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On the sampling and recovery of bandlimited functions via scattered translates of the Gaussian

Th. Schlumprecht*, N. Sivakumar

Department of Mathematics, Texas A&M University College Station, TX 77843, USA

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Abstract

Let λ be a positive number, and let $(x_j : j \in \mathbb{Z}) \subset \mathbb{R}$ be a fixed Riesz-basis sequence, namely, (x_j) is strictly increasing, and the set of functions $\{\mathbb{R} \ni t \mapsto e^{ix_j t} : j \in \mathbb{Z}\}$ is a Riesz basis (*i.e.*, unconditional basis) for $L_2[-\pi, \pi]$. Given a function $f \in L_2(\mathbb{R})$ whose Fourier transform is zero almost everywhere outside the interval $[-\pi, \pi]$, there is a unique sequence $(a_j : j \in \mathbb{Z})$ in $\ell_2(\mathbb{Z})$, depending on λ and f, such that the function

$$I_\lambda(f)(x) := \sum_{j\in\mathbb{Z}} a_j e^{-\lambda(x-x_j)^2}, \quad x \in \mathbb{R},$$

is continuous and square integrable on $(-\infty, \infty)$, and satisfies the interpolatory conditions $I_\lambda(f)(x_j) = f(x_j)$, $j \in \mathbb{Z}$. It is shown that $I_\lambda(f)$ converges to f in $L_2(\mathbb{R})$, and also uniformly on $\mathbb{R}$, as $\lambda \to 0^+$. In addition, the fundamental functions for the univariate interpolation process are defined, and some of their basic properties, including their exponential decay for large argument, are established. It is further shown that the associated interpolation operators are bounded on $\ell_p(\mathbb{Z})$ for every $p \in [1, \infty]$.
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1. Introduction

This paper, one in the long tradition of those involving the interpolatory theory of functions, is concerned with interpolation of data via the translates of a Gaussian kernel. The motivation

* Corresponding author.
E-mail addresses: thomas.schlumprecht@math.tamu.edu, schlump@math.tamu.edu (Th. Schlumprecht), sivan@math.tamu.edu (N. Sivakumar).

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for this work is twofold. The first is the theory of *Cardinal Interpolation*, which deals with the interpolation of data prescribed at the integer lattice, by means of the integer shifts of a single function. This subject has a rather long history, and it enjoys interesting connections with other branches of pure and applied mathematics, *e.g.* Toeplitz matrices, Function Theory, Harmonic Analysis, Sampling Theory. When the underlying function (whose shifts form the basis for interpolation) is taken to be the so-called *Cardinal B-Spline*, one deals with *Cardinal Spline Interpolation*, a subject championed by Schoenberg, and taken up in earnest by a host of followers. More recently, it was discovered that there is a remarkable analogy between cardinal spline interpolation and cardinal interpolation by means of the (integer) shifts of a Gaussian, a survey of which may be found in [1]. The current article may also be viewed as a contribution in this vein; it too explores further connections between the interpolatory theory of splines and that of the Gauss kernel, but does so in the context of interpolation at point sets which are more general than the integer lattice. This brings us to the second, and principal, motivating influence for our work, namely the researches of Lyubarskii and Madych [2]. This duo have considered spline interpolation at certain sets of points which are generalizations of the integer lattice, and we were prompted by their work to ask if the analogy between splines and Gaussians, very much in evidence in the context of cardinal interpolation, persists in this 'non-uniform' setting also. Our paper seeks to show that this is indeed the case. The influence of [2] on our work goes further. Besides providing us with the motivating question for our studies, it also offered us an array of basic tools which we have modified and adapted.

We shall supply more particulars – of a technical nature – concerning the present paper later in this introductory section, soon after we finish discussing some requisite general material.

Throughout this paper $L_p(\mathbb{R})$ and $L_p[a,b]$, $1 \le p \le \infty$, will denote the usual Lebesgue spaces over $\mathbb{R}$ and the interval $[a,b]$, respectively. We shall let $C(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$, and $C_0(\mathbb{R})$ will denote the space of $f \in C(\mathbb{R})$ for which $\lim_{x\to\pm\infty} f(x) = 0$. An important tool in our analysis is the *Fourier Transform*, so we assemble some of its basic facts; our sources for this material are [3,4]. If $g \in L_1(\mathbb{R})$, then the Fourier transform of g, $\widehat{g}$, is defined as follows:

$$\widehat{g}(x) := \int_{-\infty}^{\infty} g(t)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t, \quad x \in \mathbb{R}. \tag{1}$$

The Fourier transform of a $g \in L_2(\mathbb{R})$ will be denoted by $\mathcal{F}[g]$. It is known that $\mathcal{F}$ is a linear isomorphism on $L_2(\mathbb{R})$, and that the following hold:

$$\|\mathcal{F}[g]\|^2_{L_2(\mathbb{R})} = 2\pi\|g\|^2_{L_2(\mathbb{R})}, \quad g \in L_2(\mathbb{R}); \qquad \mathcal{F}[g] = \widehat{g}, \quad g \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}). \tag{2}$$

Moreover, if $g \in L_2(\mathbb{R}) \cap C(\mathbb{R})$ and $\mathcal{F}[g] \in L_1(\mathbb{R})$, then the following inversion formula holds:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}[g](x)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}x, \quad t \in \mathbb{R}. \tag{3}$$

The functions we seek to interpolate are the so-called *bandlimited* or *Paley–Wiener functions*. Specifically, we define

$$PW_\pi := \left\{g \in L_2(\mathbb{R}) \,:\, \mathcal{F}[g] = 0 \text{ almost everywhere outside } [-\pi,\pi]\right\}.$$

Let $g \in PW_\pi$. The Fourier inversion formula implies

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}[g](x)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}[g](x)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}x, \tag{4}$$

for almost all $t \in \mathbb{R}$. By (the easy implication of) the Paley–Wiener Theorem and the Riemann–Lebesgue Lemma, the last expression in (4) is an analytic function in $C_0(\mathbb{R})$. So we may assume that (4) holds for all $t \in \mathbb{R}$. Moreover, the Bunyakovskii–Cauchy–Schwarz Inequality and (2) combine to show that

$$|g(t)| \leq \frac{1}{\sqrt{2\pi}} \|\mathcal{F}[g]\|_{L_2[-\pi,\pi]} = \|g\|_{L_2(\mathbb{R})}, \quad t \in \mathbb{R}. \tag{5}$$

