A linear-programming approach to the generalized Randić index

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Abstract

The generalized Randić index $R_\alpha(G)$ of a graph $G$ is the sum of $(d_G(u)d_G(v))^\alpha$ over all edges $uv$ of $G$. Using a linear-programming approach, we establish results on graphs with a given number of vertices and edges and a bounded maximum degree that are of minimum generalized Randić index for $\alpha \in \{-1/2, -1\}$.

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1. Introduction

We will consider finite, simple and undirected graphs and use standard graph-theoretical terminology. For some real $\alpha$ and a graph $G=(V,E)$ the \textit{generalized Randić index} $R_\alpha(G)$ of $G$ is defined as the sum of $(d_G(u)d_G(v))^\alpha$ over all edges $uv$ of $G$ where $d_G(u)$ denotes the degree of $u \in V$, i.e.

$$R_\alpha(G) = \sum_{e=uv \in E(G)} (d_G(u)d_G(v))^\alpha.$$ 

Historically, $R_{-1/2}$ was introduced by Randić [9] in 1975 as one of the many graph-theoretical parameters derived from the graph underlying some molecule (cf. [6]).

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Recently, $R_x$ received considerable attention in the mathematical literature cf. e.g. [1–3,5,7,10]. In [8] Gutman et al., proved several results on the restricted class of so-called chemical graphs, i.e. graphs $G = (V,E)$ of maximum degree $\Delta(G) = \max_{u \in V} d_G(u)$ at most four, with extremal values of $R_{-1/2}$. Here, we take up their linear-programming approach to establish results on graphs with a given number of vertices and edges and a bounded maximum degree that are of minimum generalized Randić index for $x \in \{-\frac{1}{2}, -1\}$.

2. Linear-programming approach

The main result of this section is the following representation of $R_x$.

Since isolated vertices are irrelevant, we will tacitly assume from now on that all graphs have no isolated vertices.

**Theorem 2.1.** Let $G$ be a graph with $n$ vertices, $m$ edges and maximum degree at most $\Delta \geq 2$ that has exactly $e_{i,j}$ edges joining a vertex of degree $i$ and a vertex of degree $j$ for $1 \leq i, j \leq \Delta$. Then

$$R_x(G) = \sum_{i=1}^{\Delta} \sum_{j=i}^{\Delta} (ij)^x e_{i,j}$$

(1)

and

$$\Delta^{-2x}(\Delta - 1)R_x(G) = m(\Delta + 1 - 2\Delta^{-x}) + n(\Delta^{1-x} - \Delta) + \sum_{i=1}^{\Delta-1} \sum_{j=i}^{\Delta} a_{i,j} e_{i,j}$$

(2)

with

$$a_{i,j} = \begin{cases} \frac{\Delta}{i} (1 - \Delta^{-x}) - \Delta + \Delta^{-x} + (\Delta - 1) \left( \frac{i}{\Delta} \right)^x & \text{if } 1 \leq i \leq \Delta - 1 \text{ and } j = \Delta, \\ \Delta(1 - \Delta^{-x}) \left( \frac{1}{i} + \frac{1}{j} \right) - \Delta - 1 + 2\Delta^{-x} + (\Delta - 1) \left( \frac{ij}{\Delta^2} \right)^x & \text{if } 1 \leq i \leq j \leq \Delta - 1. \end{cases}$$

**Proof.** Eq. (1) is an immediate consequence of the definition and we turn to the proof of (2). If $n_i$ is the number of vertices of degree $i$ for $1 \leq i \leq \Delta$, then

$$\sum_{i=1}^{\Delta} n_i = n$$

(3)

and

$$e_{i,i} + \sum_{j=1}^{A} e_{i,j} = in_i \quad \text{for } 1 \leq i \leq \Delta.$$  

(4)
Furthermore, we have
\[
\sum_{i=1}^{A} \sum_{j=i}^{A} e_{i,j} = m,
\]
which implies that
\[
e_{A,A} = m - \sum_{i=1}^{A-1} \sum_{j=i}^{A-1} e_{i,j} - \sum_{i=1}^{A-1} e_{i,A}.
\]
(5)

