Dynamic behavior for a class of neutral functional differential equations

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1. Introduction

Recently, the following neutral functional differential equation has been considered in [1]

\[(x(t) - cx(t - r))^\prime = -F(x(t)) + G(x(t - r))\] (1.1)

where \(r > 0\), \(c \in [0, 1]\), \(F, G \in C(R^1)\) and \(F\) is strictly increasing on \(R^1\). Variants of Eq. (1.1), which have been used as models for various phenomena such as some population growth, the spread of epidemics, the dynamics of capital stocks, etc. have recently received considerable attention in the literature. For more details and more references on this subject, the reader is referred to [2–6] and the references cited therein. In paper [1], it is shown that if \(F(x) \geq G(x)\) for all \(x \in R^1\) or \(F(x) \leq G(x)\) for all \(x \in R^1\), then the \(\omega\) limit set of every bounded solution of Eq. (1.1) is an equilibrium. However, to the best of our knowledge, no results have been obtained on dynamic behavior on bounded solutions of Eq. (1.1) with \(c = 1\). Motivated by this, in this paper, we consider the following neutral functional differential equation

\[(x(t) - x(t - r))^\prime = -F(x(t)) + G(x(t - r)),\] (1.2)

where \(r > 0\) and \(F, G \in C(R^1)\). Moreover, it is assumed that \(F\) is nondecreasing on \(R^1\).

It is convenient to introduce the following assumptions.

\((A_+)\) \(G \geq F\), and for any bounded intervals \(I \subseteq R^1\) there exists a positive constant \(L = L(I) \in R^1\) such that

\(F(\alpha) - F(\beta) \leq L(\alpha - \beta)\) for any \(\alpha, \beta \in I\) with \(\alpha \geq \beta\).

\((A_-)\) \(G \leq F\), and for any bounded intervals \(I \subseteq R^1\) there exists a positive constant \(L' = L'(I) \in R^1\) such that

\(F(\alpha) - F(\beta) \geq L'(\alpha - \beta)\) for any \(\alpha, \beta \in I\) with \(\alpha \leq \beta\).

We then show that, using some comparison technique and the invariance of \(\omega\) limit set, assuming that either the condition \((A_+)\) or the condition \((A_-)\) is satisfied, then the \(\omega\) limit set of every bounded solution of Eq. (1.2) with some initial conditions is composed of \(r\)-periodic solutions. Our results are new and complement the previously known results in [1].

The paper is organized as follows. In Section 2, we establish some preliminary results, important in the proofs of our main results. In Section 3, we state and prove our main results.

\(^*\) This work was supported by Scientific Research Fund of Zhejiang Provincial Education Department (20070605).

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2. Preliminary results

In this section, we will establish several important lemmas which are essential tools in proving our main results in Section 3.

Throughout this paper, $R^1$ (or $R^1_+$) denotes the set of all (nonnegative) real numbers. Let us define

$$C = C([-r, 0], R^1), \quad C_+ = C([-r, 0], R^1_+)$$

and set $K = \{ \varphi \in C_+ : \varphi(0) - \varphi(-r) \geq 0 \}. \quad \text{(1.2)}$ One can observe that $K$ and $C_+$ are order cones in $C$. We now define several orderings as follows. $\varphi \leq_K \psi$ iff $\varphi - \psi \in K$, $\varphi <_K \psi$ iff $\varphi \in K \setminus \{ \psi \}$, $\varphi \leq_K \psi$ iff $\varphi \in \text{int} K$, $\varphi \leq_K \psi$ for any $\psi \in A$, $\varphi \ll_K \psi$ for any $\psi \in A$, $\varphi \ll_K \psi$ for any $\psi \in A$, where $\varphi, \psi \in C$ and $A \subseteq C$. Notations such as $\varphi \geq_K \psi$ and $\psi \geq_K \varphi$ can be defined analogously.

Let us define $\hat{\alpha} \in C$, where $\hat{\alpha}(\theta) = \alpha, \theta \in [-r, 0]$. In what follows, we assume that $\varphi \in C$ and use $x_t(\varphi) = x(t, \varphi)$ to denote the solution of Eq. (1.2).

**Remark 2.1.** Let $F$ be nondecreasing on $R^1$. Using a similar argument as that in proof of Lemma 2.2 in [2], we can obtain that the conclusions of Lemma 2.2 in [2] also hold.

**Lemma 2.1.** Let $\varphi \in C$. Then $x_t(\varphi)$ exists and is unique on $R^1$.\n
**Proof.** Let us show initially that $x(t, \varphi)$ exists and is unique on $[0, r]$. In fact, let $a(t) = \varphi(t - r)$ for $t \in [0, r]$, then by Lemma 2.2 in [2] the initial value problem

$$\begin{cases}
y'(t) = -F(y(t) + a(t)) + G(a(t)) \\
y(0) = \varphi(0) - \varphi(-r) \quad \text{(2.1)}
\end{cases}$$

has a unique solution $y(t)$ on $[0, r]$. Since $x(t, \varphi) = \varphi(t - r)$ satisfies (2.1) on $[0, r], x(t, \varphi)$ exists and is unique on $[0, r]$. Consequently, the Lemma follows from the induction.