Having discussed the functions which are to be interpolated, we now describe the point sets at which these functions will be interpolated; it is customary to refer to these points as *interpolation points*, or *sampling points*, or *data sites*. For the most part, though not always, we shall be concerned with data sites which give rise to *Riesz-basis sequences*. Precisely, following [2], we say that a real sequence $(x_j : j \in \mathbb{Z})$ is an Riesz-basis sequence if it satisfies the following conditions: $x_j < x_{j+1}$ for every integer j, and the sequence of functions $(e_j(t) := \mathrm{e}^{-\mathrm{i}x_j t} : j \in \mathbb{Z},\ t \in \mathbb{R})$ is a *Riesz basis* for $L_2[-\pi, \pi]$. We recall that saying that a sequence $(\varphi_j : j \in \mathbb{Z})$ in a Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ means that *every* element $h \in \mathcal{H}$ admits a unique representation of the form

$$h = \sum_{j\in\mathbb{Z}} a_j \varphi_j, \quad \sum_{j\in\mathbb{Z}} |a_j|^2 < \infty, \tag{6}$$

and that there exists a universal constant B such that

$$B^{-1} \left(\sum_{j\in\mathbb{Z}} |c_j|^2 \right)^{1/2} \leq \left\| \sum_{j\in\mathbb{Z}} c_j \varphi_j \right\|_{\mathcal{H}} \leq B \left(\sum_{j\in\mathbb{Z}} |c_j|^2 \right)^{1/2}, \tag{7}$$

for *every* square-summable sequence $(c_j : j \in \mathbb{Z})$. Classical examples of Riesz-basis sequences are given in [2], where it is also pointed out that a Riesz-basis sequence $(x_j : j \in \mathbb{Z})$ is *separated*, *i.e.*,

$$q = \inf_{j\neq k} |x_j - x_k| > 0, \tag{8}$$

and that there exists a positive number Q such that

$$x_{j+1} - x_j \leq Q, \quad j \in \mathbb{Z}. \tag{9}$$

The interpolation process we study here is one that arises from translating a fixed Gaussian. Specifically, let $\lambda > 0$ be fixed, and let $(x_j : j \in \mathbb{Z})$ be a Riesz-basis sequence. We show in the next section that given a function $f \in PW_\pi$, there exists a unique square-summable sequence $(a_j : j \in \mathbb{Z})$ – depending on λ, f, and the sampling points (x_j) – such that the function

$$I_\lambda(f)(x) := \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\lambda(x-x_j)^2}, \quad x \in \mathbb{R}, \tag{10}$$

is continuous and square integrable on $\mathbb{R}$, and satisfies the interpolatory conditions

$$I_\lambda(f)(x_k) = f(x_k), \quad k \in \mathbb{Z}. \tag{11}$$

The function $I_\lambda(f)$ is called the *Gaussian Interpolant* to f at the data sites $(x_k : k \in \mathbb{Z})$. We also observe, again in the upcoming section, that the map $f \mapsto I_\lambda(f)$ is a bounded linear operator from PW_π to $L_2(\mathbb{R})$. As expected, the norm of this operator I_λ – which we refer to as the *Gaussian Interpolation Operator* – is shown to be bounded by a constant depending on λ and the choice of the Riesz-basis sequence. However, this is not sufficient for our subsequent analysis, in which we intend to vary the scaling parameter λ. So in Section 3 we demonstrate that, if the underlying Riesz-basis sequence is fixed, and if $\lambda \le 1$ (the upper bound 1 being purely a matter of convenience), then the operator norm of I_λ can be majorized by a number which is *independent* of λ. Armed with this finding, we proceed to Section 4, wherein we establish the following main convergence result:

Theorem 1.1. *Suppose that* $(x_j : j \in \mathbb{Z})$ *is a (fixed) Riesz-basis sequence, and let* I_λ *be the associated Gaussian Interpolation Operator. Then for any* $f \in PW_\pi$*, we have* $f = \lim_{\lambda\to 0^+} I_\lambda(f)$ *in* $L_2(\mathbb{R})$ *and uniformly on* $\mathbb{R}$.

We note that, in the case when $x_j = j$, this theorem was proved in [5].

As indicated earlier, when $x_j = j \in \mathbb{Z}$, there are a number of results in the literature involving the convergence of Gaussian cardinal interpolants, as the scaling parameter λ tends to zero. The underlying reason for this is a result concerning the behaviour of the Fourier transform of the *fundamental function* associated to Gaussian cardinal interpolation (see[5, Theorem 3.2]). As pointed out in [6], it is the counterpart of this phenomenon for cardinal splines which is at the crux of the corresponding results known for cardinal splines. Now it is also natural to ask what happens to $I_\lambda(f)$ if $\lambda \nearrow \infty$. This less interesting case was discussed in [7] for equally spaced data sites. It follows from Theorem 6.1 of that paper that if $x_j = j$, for $j \in \mathbb{Z}$, then

$$\lim_{\lambda\to\infty} f(x) = \begin{cases} f(j) & \text{if } x = j \in \mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases}$$

Our proofs in Sections 2–5 rely heavily on the machinery and methods developed in [2] for cardinal splines; indeed, as mentioned earlier, our primary task in this paper has been to adapt these to the study of the Gaussian. However, most of these *arguments* do not extend *per se* to the multidimensional situation, a special case of which is presented in Section 5. The paper concludes with Section 6, in which we revisit univariate interpolation, but consider sampling points which satisfy a less restrictive condition than that of giving rise to a Riesz-basis sequence. We introduce here the fundamental functions for interpolation at such data sites, and prove that they decay exponentially for large argument. In addition to being of independent interest, this result also paves the way towards a generalization of some of the main results of Section 2.

2. Notations and basic facts

In this section we shall reintroduce the interpolation problem which concerns us, define the corresponding interpolant and interpolation operator, and establish some of their basic properties. We shall uncover these in a series of propositions.

In what follows we shall use the following notation: given a positive number λ, the Gaussian function with scaling parameter λ is defined by

$$g_\lambda(x) := \mathrm{e}^{-\lambda x^2}, \quad x \in \mathbb{R}.$$

We recall the well-known fact (see, for example, [3, p. 43]) that

$$\mathcal{F}[g_\lambda](u) = \widehat{g_\lambda}(u) = \sqrt{\frac{\pi}{\lambda}}\, \mathrm{e}^{-u^2/(4\lambda)}, \quad u \in \mathbb{R}. \tag{12}$$

The following statement is a simple consequence of [8, Lemma 2.1].