Using (4) and (5) we get from (3)
\[
\Delta n = \Delta \sum_{i=1}^{A} n_i = \Delta \sum_{i=1}^{A-1} n_i + \Delta n_A
\]
\[
= \Delta \sum_{i=1}^{A-1} n_i + e_{A,A} + \sum_{i=1}^{A} e_{i,A}
\]
\[
= \Delta \sum_{i=1}^{A-1} n_i + 2e_{A,A} + \sum_{i=1}^{A} e_{i,A}
\]
\[
= \Delta \sum_{i=1}^{A-1} n_i + 2 \left( m - \sum_{i=1}^{A-1} \sum_{j=i}^{A-1} e_{i,j} - \sum_{i=1}^{A-1} e_{i,A} \right) + \sum_{i=1}^{A-1} e_{i,A}
\]
\[
= \Delta \sum_{i=1}^{A-1} \left( \frac{e_{i,j}}{i} + \sum_{j=1}^{A} \frac{e_{i,j}}{i} \right) + 2m - 2 \sum_{i=1}^{A-1} \sum_{j=i}^{A-1} e_{i,j} - \sum_{i=1}^{A-1} e_{i,A}
\]
\[
= 2m + \sum_{i=1}^{A-1} \left( \frac{A}{i} - 1 \right) e_{i,A} + \sum_{i=1}^{A-1} \sum_{j=i}^{A-1} \left( A \left( \frac{1}{i} + \frac{1}{j} \right) - 2 \right) e_{i,j}.
\]
Solving this for \((A - 1)e_{1,A}\) yields
\[
(A - 1)e_{1,A} = \Delta n - 2m - \sum_{i=2}^{A-1} \left( \frac{A}{i} - 1 \right) e_{i,A} - \sum_{i=1}^{A-1} \sum_{j=1}^{A-1} \left( A \left( \frac{1}{i} + \frac{1}{j} \right) - 2 \right) e_{i,j}.
\]
We can now eliminate \(e_{1,A}\) in (5) and obtain
\[
(A - 1)e_{A,A} = (A + 1)m - \Delta n + \sum_{i=2}^{A-1} \left( \frac{A}{i} - A \right) e_{i,A}
\]
\[
+ \sum_{i=1}^{A-1} \sum_{j=i}^{A-1} \left( A \left( \frac{1}{i} + \frac{1}{j} \right) - A - 1 \right) e_{i,j}.
\]
The last two relations allow to eliminate $e_{1,A}$ and $e_{A,A}$ from (1) and we obtain

$$(A - 1)R_2(G)$$

$$=(A - 1)A^x e_{1,A} + (A - 1)A^{2x} e_{A,A} + (A - 1) \sum_{i=2}^{A-1} (iA)^x e_{i,A}$$

$$+(A - 1) \sum_{i=1}^{A-1} \sum_{j=i}^{A-1} (ij)^x e_{i,j}$$

$$=m(A^{2x+1} + A^{2x} - 2A^x) - n(A^{2x+1} - A^{x+1})$$

$$+ \sum_{i=2}^{A-1} \left( \frac{A^{2x+1} - A^{x+1}}{i} - A^{2x+1} + A^x + A^2(A - 1)^x \right) e_{i,A}$$

$$+ \sum_{i=1}^{A-1} \sum_{j=i}^{A-1} \left( A^{2x+1} - A^{x+1} \right) \left( \frac{1}{i} + \frac{1}{j} \right) - A^{2x} + 2A^x + (A - 1)(ij)^x e_{i,j}.$$ 

Since $((A^{2x+1} - A^{x+1})/1 - A^{2x+1} + A^x + A^2(A - 1)1^x) = 0$ we obtain (2) by multiplying both sides with $A^{-2x}$. □