For $\varphi \in C$, we define $O(\varphi) = \{ x_t(\varphi) : t \geq 0 \}$. If $O(\varphi)$ is bounded, then $\overline{O(\varphi)}$ is compact in $C$, where $\overline{O(\varphi)}$ denotes the closure of $O(\varphi)$, and in this case we define

$$\omega(\varphi) = \bigcap_{t \geq 0} \overline{O(x_t(\varphi))}.$$

One can observe that $\omega(x)$ is nonempty, compact, invariant and connected. \hfill $\square$

We now state the following monotonicity properties with (1.2)

**Lemma 2.2.** Assume that $(A_+)$ holds, $\varphi \in C$ and $\alpha \in R^1$ such that $\varphi \geq_K \hat{\alpha}$. Then $x_t(\varphi) \geq_K \hat{\alpha}$ for all $t \in R^1$.\n
**Proof.** Let $y(t) = x(t, \varphi) - x(t - r, \varphi)$, for all $t \in R^1$. Let us claim $y(t) \geq 0$, for all $t \in [0, r]$. Otherwise, there exists $t_0 \in (0, r)$ such that $y(t_0) < 0$ and $y'(t_0) < 0$. Thus,

$$x(t_0, \varphi) < x(t_0 - r, \varphi).$$

It follows from (1.2) that

$$y'(t_0) = -F(x(t_0, \varphi)) + G(x(t_0 - r, \varphi)) \geq -F(x(t_0, \varphi)) + F(x(t_0 - r, \varphi)) \geq 0,$$

which yields a contradiction. Then, we get for any $t \in [0, r], x_t(\varphi) \geq_K \hat{\alpha}$, so that the lemma follows by an induction argument. \hfill $\square$

Arguing as in the proof of Lemma 2.2, we can get the following result:

**Lemma 2.3.** Assume that $(A_-)$ holds, $\varphi \in C$ and $\alpha \in R^1$ such that $\varphi \leq_K \hat{\alpha}$. Then $x_t(\varphi) \leq_K \hat{\alpha}$ for all $t \in R^1$.\n
**Lemma 2.4.** Let $(A_+)$ hold, $\varphi \in C$ and $\alpha \in R^1$ such that $\varphi \geq_K \hat{\alpha}$. Then one of the following conclusions holds:

(i) there exists a constant $T > 0$ such that $x_t(\varphi) \gg_K \hat{\alpha}$ for $t \geq T$;

(ii) $x_t(\varphi) = x_{t+T}(\varphi)$, for $t \geq 0$.

**Proof.** Let $y(t) = x(t, \varphi) - x(t - r, \varphi)$. We consider the following two cases to finish the proof.

**Case 1.** $y(t_1) > 0$ for some $t_1 \geq 0$. Next we will prove that $y(t) > 0$ for $t \in [t_1, \infty)$. Otherwise, $t_2 = \inf\{ t \geq t_1 : y_1(t) = 0 \} < +\infty$.

Let

$$\eta = \frac{1}{2}(t_2 - t_1), \quad I = \left[ \min_{t \in [t_2 - \eta, t_2]} \{ x(t, \varphi), x(t - r, \varphi) \}, \max_{t \in [t_2 - \eta, t_2]} \{ x(t, \varphi), x(t - r, \varphi) \} \right].$$
By \((A_+)\), there exists a positive constant \(L = L(l) \in \mathbb{R}^1\) such that
\[
F(\alpha) - F(\beta) \leq L(\alpha - \beta) \quad \text{for any } \alpha, \beta \in l \text{ with } \alpha \geq \beta.
\]
Then, for \(t \in [t_2 - \eta, \ t_2)\) with \(x(t, \varphi) - x(t - r, \varphi) > 0\), we have
\[
y'(t) = -F(x(t, \varphi)) + G(x_2(t - r, \varphi)) \\
\geq -F(x(t, \varphi)) + F(x(t - r, \varphi)) \\
\geq -L(x(t, \varphi) - x(t - r, \varphi)) \\
= -Ly(t).
\]
(2.2)
Thus, from (2.2), we obtain
\[
[y(t)e^{lt}]' = [y'(t) + Ly(t)]e^{lt} \geq 0, \quad \text{where } t \in [t_2 - \eta, \ t_2).
\]
Hence,
\[
y(t_2) \geq y(t_2 - \eta)e^{-lt} > 0,
\]
which contradicts the definition of \(t_2\). Therefore,
\[
y(t) > 0 \quad \text{for } t \geq t_1.
\]
From Lemma 2.2, we obtain
\[
x(t, \varphi) > x(t - r, \varphi) \geq \alpha \quad \text{for } t \geq t_1.
\]
It follows that
\[
x_t(\varphi) \gg_k \alpha \quad \text{for } t \geq T = t_1 + r.
\]
Case 2. \(y(t) = 0\) for all \(t \geq 0\). Then,
\[
x(t, \varphi) = x(t - r, \varphi) \quad \text{for } t \geq 0.
\]
Thus
\[
x(t, \varphi) = x(t + r, \varphi) \quad \text{for } t \geq -r.
\]
Hence,
\[
x_t(\varphi) = x_{t+r}(\varphi), \quad \text{for } t \geq 0,
\]
which implies that \(x_t(\varphi)\) is a periodic solution with periodic \(r\).