Proposition 2.1. *Suppose that $(x_j : j \in \mathbb{Z})$ is a sequence of real numbers satisfying the following condition: there exists a positive number q such that $x_{j+1} - x_j \geq q$ for every integer j. Let $\lambda > 0$ be fixed, and let $(a_j : j \in \mathbb{Z})$ be a bounded sequence of complex numbers. Then the function $\mathbb{R} \ni x \mapsto \sum_{j\in\mathbb{Z}} a_j g_\lambda(x - x_j)$ is continuous and bounded throughout the real line.*

This next result is an important finding in the theory of radial-basis functions.

Theorem 2.2 (*cf. [9, Theorem 2.3]*). *Let λ and q be fixed positive numbers, and let $\|\cdot\|_2$ denote the Euclidean norm in $\mathbb{R}^d$. There exists a number θ, depending only on d, λ, and q, such that the following holds: if (x_j) is any sequence in $\mathbb{R}^d$ with $\|x_j - x_k\|_2 \geq q$ for $j \neq k$, then $\sum_{j,k} \xi_j \overline{\xi}_k g_\lambda(\|x_j - x_k\|_2) \geq \theta \sum_j |\xi_j|^2$, for every sequence of complex numbers (ξ_j).*

Remark 2.3. Suppose that λ is a fixed positive number, and let $(x_j : j \in \mathbb{Z})$ be a sequence satisfying the conditions of Proposition 2.1. The latter conditions imply that $|x_j - x_k| \geq |j-k|q$ for every pair of integers j and k, so the entries of the bi-infinite matrix $(g_\lambda(x_k - x_j))_{k,j\in\mathbb{Z}}$ decay exponentially away from its main diagonal. So the matrix is realizable as the sum of a uniformly convergent series of diagonal matrices. Hence it acts as a bounded operator on every $\ell_p(\mathbb{Z})$, $1 \leq p \leq \infty$. Moreover, as the matrix is also symmetric, Theorem 2.2 ensures that it is boundedly invertible on $\ell_2(\mathbb{Z})$. In particular, given a square-summable sequence $(d_k : k \in \mathbb{Z})$, there exists a unique square-summable sequence $(a(j, \lambda) : j \in \mathbb{Z})$ such that

$$\sum_{j\in\mathbb{Z}} a(j, \lambda) g_\lambda(x_k - x_j) = d_k, \quad k \in \mathbb{Z}.$$

Remark 2.4. Suppose that $(x_j : j \in \mathbb{Z})$ is a Riesz-basis sequence. Thus, given $h \in L_2[-\pi, \pi]$, there exists a square-summable sequence $(a_j : j \in \mathbb{Z})$ such that $h(t) = \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\mathrm{i}x_j t}$ for almost every $t \in [-\pi, \pi]$, and a constant $B > 0$ such that

$$B^{-2} \sum_{j\in\mathbb{Z}} |a_j|^2 \leq \int_{-\pi}^{\pi} \left| \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\mathrm{i}x_j t} \right|^2 \mathrm{d}t \leq B^2 \sum_{j\in\mathbb{Z}} |a_j|^2. \tag{13}$$

Then the extension $H(u) := \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\mathrm{i}x_j u}$ is locally square integrable on all of $\mathbb{R}$; in particular it is well defined for almost every real number u.

Moreover, the following holds:

$$\int_{(2l-1)\pi}^{(2l+1)\pi} |H|^2 \leq B^2 \sum_{j\in\mathbb{Z}} |a_j|^2, \quad l \in \mathbb{Z}, \tag{14}$$

whence the Bunyakovskii–Cauchy–Schwarz Inequality implies that

$$\int_{(2l-1)\pi}^{(2l+1)\pi} |H| \leq \sqrt{2\pi}\, B \left(\sum_{j\in\mathbb{Z}} |a_j|^2 \right)^{1/2}, \quad l \in \mathbb{Z}. \tag{15}$$

The next result is the first of the two main offerings of the current section.

Theorem 2.5. *Suppose that λ is a fixed positive number, and let $(x_j : j \in \mathbb{Z})$ be a Riesz-basis sequence. Assume that $\overline{a} := (a_j : j \in \mathbb{Z})$ is a square-summable sequence. The following hold:*

(i) *The function*

$$s(\overline{a}, x) = s(x) := \sum_{j\in\mathbb{Z}} a_j g_\lambda(x - x_j), \quad x \in \mathbb{R},$$

belongs to $C(\mathbb{R}) \cap L_2(\mathbb{R})$.

(ii) *The function*

$$\tilde{s}(\overline{a}, u) = \tilde{s}(u) := \mathrm{e}^{-u^2/(4\lambda)} \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\mathrm{i}x_j u}$$

is well defined for almost every real number u, and $\tilde{s} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$.

(iii)

$$\mathcal{F}[s] = \sqrt{\frac{\pi}{\lambda}}\, \tilde{s}.$$

(iv) *The map $\overline{a} := (a_j : j \in \mathbb{Z}) \mapsto s(\overline{a}, x) := \sum_{j\in\mathbb{Z}} a_j g_\lambda(x - x_j)$, $x \in \mathbb{R}$, is a bounded linear transformation from $\ell_2(\mathbb{Z})$ into $L_2(\mathbb{R})$.*

Proof. (i) The continuity of s on $\mathbb{R}$ follows at once from (8) and Proposition 2.1. Define $s_N(x) := \sum_{j=-N}^{N} a_j g_\lambda(x - x_j)$, $x \in \mathbb{R}$, $N \in \mathbb{N}$. As $s_N \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ for every positive integer N, Eq. (2), (12), and a standard Fourier-transform calculation provide the following relations: if $N > M$ are positive integers, then

$$\begin{aligned} 2\pi \|s_N - s_M\|^2_{L_2(\mathbb{R})} &= \|\widehat{s_N} - \widehat{s_M}\|^2_{L_2(\mathbb{R})} \\ &= \frac{\pi}{\lambda} \int_{-\infty}^{\infty} \mathrm{e}^{-u^2/(2\lambda)} \left| \sum_{|j|=M+1}^{N} a_j \mathrm{e}^{-\mathrm{i}x_j u} \right|^2 \mathrm{d}u \\ &= \frac{\pi}{\lambda} \sum_{l\in\mathbb{Z}} \int_{(2l-1)\pi}^{(2l+1)\pi} \mathrm{e}^{-u^2/(2\lambda)} \left| \sum_{|j|=M+1}^{N} a_j \mathrm{e}^{-\mathrm{i}x_j u} \right|^2 \mathrm{d}u. \end{aligned} \tag{16}$$