3. Graphs with minimum $R_{-1/2}(G)$

If $x = -\frac{1}{2}$, then (2) simplifies to

$$A(A - 1)R_{-1/2}(G) = m(A + 1 - 2\sqrt{A}) + nA(\sqrt{A} - 1) + \sum_{i=1}^{A-1} \sum_{j=i}^{A} a_{i,j} e_{i,j} \quad (6)$$

with

$$a_{i,j} = \begin{cases} 
\frac{1}{i} A(1 - \sqrt{A}) - (A - \sqrt{A}) + \sqrt{A} \sqrt{i} (A - 1) & \text{if } 1 \leq i \leq A - 1 \text{ and } j = A, \\
A(1 - \sqrt{A}) \left( \frac{1}{i} + \frac{1}{j} \right) + A(A - 1) \frac{1}{\sqrt{ij}} - (A + 1 - 2\sqrt{A}) & \text{if } 1 \leq i \leq j \leq A - 1. 
\end{cases}$$

The next lemmata provide useful information about the coefficients.
Lemma 3.1. If $\Delta \geq 2$, $1 \leq i \leq \Delta - 1$ and $x = -\frac{1}{2}$, then $a_{i,j} > a_{i,\Delta} + a_{j,\Delta}$.

Proof. We have

$$a_{i,j} > a_{i,\Delta} + a_{j,\Delta} \iff \Delta(A - 1) \frac{\sqrt{ij}}{ij} + (A - 1) > \sqrt{\Delta}(A - 1) \left( \frac{\sqrt{i}}{i} + \frac{\sqrt{j}}{j} \right)$$

$$\iff \Delta \sqrt{ij} + ij > \Delta(j \sqrt{i} + i \sqrt{j})$$

$$\iff \Delta > \sqrt{\Delta}(\sqrt{i} + \sqrt{j}) - \sqrt{ij}$$

$$\iff \left( \sqrt{\Delta} - \frac{\sqrt{i} + \sqrt{j}}{2} \right)^2 > \left( \frac{\sqrt{i} + \sqrt{j}}{2} \right)^2 - \sqrt{ij}$$

$$\iff \left( \sqrt{\Delta} - \frac{\sqrt{i} + \sqrt{j}}{2} \right)^2 > \left( \frac{\sqrt{j} - \sqrt{i}}{2} \right)^2$$

$$\iff \sqrt{\Delta} > \sqrt{j}.$$ 

The result follows, since the last statement is again true by assumption.

Lemma 3.2. If $\Delta \geq 2$, $1 \leq i \leq \Delta - 1$, $(i,j) \not\in \{(1,\Delta), (\Delta,\Delta)\}$ and $x = -\frac{1}{2}$, then $a_{i,j} > 0$.

(Note that $a_{1,\Delta} = 0$.)

Proof. Firstly, let $2 \leq i \leq \Delta - 1$ and $j = \Delta$. We have

$$a_{i,\Delta} > 0 \iff \sqrt{\Delta} + i - \sqrt{i}(\sqrt{\Delta} + 1) < 0$$

$$\iff \left( \sqrt{i} - \frac{\sqrt{\Delta} + 1}{2} \right)^2 < \left( \frac{\sqrt{\Delta} + 1}{4} \right) - \sqrt{\Delta}$$

$$\iff 1 < i < \Delta.$$ 

(The last equivalence is most easily seen by considering the given expressions for $i = 1$ and $i = \Delta$. For these values both sides of are equal.) Since the last statement is true by assumption, we have $a_{i,\Delta} > 0$.

Now let $1 \leq i \leq \Delta - 1$. By Lemma 3.1, we have $a_{i,j} > a_{i,\Delta} + a_{j,\Delta} \geq 0$ which completes the proof.

We are now in a position to derive the desired results about graphs of minimal $R_{-1/2}$. As an immediate consequence of Lemma 3.2 and (6) we get the following.

Theorem 3.3. For $n$, $m$ and $\Delta \geq 2$ let $G$ and $H$ be two graphs with $n$ vertices, $m$ edges and maximum degree at most $\Delta$ such that $\{d_G(u),d_G(v)\} \in \{\{1,\Delta\},\{\Delta,\Delta\}\}$
holds for all edges $uv$ of $G$ and $\{d_H(u), d_H(v)\} \in \{\{1, A\}, \{A, A\}\}$ does not hold for all edges $uv$ of $H$. Then

$$\frac{\Delta n + \left(\sqrt{A} - 1\right) m}{\Delta \left(\sqrt{A} + 1\right)} = R_{-1/2}(G) < R_{-1/2}(H).$$

It is a natural question under which assumptions on $n, m$ and $\Delta$ a graph $G$ as in the statement of Theorem 3.3 exists. The next result answers this question completely.

