k(p)$ and $k(x)$ are adjacent.

Claim 2. *Let $p, x \in R(X)$ be adjacent vertices of X. Then the number of edges of X that join p and x coincides with the number of edges of Y that join $k(p)$ and $k(x)$.*

In order to prove Claim 2, let $I_1, \ldots, I_s$ be the different edges of X that join p and x.

Let $\mathcal{C} = \langle \mathrm{int}_X(L_1), \ldots, \mathrm{int}_X(L_r)\rangle_3$, be a component of $F_3(X) - R_3(X)$ such that $\{p\}, \{x\} \in \mathrm{cl}_{F_3(X)}(\mathcal{C})$. Proceeding as in the proof of Claim 1, $\{p, x\} \in L_1 \cap \cdots \cap L_r$. Hence, $\{L_1, \ldots, L_r\}$ is a nonempty subset of $\{I_1, \ldots, I_s\}$ with at most 3 elements.

On the other hand, if $\{L_1, \ldots, L_r\}$ is a nonempty subset of $\{I_1, \ldots, I_s\}$ with at most 3 elements, then $\{p\}, \{x\} \in \mathrm{cl}_{F_3(X)}(\langle \mathrm{int}_X(L_1), \ldots, \mathrm{int}_X(L_r)\rangle_3)$.

Therefore, the number of components $\mathcal{C}$ of $F_3(X) - R_3(X)$ such that $\{p\}, \{x\} \in \mathrm{cl}_{F_3(X)}(\mathcal{C})$ is equal to $s + \binom{s}{2} + \binom{s}{3}$.

Since h is a homeomorphism, this number must coincide with the number of components of $h(F_3(X) - R_3(X)) = F_3(Y) - R_3(Y)$ that contain $h(\{p\}) = \{k(p)\}$ and $h(\{x\}) = \{k(x)\}$ in its closure. Which is equal to $m + \binom{m}{2} + \binom{m}{3}$, where m is the number of edges of Y that joins $k(p)$ and $k(x)$. Thus $s + \binom{s}{2} + \binom{s}{3} = m + \binom{m}{2} + \binom{m}{3}$. Hence, $s = m$. This completes the proof of Claim 2.

Claim 3. *If $p \in R(X)$ and $\mathrm{ord}(p, X) = r$, then $\mathrm{ord}(k(p), Y) = r$.*

We prove Claim 3. By Lemmas 3.1 and 5.1, $h(R_3(X)) = R_3(Y)$ and $h(\mathcal{E}_3(X)) = \mathcal{E}_3(Y)$. Thus, for each $A \in F_3(X)$, $v_X(A) = v_Y(h(A))$. In particular, $v_X(\{p\}) = v_Y(\{k(p)\})$.

If $m = \operatorname{ord}(k(p), Y)$, by Lemma 5.3, then $r + \binom{r}{2} + \binom{r}{3} = v_X(\{p\}) = v_Y(\{k(p)\}) = m + \binom{m}{2} + \binom{m}{3}$. Thus $r = m$. Therefore, Claim 3 is proved.

Claim 4. *Let $p \in R(X)$ be such that $\operatorname{ord}(p, X) = r$. Suppose that the number of loops of X (respectively, Y) containing p (respectively, $k(p)$) is m (respectively, m'), the number of end points of X (respectively, Y) adjacent to p (respectively, $k(p)$) is t (respectively, t') and the number of edges of X (respectively, Y) joining p (respectively, $k(p)$) to another ramification point of X (respectively, Y) is s (respectively, s'). Then $m = m'$, $t = t'$ and $s = s'$.*

We prove Claim 4. By Claim 3, $\operatorname{ord}(k(p), Y) = r$. Thus $2m + t + s = r = 2m' + t' + s'$.

Let $I_1, \ldots, I_u$ be the different edges of X that contain p. Then $u = m + t + s$.

Proceeding as in the proof of Claim 2, the number of components $\mathcal{C}$ of $F_3(X) - R_3(X)$ that satisfy $\{p\} \in \operatorname{cl}_{F_3(X)}(\mathcal{C})$ is equal to the number of nonempty subsets of $\{I_1, \ldots, I_u\}$ with at most 3 elements. Hence, this number of components is equal to $u + \binom{u}{2} + \binom{u}{3}$.

Since h is a homeomorphism, this number is equal to the number of components of $h(F_3(X) - R_3(X)) = F_3(Y) - R_3(Y)$ that contain $h(\{p\}) = \{k(p)\}$ in its closure which, similarly, is equal to $u' + \binom{u'}{2} + \binom{u'}{3}$, where u' is the number of edges of Y that contain $k(p)$. Thus $u + \binom{u}{2} + \binom{u}{3} = u' + \binom{u'}{2} + \binom{u'}{3}$. Hence, $u = u'$. Since $u' = m' + t' + s'$, $m + t + s = m' + t' + s'$. Since, we knew that $2m + t + s = 2m' + t' + s'$, we obtain that $m = m'$ and $t + s = t' + s'$.

Using Claims 1 and 2 it follows that $s = s'$. Therefore, $t = t'$.