Let $H_{M,N}(u) := \sum_{|j|=M+1}^{N} a_j \mathrm{e}^{-\mathrm{i}x_j u}$, $u \in \mathbb{R}$. Using the estimates $\mathrm{e}^{-u^2/(2\lambda)} \le 1$ for $u \in [-\pi, \pi]$, and $\mathrm{e}^{-u^2/(2\lambda)} \le \mathrm{e}^{-(2|l|-1)^2\pi^2/(2\lambda)}$ for $(2l-1)\pi \le u \le (2l+1)\pi$, $l \in \mathbb{Z} \setminus \{0\}$, we find from (16) that

$$\begin{aligned} 2\pi \|s_N - s_M\|^2_{L_2(\mathbb{R})} \le \frac{\pi}{\lambda} \Bigg[& \left\| H_{M,N} \right\|^2_{L_2[-\pi,\pi]} \\ & + \sum_{l\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{-(2|l|-1)^2\pi^2/(2\lambda)} \left\| H_{M,N} \right\|^2_{L_2[-(2l-1)\pi,(2l+1)\pi]} \Bigg]. \end{aligned} \tag{17}$$

Now let B be the constant satisfying (13). The integrals on the right-hand side of (17) may be estimated as follows:

$$2\pi \|s_N - s_M\|^2_{L_2(\mathbb{R})} \le \left(\frac{\pi}{\lambda}\right) B^2 \left(\sum_{|j|=M+1}^{N} |a_j|^2 \right) \left[1 + \sum_{l\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{-(2|l|-1)^2\pi^2/(2\lambda)} \right]$$

$$\leq \left(\frac{B^2\pi}{\lambda}\right)\left(\sum_{|j|=M+1}^{N}|a_j|^2\right)\left[1+\frac{2\mathrm{e}^{-\pi^2/(2\lambda)}}{1-\mathrm{e}^{-\pi^2/(2\lambda)}}\right]$$

$$=: \left(\frac{B^2\pi}{\lambda}\right)\left(\sum_{|j|=M+1}^{N}|a_j|^2\right)\left[1+\kappa(\pi^2/(2\lambda))\right]. \tag{18}$$

As $(a_j : j \in \mathbb{Z})$ is square summable, we find that $(s_N : N \in \mathbb{N})$ is a Cauchy sequence in $L_2(\mathbb{R})$, and hence that $s \in L_2(\mathbb{R})$ as promised.

(ii) Let $H(u) := \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\mathrm{i}x_j u}$. As observed in Remark 2.4, H is defined almost everywhere on $\mathbb{R}$, so the same is true of $\tilde{s}$ as well. Now the argument in (i), combined with (14), shows that

$$\|\tilde{s}\|^2_{L_2(\mathbb{R})} \leq B^2\left(\sum_{j\in\mathbb{Z}}|a_j|^2\right)\left[1+\kappa(\pi^2/(2\lambda))\right], \tag{19}$$

whilst a slight, but obvious, variation on the theme, coupled with (15), demonstrates that

$$\|\tilde{s}\|_{L_1(\mathbb{R})} \leq \sqrt{2\pi}\, B\left(\sum_{j\in\mathbb{Z}}|a_j|^2\right)^{1/2}\left[1+\kappa(\pi^2/(2\lambda))\right], \tag{20}$$

and this completes the proof.

(iii) Let $(s_N : N \in \mathbb{N})$ be the sequence defined in the proof of (i). As each $s_N \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and $\lim_{N\to\infty}\|s_N - s\|_{L_2(\mathbb{R})} = 0$, it suffices to show, thanks to (2), that $\lim_{N\to\infty}\|\widehat{s_N} - \sqrt{(\pi/\lambda)}\ \tilde{s}\|_{L_2(\mathbb{R})} = 0$. Calculations similar to the one carried out in (i) show that

$$\|\widehat{s_N} - \sqrt{(\pi/\lambda)}\ \tilde{s}\|^2_{L_2(\mathbb{R})} = \frac{\pi}{\lambda}\int_{-\infty}^{\infty}\mathrm{e}^{-u^2/(2\lambda)}\left|\sum_{|j|=N+1}^{\infty} a_j\mathrm{e}^{-\mathrm{i}x_j u}\right|^2 \mathrm{d}u$$

$$\leq \frac{B^2\pi}{\lambda}\left(\sum_{|j|=N+1}^{\infty}|a_j|^2\right)^2\left[1+\kappa(\pi^2/(2\lambda))\right],$$

and the last term approaches zero as N tends to infinity, because the sequence $(a_j : j \in \mathbb{Z})$ is square summable.

(iv) The linearity of the map is evident, and that it takes $\ell_2(\mathbb{Z})$ into $L_2(\mathbb{R})$ is the content of part (i). Now (2), part (iii) above, and (19) combine to yield the relations

$$2\pi\,\|s(\overline{a},\cdot)\|^2_{L_2(\mathbb{R})} = \left\|\sqrt{\frac{\pi}{\lambda}}\,\tilde{s}(\overline{a},\cdot)\right\|^2_{L_2(\mathbb{R})} \leq \frac{B^2\pi}{\lambda}\left[1+\kappa(\pi^2/(2\lambda))\right]\|\overline{a}\|^2_{\ell_2(\mathbb{Z})}. \quad \square$$

This next result points to a useful interplay between Riesz-basis sequences and bandlimited functions (see, for example, [10, pp. 29–32]). It serves as a prelude to the second main theorem of this section.