**Theorem 3.4.** Let $n, m$ and $\Delta \geq 2$ be positive integers. There is a graph $G = (V, E)$ with $n$ vertices and $m$ edges such that $\{d_G(u), d_G(v)\} \in \{\{1, A\}, \{A, A\}\}$ for all $uv$ of $G$, if and only if

$$n/A = (2m - n)/(A - 1)$$

is an integer and

$$\Delta n_A \geq m \geq \begin{cases} n_A \left(A - \frac{n_A - 1}{2}\right) & \text{if } n_A \leq A, \\ \left\lfloor \frac{\Delta n_A}{2}\right\rfloor & \text{if } n_A > A. \end{cases}$$

(7)

**Proof.** It is easy to see that a graph as in the statement of the theorem has exactly $n_A = (2m - n)/(A - 1)$ vertices of degree $A$.

Let $G$ be a graph as in the statement of the theorem. Let $n_1 \geq 0$ denote the number of vertices of degree 1 in $G$. We have $2m = \Delta n_A + n_1$ and $n_1 \leq m \leq \Delta n_A$. If $n_A \leq A$, then each vertex of degree $A$ has at least $A - (n_A - 1)$ neighbours of degree 1 and $n_1 \geq n_A(A - (n_A - 1))$. Hence

$$2\Delta n_A \geq 2m \geq \begin{cases} n_A(2A - (n_A - 1)) & \text{if } n_A \leq A, \\ \Delta n_A & \text{if } n_A > A. \end{cases}$$

which implies the desired bounds on $m$.

Now let $n, m$ and $A \geq 2$ be such that $n_A$ is an integer and (7) holds. We have to show the existence of a graph $G = (V, E)$ as in the statement of the theorem.

We use the existence of a special graph $H$. If $n_A \leq A$, then let $H = K_{n_1}$. If $n_A > A$, then we proceed as follows. It is a well-known fact (cf., e.g. [4]) that for $l \geq 1$ the complete graph of even order $K_{2l}$ is the disjoint union of perfect matchings and that the complete graph of odd order $K_{2l+1}$ is the disjoint union of hamiltonian cycles.

Let $H$ be a $\Delta$-regular graph of order $n_A$ if $\Delta n_A$ is even (consider the disjoint union of some perfect matchings or some hamiltonian cycles) and a graph with $n_A - 1$ vertices of degree $\Delta$ and one vertex of degree $\Delta - 1$ if $\Delta n_A$ is odd (consider the disjoint union of some hamiltonian cycles and an ‘almost’ perfect matching arising from one hamiltonian cycle).
Let $G_0$ arise from $H$ by adding vertices of degree 1 that are adjacent to vertices of $H$ such that $d_{G_0}(u) = \Delta$ for all $u \in V(H)$. Then

$$m(G_0) = \begin{cases} n_\Delta \left( A - \frac{n_\Delta - 1}{2} \right) & \text{if } n_\Delta \leq A, \\ \left\lceil \frac{A n_\Delta}{2} \right\rceil & \text{if } n_\Delta > A. \end{cases}$$

For $i = 1, 2, \ldots, |E(H)|$ let the graph $G_i$ arise from $G_{i-1}$ by deleting some edge $e = xy \in E(G_{i-1}) \cap E(H)$, adding two new vertices $x'$ and $y'$ to $V(G_{i-1})$ and the two new edges $xx'$ and $yy'$ to $E(G_{i-1})$. We have $d_{G_i}(u) \in \{1, \Delta\}$ for all $u \in V(G_i)$ and $|E(G_i)| = |E(G_{i-1})| + 1$. Note that graph $G_{|E(H)|}$ has $\frac{n_\Delta}{2}$ edges. The constructed graphs imply the desired result. 