We are ready to show that X and Y are equivalent graphs, and thus X and Y are homeomorphic continua.

Given two different adjacent ramification points p and x in X, let $\mathcal{A}(p, x) = \{J\colon J$ is an edge of X and J joins p and $x\}$ and let $\mathcal{A}'(p, x) = \{L\colon L$ is an edge of Y and L joins $k(p)$ and $k(x)\}$. By Claim 2, we can choose a bijection $k(p, x)$ from $\mathcal{A}(p, x)$ onto $\mathcal{A}'(p, x)$. Given a ramification point p of X, let $\mathcal{B}(p) = \{J\colon J$ is a loop of X and $p \in J\}$, $\mathcal{B}'(p) = \{L\colon L$ is a loop of Y and $k(p) \in L\}$, $\mathcal{C}(p) = \{J\colon J$ is an edge of X that joins p and an end point of $X\}$ and $\mathcal{C}'(p) = \{L\colon L$ is an edge of Y that joins $k(p)$ and an end point of $Y\}$. By Claim 4, it is possible to choose bijections $k_1(p)\colon \mathcal{B}(p) \to \mathcal{B}'(p)$ and $k_2(p)\colon \mathcal{C}(p) \to \mathcal{C}'(p)$.

Let $\mathcal{S}(X)$ (respectively, $\mathcal{S}(Y)$) be the set of edges of X (respectively, Y). Since varying the points p and x we obtain disjoint sets $\mathcal{A}(p, x)$, $\mathcal{B}(p)$ and $\mathcal{C}(p)$ and the union of all of them is $\mathcal{S}(X)$, we can define a common extension $K : \mathcal{S}(X) \to \mathcal{S}(Y)$ of all the functions of the form $k(p, x)$, $k_1(p)$ and $k_2(p)$, and K is a bijection.

Let $\mathcal{V}(X)$ (respectively, $\mathcal{V}(Y)$) be the set of vertices of X (respectively, Y). Now, we extend the function k (defined on the ramification points of X) to $\mathcal{V}(X)$. Given an end point x of X, there exists an edge J of X that joins x and a ramification point p of X. Then $K(J)$ contains exactly one end point y of Y. Then define $k(x) = y$. Hence k is a bijection.

Therefore, we have defined a bijection $K : \mathcal{S}(X) \to \mathcal{S}(Y)$ and a bijection $k : \mathcal{V}(X) \to \mathcal{V}(Y)$ such that $p \in J$ if and only if $k(p) \in K(J)$, and for each loop L of X, $K(L)$ is a loop of Y.

This proves that the graphs X and Y are isomorphic as graphs. Therefore, X is homeomorphic to Y. □

Theorem 5.7. *Let X be an arc or a simple closed curve. Let Y be a finite graph and $n \in \{2, 3\}$. Suppose that $F_n(X)$ is homeomorphic to $F_n(Y)$. Then X is homeomorphic to Y.*

Proof. Since X does not contain ramification points, $R(X) = \emptyset$. Let $h : F_n(X) \to F_n(Y)$ be a homeomorphism. Then $h(\mathcal{E}_n(X)) = \mathcal{E}_n(Y)$. By Lemma 5.1, $F_n(X) = \mathcal{E}_n(X)$, by Lemma 3.1, Y does not contain ramification points, so Y is an arc or a simple closed curve. If X is an arc, then $F_2(X)$ is a 2-cell and $F_3(X)$ is a 3-cell [6, Theorem 6]. If X is a simple closed curve, then $F_2(X)$ is homeomorphic to a Möbius strip and $F_3(X)$ is homeomorphic to the 3-dimensional sphere in $\mathbb{R}^4$ (see [7]). From these facts, the theorem is now immediate. □

Corollary 5.8. *Let X and Y finite graphs. Suppose that $F_n(X)$ is homeomorphic to $F_n(Y)$. Then X is homeomorphic to Y.*

Using Corollary 3.5, we obtain the following result.

Corollary 5.9. *Let X be a finite graph and Z a continuum. Suppose that $F_n(X)$ is homeomorphic to $F_n(Z)$. Then X is homeomorphic to Z.*

Question 5.10. Do there exist a finite graph X, a continuum Z and numbers $n, m \in \mathbb{N}$ such that $F_n(X)$ is homeomorphic to $F_m(Z)$ and $m > 1$ but X is not homeomorphic to Z? By Corollary 5.9 if such continua X and Z do exist, then $n \neq m$, moreover Z is a Peano continuum and every nonempty open subset of Z has dimension greater than 1. Indeed, since Z contains arcs, $F_m(Z)$ contains m-cells. By the proof of Lemma 3.1 of [8] and Theorem 3.4, each nonempty open subset of $F_n(X)$ is n-dimensional. Thus $m < n$. If there exists an open subset of Z of dimension 1, then there exists (again, by the proof of Lemma 3.1 of [8]) a nonempty open subset $\mathcal{U}$ of $F_m(Z)$ such that $\dim[\mathcal{U}] \leqslant m$, which is a contradiction. Therefore, every nonempty open subset of Z has dimension greater than 1.

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