Proposition 2.6. *Suppose that $(x_j : j \in \mathbb{Z})$ is a Riesz-basis sequence, and that $f \in PW_\pi$. Then the (sampled) sequence $(f(x_j) : j \in \mathbb{Z})$ is square summable. Moreover, there is an absolute constant C– depending only on (x_j), but not on f– such that $\sum_{j\in\mathbb{Z}}|f(x_j)|^2 \leq C^2\|f\|^2_{L_2(\mathbb{R})}$.*

Proof. Let

$$\langle h_1, h_2\rangle := \int_{-\pi}^{\pi} h_1\overline{h}_2, \quad h_1, h_2 \in L_2[-\pi, \pi], \tag{21}$$

denote the standard inner product in $L_2[-\pi, \pi]$. Let $e_j(t) := \mathrm{e}^{-\mathrm{i}x_j t}$, $j \in \mathbb{Z}$, $t \in [-\pi, \pi]$, so that (4) implies the identities

$$2\pi f(x_j) = \langle \mathcal{F}[f], e_j\rangle, \quad j \in \mathbb{Z}. \tag{22}$$

Letting $(\tilde{e}_j : j \in \mathbb{Z})$ be the coordinate functionals of $(e_j : j \in \mathbb{Z})$ (which means that $h = \sum_j \langle h, \tilde{e}_j\rangle e_j$ for any $h \in L_2[\pi, \pi]$), it follows that $(\tilde{e}_j : j \in \mathbb{Z})$ is also a Riesz basis whose coordinate functionals are $(e_j : j \in \mathbb{Z})$. Thus,

$$g = \sum_{j\in\mathbb{Z}} \langle g, e_j\rangle \tilde{e}_j, \quad g \in L_2[-\pi, \pi]. \tag{23}$$

So (7) provides a universal constant $\tilde{B}$ such that

$$\sum_{j\in\mathbb{Z}} |c_j|^2 \le \tilde{B}^2 \left\| \sum_{j\in\mathbb{Z}} c_j \tilde{e}_j \right\|^2_{L_2[-\pi,\pi]}$$

for every square-summable sequence $(c_j : j \in \mathbb{Z})$. Hence (22) and (23) imply that

$$4\pi^2 \sum_{j\in\mathbb{Z}} |f(x_j)|^2 \le \tilde{B}^2 \, \|\mathcal{F}[f]\|^2_{L_2[-\pi,\pi]} = 2\pi \tilde{B}^2 \|f\|^2_{L_2(\mathbb{R})},$$

the final equation stemming from (2), and the fact that $\mathcal{F}[f] = 0$ almost everywhere in $\mathbb{R} \setminus [-\pi, \pi]$. □

We now state the second of the two main results in this section. Most of our work has already been accomplished; what remains is to recast the findings in the context of our interpolation problem.

Theorem 2.7. *Let λ be a fixed positive number, and let $(x_j : j \in \mathbb{Z})$ be a Riesz-basis sequence. The following hold:*

(i) *Given $f \in PW_\pi$, there exists a unique square-summable sequence $(a(j, \lambda) : j \in \mathbb{Z})$ such that*

$$\sum_{j\in\mathbb{Z}} a(j, \lambda) g_\lambda(x_k - x_j) = f(x_k), \quad k \in \mathbb{Z}.$$

(ii) *Let f and $(a(j, \lambda) : j \in \mathbb{Z})$ be as in* (i). *The Gaussian interpolant to f at the points $(x_j : j \in \mathbb{Z})$, to wit,*

$$I_\lambda(f)(x) = \sum_{j\in\mathbb{Z}} a(j, \lambda) g_\lambda(x - x_j), \quad x \in \mathbb{R},$$

belongs to $C(\mathbb{R}) \cap L_2(\mathbb{R})$.

(iii) *Let f and $I_\lambda(f)$ be as above. The Fourier transform of $I_\lambda(f)$ is given by*

$$\mathcal{F}[I_\lambda(f)](u) = \sqrt{\frac{\pi}{\lambda}}\, \mathrm{e}^{-u^2/(4\lambda)} \sum_{j\in\mathbb{Z}} a(j, \lambda) \mathrm{e}^{-\mathrm{i}x_j u}$$

for almost every real number u. Moreover, $\mathcal{F}[I_\lambda(f)] \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$.

(iv) *If f and $I_\lambda(f)$ are as above, then*

$$I_\lambda(f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}[I_\lambda(f)](u)\mathrm{e}^{\mathrm{i}xu}\mathrm{d}u, \quad x \in \mathbb{R}.$$

In particular, $I_\lambda(f) \in C_0(\mathbb{R})$.

(v) *The Gaussian interpolation operator I_λ is a bounded linear operator from PW_π to $L_2(\mathbb{R})$. That is, the map $PW_\pi \ni f \mapsto I_\lambda(f)$ is linear, and there exists a positive constant D, depending only on λ and $(x_j : j \in \mathbb{Z})$, such that*

$$\|I_\lambda(f)\|_{L_2(\mathbb{R})} \le D\,\|f\|_{L_2(\mathbb{R})}$$

for every $f \in PW_\pi$.

Proof. Assertion (i) follows from (8), Remark 2.3, and Proposition 2.6. Assertion (ii) is obtained from part (i) of Theorem 2.5, whilst assertion (iii) is a consequence of parts (ii) and (iii) of Theorem 2.5. Assertion (iv) follows from the fact that $I_\lambda(f)$ is continuous throughout $\mathbb{R}$, that $\mathcal{F}[I_\lambda(f)] \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$, and Eq. (3). Moreover, this representation and Eq. (1) show that $2\pi I_\lambda(f)(x) = \mathcal{F}[\widehat{I_\lambda(f)}](-x)$ for every real number x. So the Riemann–Lebesgue Lemma ensures that $I_\lambda \in C_0(\mathbb{R})$. As to (v), let T be the matrix $(g_\lambda(x_k - x_j))_{j,k\in\mathbb{Z}}$, let $f \in PW_\pi$, and let $\overline{d} := (f(x_j) : j \in \mathbb{Z})$. Then $a(j,\lambda)$ is the jth component of the vector $T^{-1}\overline{d}$, and this demonstrates that I_λ is linear. Now part (iv) of Theorem 2.5 asserts that

$$\|I_\lambda(f)\|_{L_2(\mathbb{R})} = O\left(\left\|T^{-1}\overline{d}\right\|_{\ell_2(\mathbb{Z})}\right) = O\left(\|\overline{d}\|_{\ell_2(\mathbb{Z})}\right), \tag{24}$$

where the Big-O constant depends only on λ and the Riesz-basis sequence $(x_j : j \in \mathbb{Z})$. Furthermore, Proposition 2.6 reveals that

$$\left\|\overline{d}\right\|_{\ell_2(\mathbb{Z})} = O\left(\|f\|_{L_2(\mathbb{R})}\right), \tag{25}$$

with the Big-O constant here depending only on $(x_j : j \in \mathbb{Z})$. Combining (24) with (25) finishes the proof. □