The proof of the lemma is now complete. \(\Box\)

Arguing as in the proof of Lemma 2.4, we can get the following result:

Lemma 2.5. Let \((A_-)\) hold, \(\varphi \in C\) and \(\alpha \in \mathbb{R}^1\) such that \(\varphi \leq_k \alpha\). Then one of the following conclusions holds:
(i) there exists a constant \(T > 0\) such that \(x_t(\varphi) \ll_k \alpha\) for \(t \geq T\);
(ii) \(x_t(\varphi) = x_{t+r}(\varphi), \quad \text{for } t \geq 0\).

3. Main results and their proofs

Our main results are the following.

Theorem 3.1. Let \((A_+)\) hold and \(\varphi \in C\) satisfies \(\varphi(0) - \varphi(-r) \geq 0\). If \(O(\varphi)\) is bounded, then \(\omega(\varphi)\) is composed of \(r\)-periodic solutions of Eq. (1.2).

Proof. Since \(\varphi(0) - \varphi(-r) \geq 0\), we can choose a constant \(\eta \in \mathbb{R}^1\) such that \(\varphi \leq_k \eta\). Then, it follows from Lemmas 2.2 and 2.4 that \(\eta \leq_k \omega(\varphi)\). This implies that the set \(\{\alpha \in \mathbb{R}^1 : \hat{\alpha} \leq_k \omega(\varphi)\}\) is nonempty. Let \(\alpha^* = \sup\{\alpha \in \mathbb{R}^1 : \hat{\alpha} \leq_k \omega(\varphi)\}\). Since \(\omega(\varphi)\) is compact, we obtain \(\alpha^* \in \mathbb{R}^1\).

We will show that the following claim is true.

Claim. \(\forall \psi \in \omega(\varphi), \text{ there exists } \theta \in [-r, 0] \text{ such that } \psi(\theta) = \alpha^*\).

Otherwise, there exists \(\psi \in \omega(\varphi)\) such that \(\psi \gg_k \hat{\alpha}^*\). Then, there exists \(\alpha^* > \alpha^*\) such that
\[
\psi \gg_k \hat{\alpha}^*.
\]
By the definition of \(\omega(\varphi)\), there exists \(t_3 > 0\) such that
\[
x_{t_3}(\varphi) \geq_k \hat{\alpha}^* \gg_k \hat{\alpha}^*.
\]
Thus, \( \omega(\varphi) \geq_K \Gamma^* \gg_K \Gamma^* \).

This contradicts the definition of \( \omega^* \). Thus, the above Claim is true. By Lemma 2.4 and the invariance of \( \omega(\varphi) \), we get:

\[ \forall \psi \in \omega(\varphi), x_t(\psi) \text{ is } r\text{-periodic for } t \in [0, +\infty). \]

Again from the invariance of \( \omega(\varphi) \), it follows that \( \omega(\varphi) \) is composed of \( r \)-periodic solutions of Eq. (1.2).

The proof of the theorem is now complete. □

**Theorem 3.2.** Let \((A)\) hold and \( \varphi \in C \) satisfies \( \varphi(0) - \varphi(-r) \leq 0 \). If \( O(\varphi) \) is bounded, then \( \omega(\varphi) \) is composed of \( r \)-periodic solutions of Eq. (1.2).

**Proof.** By a similar argument as that in the proof of Theorem 3.1, the conclusion of Theorem 3.2 follows immediately by applying Lemmas 2.3 and 2.5. □

Putting Theorems 3.1 and 3.2 together, we obtain the following results.

**Corollary 3.1.** Let \((A_+)\) and \((A_-)\) hold, and \( \varphi \in C \). If \( O(\varphi) \) is bounded, then \( \omega(\varphi) \) is composed of \( r \)-periodic solutions of Eq. (1.2).

**Proof.** Since \((A_+)\) and \((A_-)\) hold, we have \( F = G \). Then, if \( \varphi(0) - \varphi(-r) \geq 0 \), the conclusion of Corollary 3.1 follows immediately by applying Theorem 3.1; if \( \varphi(0) - \varphi(-r) \leq 0 \), the conclusion of Corollary 3.1 follows immediately by applying Theorem 3.2; □

**Corollary 3.2.** Let \((A_+)\) and \((A_-)\) hold, and \( \varphi \in C \) satisfies \( \varphi(0) - \varphi(-r) = 0 \). Then \( \omega(\varphi) \) is composed of \( r \)-periodic solutions of Eq. (1.2).

**Proof.** From Lemmas 2.4 and 2.5, it follows that \( O(\varphi) \) is bounded. Therefore, by Theorems 3.1 or 3.2, the conclusion of Corollary 3.2 holds. □

**Remark 3.1.** Since Eq. (1.2) is the case of Eq. (1.1) with \( c = 1 \), and \( F \) is nondecreasing on \( R^1 \). Our results are new and complement some corresponding ones already known.

**References**