The next theorem provides information about the structure of graphs with minimum $R_{-1/2}$ and a given degree sequence.

**Theorem 3.5.** Let $G$ and $H$ be two graphs with $n_i$ vertices of degree $i$ for $1 \leq i \leq \Delta$ and some $\Delta \geq 2$. If the set $\{u \in V(G) | d_G(u) < \Delta\}$ is independent and $\{u \in V(H) | d_H(u) < \Delta\}$ is not independent, then $R_{-1/2}(G) < R_{-1/2}(H)$.

**Proof.** Let $e_{i,j}$ denote the number of edges in $G$ that join a vertex of degree $i$ and a vertex of degree $j$ for $1 \leq i, j \leq \Delta$. Using Lemma 3.1, we deduce

$$\sum_{i=1}^{A-1} \sum_{j=i}^{A} a_{i,j} e_{i,j} = \sum_{i=1}^{A-1} a_{i,i} e_{i,i} + \sum_{i=1}^{A-1} a_{i,i} e_{i,i} + \sum_{i=1}^{A-1} a_{i,j} e_{i,j}$$

$$> \sum_{i=1}^{A-1} 2 a_{i,i} e_{i,i} + \sum_{i=1}^{A-1} a_{i,i} e_{i,i} + \sum_{i=1}^{A-1} a_{i,i} e_{i,i}$$

$$= \sum_{i=1}^{A-1} a_{i,i} \left( e_{i,i} + \sum_{j=1}^{\Delta} e_{i,j} \right)$$

$$= \sum_{i=1}^{A-1} a_{i,i} (i n_i).$$

Now (6) implies the desired result. 

The next result deals again with the existence problem.

**Theorem 3.6.** Let $G$ be a graph with $n_i$ vertices of degree $i$ for $1 \leq i \leq \Delta$ and some $\Delta \geq 2$. If $n_\Delta > 2 \sum_{i=1}^{\Delta-1} i n_i$, then there is a graph $G_0$ with $n_i$ vertices of degree $i$ for $1 \leq i \leq \Delta$ such that $\{u \in V(G) | d_{G_0}(u) < \Delta\}$ is independent.
Proof. We assume that \( uv \in E(G) \) such that \( d_G(u), d_G(v) \leq \Delta - 1 \). Let \( V_{<A} \) be the set of vertices of degree between 1 and \( \Delta - 1 \) in \( G \) and let \( n_{<A} = |V_{<A}| = n_1 + n_2 + \cdots + n_{\Delta - 1} \).

Since \( G \) has at least \( \Delta n_{\Delta/2} \) edges and there are at most \( \Delta \sum_{i=1}^{\Delta - 1} n_i \) edges in \( G \) that are incident with a vertex in \( V_{<A} \cup N_G(V_{<A}) \), there is some edge \( u'v' \) in \( G \) that is not incident with a vertex in \( V_{<A} \cup N_G(V_{<A}) \). Let \( G' \) arise from \( G \) by deleting the edges \( uv, u'v' \) and adding the edges \( uu', vv' \). Repeating this construction, the desired result follows.

Before we proceed to the next section, where we consider the case \( \alpha = -1 \), we enumerate—without proof and for the convenience of the reader—some relations among the coefficients for \( \alpha = -1 \) that we found during our research without using them in a proof: If \( 1 \leq i \leq j \leq \Delta - 1 \) is such that \( (i, j) \neq (1,1) \), then \( a_{i,j} > a_{i,j+1}, a_{1,j} > a_{j,j} \) and \( i a_{i,\Delta} > (\Delta - 1) a_{\Delta-1,\Delta} \).

4. Graphs with minimum \( R_{-1}(G) \)

If \( \alpha = -1 \), then (2) simplifies to
\[
\Delta^2(\Delta - 1)R_{-1}(G) = m(1 - \Delta) + n\Delta(\Delta - 1) + \sum_{i=1}^{\Delta - 1} \sum_{j=i}^{\Delta} a_{i,j} e_{i,j}
\]
with
\[
a_{i,j} = \begin{cases} 
0 & \text{if } 1 \leq i \leq \Delta - 1 \text{ and } j = \Delta, \\
\frac{1}{ij} (\Delta - 1)(\Delta - i)(\Delta - j) & \text{if } 1 \leq i \leq j \leq \Delta - 1.
\end{cases}
\]

Obviously, \( a_{i,j} > 0 \) for \( 1 \leq i \leq j \leq \Delta - 1 \) and we immediately get the following theorem.