3. Uniform boundedness of the interpolation operators

In the final result of the previous section, it was shown that, for a fixed scaling parameter λ, and a fixed Riesz-basis sequence $(x_j : j \in \mathbb{Z})$, the associated interpolation operator I_λ is a continuous linear map from PW_π into $L_2(\mathbb{R})$. As expected, the norm of this operator was shown to be bounded by a number which depends on both the scaling parameter and the choice of the Riesz-basis sequence. The goal in the current section is to demonstrate that, if the scaling parameter is bounded above by a fixed number (taken here to be 1 for convenience), then the norm of I_λ can be bounded by a number which depends only on $(x_j : j \in \mathbb{Z})$. The proofs in this section (as well as in the next) are patterned after [2].

Let $(x_j : j \in \mathbb{Z})$ be a Riesz-basis sequence, and let B be the associated constant satisfying the inequalities in (13). Given $h \in L_2[-\pi,\pi]$, there is a square-summable sequence $(a_j : j \in \mathbb{Z})$ such that $h(t) = \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\mathrm{i}x_j t}$ for almost every $t \in [-\pi,\pi]$. Let H denote the extension of h to almost all of $\mathbb{R}$, as considered in Remark 2.4. Given an integer l, we define the following linear map A_l on $L_2[-\pi,\pi]$:

$$A_l(h)(t) := H(t + 2\pi l) = \sum_{j\in\mathbb{Z}} a_j \mathrm{e}^{-\mathrm{i}x_j(t+2\pi l)} \tag{26}$$

for almost every $t \in [-\pi, \pi]$. We see from (14) and (13) that

$$\|A_l h\|^2_{L_2[-\pi,\pi]} = \|H\|^2_{L_2[(2l-1)\pi,(2l+1)\pi]} \le B^2 \sum_{j\in\mathbb{Z}} |a_j|^2 \le B^4 \|h\|^2_{L_2[-\pi,\pi]} . \tag{27}$$

Thus every A_l is a bounded operator from $L_2[-\pi, \pi]$ into itself; moreover, the associated operator norms of these operators are uniformly bounded:

$$\|A_l\| \le B^2. \tag{28}$$

In what follows, we shall assume that $(x_j : j \in \mathbb{Z})$ is a (fixed) Riesz-basis sequence, and let $e_j(t) := \mathrm{e}^{-\mathrm{i}x_j t}$, $t \in \mathbb{R}$, $j \in \mathbb{Z}$. We also denote by $\langle\cdot,\cdot\rangle$ the standard inner product in $L_2[-\pi, \pi]$, as defined via (21). Our first main task now is to exploit the presence of the Riesz basis $(e_j : j \in \mathbb{Z})$ in $L_2[-\pi, \pi]$ to find an effective representation for the Fourier transform of the Gaussian interpolant to a given bandlimited function, on the interval $[-\pi, \pi]$. We begin with a pair of preliminary observations:

Lemma 3.1. *Let (x_j) and (e_j) be as above, and let f be a given function in PW_π. If $\phi \in L_2[-\pi, \pi]$ satisfies the conditions*

$$2\pi f(x_k) = \int_{-\pi}^{\pi} \phi(t)\mathrm{e}^{\mathrm{i}x_k t}\mathrm{d}t, \quad k \in \mathbb{Z}, \tag{29}$$

then $\mathcal{F}[f]$ agrees with ϕ in $L_2[-\pi, \pi]$.

Proof. Eqs. (22) and (29) reveal that $\langle\mathcal{F}[f], e_k\rangle = \langle\phi, e_k\rangle$ for every integer k, and the required result follows from (23). □

Lemma 3.2. *Let (x_j) and (e_j) be as above, and let B be the constant satisfying (13). Let $h \in L_2[-\pi, \pi]$, and let $\alpha > 0$. For $l \in \mathbb{Z}$, let ϕ_l denote the function $\phi_l = A_l^*\left(\mathrm{e}^{-\alpha(\cdot+2\pi l)^2} A_l(h)\right)$, where A_l^* denotes the adjoint of A_l. Then*

$$\|\phi_0\|_{L_2[-\pi,\pi]} \le \|h\|_{L_2[-\pi,\pi]} \quad \textit{and} \quad \sum_{l\in\mathbb{Z}\setminus\{0\}} \|\phi_l\|_{L_2[-\pi,\pi]} \le \|h\|_{L_2[-\pi,\pi]} B^4\kappa(\pi^2\alpha), \tag{30}$$

where κ is the familiar function from (18). In particular, the series $\sum_{l\in\mathbb{Z}} \phi_l$ converges in $L_2[-\pi, \pi]$.

Proof. We note that

$$\|\phi_0\|_{L_2[-\pi,\pi]} = \left\|\mathrm{e}^{-\alpha(\cdot)^2} h\right\|_{L_2[-\pi,\pi]} \le \|h\|_{L_2[-\pi,\pi]}, \tag{31}$$

whereas (28) gives rise to the following estimates for every $l \in \mathbb{Z} \setminus \{0\}$:

$$\begin{aligned} \|\phi_l\|_{L_2[-\pi,\pi]} &\le B^2 \left\|\mathrm{e}^{-\alpha(\cdot+2\pi l)^2} A_l(h)\right\|_{L_2[-\pi,\pi]} \\ &\le B^2\mathrm{e}^{-\pi^2\alpha(2|l|-1)^2} \|A_l h\|_{L_2[-\pi,\pi]} \\ &\le B^4\mathrm{e}^{-\pi^2\alpha(2|l|-1)^2} \|h\|_{L_2[-\pi,\pi]}. \end{aligned} \tag{32}$$

This completes the proof. □

We are now ready for the first main result of this section.

Theorem 3.3. *Let $\lambda > 0$ be fixed, and let $f \in PW_\pi$. Let ψ_λ denote the restriction, to the interval $[-\pi, \pi]$ of the function*

$$\Psi_\lambda := \sqrt{\frac{\lambda}{\pi}}\, e^{(\cdot)^2/(4\lambda)} \mathcal{F}[I_\lambda(f)].$$

$$\mathcal{F}[f] = \mathcal{F}[I_\lambda(f)] + \sqrt{\frac{\pi}{\lambda}} \sum_{l \in \mathbb{Z}\setminus\{0\}} A_l^* \left(e^{-(\cdot+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda) \right) \quad \textit{on } [-\pi, \pi]. \tag{33}$$

Proof. Let ϕ denote the function on the right-hand side of (33). In view of Lemma 3.1, it suffices to show that

$$2\pi f(x_k) = \langle \phi, e_k \rangle, \quad k \in \mathbb{Z}. \tag{34}$$

Now Theorem 2.7 implies the relations

$$2\pi f(x_k) = 2\pi I_\lambda(f)(x_k) = \sqrt{\frac{\pi}{\lambda}} \int_{-\infty}^{\infty} e^{-u^2/(4\lambda)}\, \Psi_\lambda(u) e^{i x_k u} du, \quad k \in \mathbb{Z}, \tag{35}$$

whilst

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-u^2/(4\lambda)}\, \Psi_\lambda(u) e^{i x_k u} du &= \sum_{l \in \mathbb{Z}} \int_{(2l-1)\pi}^{(2l+1)\pi} e^{-u^2/(4\lambda)}\, \Psi_\lambda(u) e^{i x_k u} du \\
&= \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{-(t+2\pi l)^2/(4\lambda)}\, \Psi_\lambda(t + 2\pi l) e^{i x_k (t+2\pi l)} dt \\
&= \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{-(t+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda)(t) \overline{A_l(e_k)(t)} dt \\
&= \sum_{l \in \mathbb{Z}} \left\langle e^{-(\cdot+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda), A_l(e_k) \right\rangle \\
&= \sum_{l \in \mathbb{Z}} \left\langle A_l^* \left(e^{-(\cdot+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda) \right), e_k \right\rangle \\
&= \left\langle \sum_{l \in \mathbb{Z}} A_l^* \left(e^{-(\cdot+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda) \right), e_k \right\rangle,
\end{aligned} \tag{36}$$

the final step being justified by Lemma 3.2. Noting that

$$\sum_{l \in \mathbb{Z}} A_l^* \left(e^{-(\cdot+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda) \right) = e^{-(\cdot)^2/(4\lambda)} \psi_\lambda + \sum_{l \in \mathbb{Z}\setminus\{0\}} A_l^* \left(e^{-(\cdot+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda) \right),$$

we find that (36), (35), and part (iii) of Theorem 2.7 yield (34), and with it the proof. □

Combining (33) and (30) leads directly to the following:

Corollary 3.4. *Suppose that κ, λ, f, ψ_λ, and B are as before. Then*

$$\|\mathcal{F}[I_\lambda(f)]\|_{L_2[-\pi,\pi]} \le \|\mathcal{F}[f]\|_{L_2[-\pi,\pi]} + \sqrt{\frac{\pi}{\lambda}}\, B^4 \kappa(\pi^2/(4\lambda))\, \|\psi_\lambda\|_{L_2[-\pi,\pi]}.$$

The preceding corollary shows that, if f is bandlimited, then the energy of the Fourier transform of (its Gaussian interpolant) $I_\lambda(f)$ – on the interval $[-\pi, \pi]$ – is controlled by that of the Fourier transform of f on that interval, plus another term which involves the energy of ψ_λ on the interval. Our next task is to show that this second term can also be bounded effectively via the energy of $\mathcal{F}[f]$ on $[-\pi, \pi]$. Before proceeding with this, however, we pause to consider formally, an operator which has already made its debut, albeit indirectly, in Theorem 3.3. This operator will also play a role later in this section, and a larger one in the next.

Given a positive number α, we define the operator $\mathcal{T}_\alpha$ on $L_2[-\pi, \pi]$ as follows:

$$\mathcal{T}_\alpha(h) := e^{\pi^2\alpha} \sum_{l\in\mathbb{Z}\setminus\{0\}} A_l^*\left(e^{-\alpha(\cdot+2\pi l)^2} A_l(h)\right), \quad h \in L_2[-\pi, \pi]. \tag{37}$$

That this operator is well defined is guaranteed by Lemma 3.2, whilst its linearity is plain. The following properties of $\mathcal{T}_\alpha$ are easy to verify using Lemma 3.2.

Proposition 3.5. *The operator $\mathcal{T}_\alpha$ is self-adjoint, positive, and its norm is no larger than $e^{\pi^2\alpha}B^4\kappa(\pi^2\alpha)$, where κ is the function defined through (18), and B is the familiar constant associated to the given Riesz-basis sequence (x_j).*

We now return to the task of carrying forward the estimate in Corollary 3.4. The first order of business is to attend to $\|\psi_\lambda\|_{L_2[-\pi,\pi]}$:

Proposition 3.6. *The following holds:*

$$\|\psi_\lambda\|_{L_2[-\pi,\pi]} \le \sqrt{\frac{\lambda}{\pi}}\, e^{\pi^2/(4\lambda)}\, \|\mathcal{F}[f]\|_{L_2[-\pi,\pi]}\,.$$

Proof. Eq. (33) asserts that

$$\mathcal{F}[f] = \mathcal{F}[I_\lambda(f)] + e^{-\pi^2/(4\lambda)}\sqrt{\frac{\pi}{\lambda}}\,\mathcal{T}_{1/(4\lambda)}(\psi_\lambda). \tag{38}$$

As

$$\langle \mathcal{F}[I_\lambda(f)], \psi_\lambda\rangle = \sqrt{\frac{\pi}{\lambda}} \int_{-\pi}^{\pi} e^{-u^2/(4\lambda)}\, |\psi_\lambda(u)|^2\, du \ge 0, \tag{39}$$

and $\mathcal{T}_{1/(4\lambda)}$ is a positive operator, we find from (38) that $\langle \mathcal{F}[f], \psi_\lambda\rangle$ is nonnegative, and also that $\langle \mathcal{F}[f], \psi_\lambda\rangle \ge \langle \mathcal{F}[I_\lambda(f)], \psi_\lambda\rangle$. Hence the Bunyakovskii–Cauchy–Schwarz inequality and (39) lead to the relations

$$\begin{aligned}\|\mathcal{F}[f]\|_{L_2[-\pi,\pi]}\, \|\psi_\lambda\|_{L_2[-\pi,\pi]} &\ge \sqrt{\frac{\pi}{\lambda}} \int_{-\pi}^{\pi} e^{-u^2/(4\lambda)}\, |\psi_\lambda(u)|^2\, du \\ &\ge \sqrt{\frac{\pi}{\lambda}} e^{-\pi^2/(4\lambda)}\, \|\psi_\lambda\|^2_{L_2[-\pi,\pi]}\,,\end{aligned}$$

and the required result follows directly. □

The upcoming corollary is obtained via a combination of Corollary 3.4, Proposition 3.6, Eq. (2) and the fact that $e^{\pi^2/4\lambda}\kappa(\pi^2/(4\lambda)) = (1 - e^{-\pi^2/4\lambda})^{-1} \le 2$, whenever $\lambda \le 1$.