Theorem 4.1. For \( n, m \) and \( \Delta \geq 2 \) let \( G \) and \( H \) be two graphs with \( n \) vertices, \( m \) edges and maximum degree at most \( \Delta \) such that \( \{u \in V(G) | d_G(u) < \Delta \} \) is independent and \( \{u \in V(H) | d_H(u) < \Delta \} \) is not independent, then
\[
\frac{\Delta n - m}{\Delta^2} = R_{-1}(G) < R_{-1}(H).
\]

Of course, the realization results—Theorems 3.4 and 3.6—still apply in this situation.

For the case of trees we can actually be more specific. Similarly as in the proof of Theorem 3.4, it is easy to see that there is a tree \( T \) with \( n \) vertices, \( m = n - 1 \) edges and maximum degree \( \Delta \) such that \( \{d_T(u), d_T(v) \} \in \{\{1, \Delta\}, \{\Delta, \Delta\}\} \), if and only if \( (m - 1)/(\Delta - 1) \) is an integer. If \( (m - 1)/(\Delta - 1) \) is not an integer, then there is a tree which has exactly one vertex \( v \) of degree \( d \not\in \{1, \Delta\} \). It is now easy to see that we can assume that \( v \) is not adjacent to any vertex of degree 1, i.e. the set
\{u \in V(T) \mid d_T(u) < A\} \text{ is independent, if and only if}  \\
A + k(A - 1) \leq m \leq A + k(A - 1) + k  

for some \(0 \leq k \leq A - 1\) or \(m \geq A + (A - 1)^2\).

In the following final lemma, we leave our linear-programming approach from Theorem 2.1 and investigate under which assumptions \textit{local changes} decrease the Randić index \(R_{-1}\).

**Lemma 4.2.** Let \(G\) be a graph with \(n\) vertices, \(m\) edges and maximum degree at most \(A\). For \(v \in V(G)\) let 
\[
w_G(v) = \sum_{u \in N_G(v)} \frac{1}{d_G(u)d_G(v)}.\]

Let \(x\) and \(y\) be two vertices of \(G\) at distance at least three such that \(d_G(x) < A\) and \(N_G(y) = \{z_1, z_2, \ldots, z_t\}\) where 
\[
d_G(z_1) \leq d_G(z_2) \leq \cdots \leq d_G(z_t).\]

If either \(w_G(x) = w_G(y)\) and \(d_G(z_1) < d_G(z_t)\), or \(w_G(x) > w_G(y)\), then either for \(z = z_1\) or for \(z = z_t\) 
\[R_{-1}(G - yz + xz) < R_{-1}(G).\]

**Proof.** Let \(x\) and \(y\) be as in the statement of the lemma, i.e. \(w_G(x) \geq w_G(y)\). Furthermore, let \(s = d_G(x)\), \(t = d_G(y)\) and \(N(x) = \{u_1, u_2, \ldots, u_s\}\). If \(s \geq t - 2\), then let \(z = z_1\), and if \(s < t - 2\), then let \(z = z_t\). Let \(H = G - yz + xz\). We have 
\[
R_{-1}(G) - R_{-1}(H) = w_G(x) + w_G(y) - w_H(x) - w_H(y)  
= \sum_{i=1}^{s} \frac{1}{3d_G(u_i)} + \sum_{j=1}^{t} \frac{1}{td_G(z_j)}  
- \left( \sum_{i=1}^{s} \frac{1}{(s+1)d_G(u_i)} + \frac{1}{(s+1)d_G(z)} \right)  
- \left( \sum_{j=1}^{t} \frac{1}{(t-1)d_G(z_j)} - \frac{1}{(t-1)d_G(z)} \right).\]

Thus, 
\[
R_{-1}(H) < R_{-1}(G)  
\iff \sum_{j=1}^{t} \frac{1}{(t-1)d_G(z_j)} - \sum_{j=1}^{t} \frac{1}{td_G(z_j)} + \frac{1}{(s+1)d_G(z)} - \frac{1}{(t-1)d_G(z)}.\]
\[
< \sum_{i=1}^{s} \frac{1}{sd_G(u_i)} - \sum_{i=1}^{s} \frac{1}{(s+1)d_G(u_i)}
\]
\[
\Leftrightarrow (s+1)st \sum_{j=1}^{t} \frac{1}{d_G(z_j)} - (s+1)s(t-1) \sum_{j=1}^{t} \frac{1}{d_G(z_j)}
\]
\[
+ (st(t-1) - (s+1)st) \frac{1}{d_G(z)}
\]
\[
< (s+1)t(t-1) \sum_{i=1}^{s} \frac{1}{d_G(u_i)} - st(t-1) \sum_{i=1}^{s} \frac{1}{d_G(u_i)}
\]
\[
\Leftrightarrow (s+1)s \sum_{j=1}^{t} \frac{1}{d_G(z_j)} + st(t-2-s) \frac{1}{d_G(z)}
\]
\[
< t(t-1) \sum_{i=1}^{s} \frac{1}{d_G(u_i)}.
\]

Now, we prove this last inequality. Since \(t/d_G(z_1) \geq \sum_{i=1}^{t} 1/d_G(z_i) \geq \frac{t}{d_G(z_1)}\) and, by the choice of \(z\), we have

\[
s(t-2-s) \frac{t}{d_G(z)} \leq s(t-2-s) \sum_{j=1}^{t} \frac{1}{d_G(z_j)}.
\]

This together with \(w_G(x) \geq w_G(y)\) implies

\[
(s+1)s \sum_{j=1}^{t} \frac{1}{d_G(z_j)} + st(t-2-s) \frac{1}{d_G(z)}
\]
\[
\leq (s+1)s \sum_{j=1}^{t} \frac{1}{d_G(z_j)} + s(t-2-s) \sum_{j=1}^{t} \frac{1}{d_G(z_j)}
\]
\[
= (t-1)st \sum_{j=1}^{t} \frac{1}{td_G(z_j)}
\]
\[
\leq (t-1)st \sum_{i=1}^{s} \frac{1}{sd_G(u_i)}
\]

with equality if and only if \(d_G(z_1) = d_G(z_t)\) and \(w_G(x) = w_G(y)\). This completes the proof. \(\square\)

For some graph \(G\) let \(m_{<A}(G) = \sum_{1 \leq i \leq j < A} e_{i,j}\) denote the number of edges that join vertices in \(\{u \in V(G) | d_G(u) < A\}\). Although a graph \(G\) with \(m_{<A}(G) = 0\) has minimum Randić index \(R_{\gamma-1}\) for a fixed number of vertices and edges and a fixed
maximum degree, the following example shows that \(m_{<A}(G) < m_{<A}(H)\) does not imply \(R_{-1}(G) \leq R_{-1}(H)\).

**Example 4.3.** Let \(\Delta \geq 4\) and let \(G\) arise from the complete bipartite graph \(K_{\Delta, \Delta-1}\) with partite sets \(Y = \{y_1, y_2, \ldots, y_{\Delta-1}\}\) and \(Z = \{z_1, z_2, \ldots, z_{\Delta}\}\) by replacing the edge \(y_1z_1\) by the path \(y_1x_1x_2x_3z_1\). We have \(d_G(x_2) < \Delta\), \(x_2\) and \(y_2\) are at distance three in \(G\) and \(w_G(x_2) = \frac{1}{2} > w_G(y_2) = 1/(\Delta - 1)\). By Lemma 4.2, we know that \(R_{-1}(G - y_2z_1 + x_2z_1) < R_{-1}(G)\). On the other hand, \(m_{<A}(G) = 3 < m_{<A}(G - y_2z_1 + x_2z_1) = 3 + \Delta - 1\).

**References**