Corollary 3.7. *Assume that* $0 < \lambda \le 1$, *and let* (x_j) *and* f *be as above. The following holds:*

$$\|\mathcal{F}[I_\lambda(f)]\|_{L_2[-\pi,\pi]} \le \sqrt{2\pi}\big[1 + 2B^4\big]\|f\|_{L_2(\mathbb{R})}.$$

The preceding result accomplishes the first half of what we set about to do in this section. The second part will be dealt with next; our deliberations will be quite brief, for the proof is now familiar terrain.

Proposition 3.8. *Let* $0 < \lambda \le 1$. *The following holds:*

$$\|\mathcal{F}[I_\lambda(f)]\|_{L_2(\mathbb{R}\setminus[-\pi,\pi])} \le B^2\sqrt{\frac{\pi}{\lambda}}\|\psi_\lambda\|_{L_2[-\pi,\pi]}\sqrt{\kappa(\pi^2/(2\lambda))} \le \sqrt{8\pi}B^2\|f\|_{L_2(\mathbb{R})}.$$

Proof. We begin by noting that

$$\begin{aligned}\|\mathcal{F}[I_\lambda(f)]\|^2_{L_2(\mathbb{R}\setminus[-\pi,\pi])} &= \frac{\pi}{\lambda}\int_{\mathbb{R}\setminus[-\pi,\pi]} e^{-u^2/(2\lambda)}|\Psi_\lambda(u)|^2 du\\ &= \frac{\pi}{\lambda}\sum_{l\in\mathbb{Z}\setminus\{0\}}\int_{(2l-1)\pi}^{(2l+1)\pi} e^{-u^2/(2\lambda)}|\Psi_\lambda(u)|^2 du. \end{aligned} \tag{40}$$

The last term in (40) may be bounded as follows:

$$\begin{aligned}&\frac{\pi}{\lambda}\sum_{l\in\mathbb{Z}\setminus\{0\}}\int_{-\pi}^{\pi} e^{-(t+2\pi l)^2/(2\lambda)}\,|(A_l(\psi_\lambda))\,(t)|^2\,dt\\ &\quad\le \frac{\pi}{\lambda}\sum_{l\in\mathbb{Z}\setminus\{0\}} e^{-(2|l|-1)^2\pi^2/(2\lambda)}\,\|A_l(\psi_\lambda)\|^2_{L_2[-\pi,\pi]}\\ &\quad\le \frac{B^4\pi}{\lambda}\,\|\psi_\lambda\|^2_{L_2[-\pi,\pi]}\,\kappa(\pi^2/(2\lambda)), \end{aligned} \tag{41}$$

the last inequality being consequent upon (28). This proves the first of the two stated inequalities.

The second inequality follows from the first, by way of Proposition 3.6, (2) and the fact that $(1 - e^{-\pi^2/4\lambda})^{-1} \le 2$, whenever $\lambda \le 1$. □

We close this section by summarizing the findings of Corollary 3.7 and Proposition 3.8:

Theorem 3.9. *Suppose that* $(x_j : j \in \mathbb{Z})$ *is a fixed Riesz-basis sequence. Then* $\{I_\lambda : 0 < \lambda \le 1\}$ *is a uniformly-bounded family of linear operators from* PW_π *to* $L_2(\mathbb{R})$.

4. Convergence of I_λ

This section is devoted to the proof of the convergence result stated in the introduction (Theorem 1.1). We begin by laying some requisite groundwork. Let α be a fixed positive number. Recall the linear operator $\mathcal{T}_\alpha$ from (37):

$$\mathcal{T}_\alpha(h) := e^{\pi^2\alpha}\sum_{l\in\mathbb{Z}\setminus\{0\}} A_l^*\left(e^{-\alpha(\cdot+2\pi l)^2}A_l(h)\right), \qquad h \in L_2[-\pi,\pi].$$

We now define the following (multiplier) operator on $L_2[-\pi,\pi]$:

$$\mathcal{M}_\alpha(h) := e^{-\alpha(\pi^2-(\cdot)^2)}h, \quad h \in L_2[-\pi,\pi]. \tag{42}$$

The following properties of $\mathcal{M}_\alpha$ are easy to verify.

Proposition 4.1. *The operator $\mathcal{M}_\alpha$ is a bounded linear operator on $L_2[-\pi, \pi]$, whose norm does not exceed 1. Moreover, it is self-adjoint, strictly positive, and invertible.*

In what follows, we let $(x_j : j \in \mathbb{Z})$ be a fixed Riesz-basis sequence, and let $r : L_2(\mathbb{R}) \to L_2[-\pi, \pi]$ denote the map which sends a function in $L_2(\mathbb{R})$ to its restriction to the interval $[-\pi, \pi]$; note that r is a bounded linear map with unit norm.

Let $f \in PW_\pi$. Recall (from part (iii) of Theorem 2.7) that, if $I_\lambda(f)$ denotes the Gaussian interpolant to f at the points (x_j), then

$$\psi_\lambda(t) = \sqrt{(\lambda/\pi)}\, \mathrm{e}^{t^2/(4\lambda)} \mathcal{F}[I_\lambda(f)](t) \tag{43}$$

for almost every t in $[-\pi, \pi]$ (remembering that ψ_λ is the restriction of Ψ_λ to the interval $[-\pi, \pi]$). With all this in mind, we find from the definitions of $\mathcal{T}_\alpha$ and $\mathcal{M}_\alpha$ that Eq. (33) may be cast in the following form:

$$\left(\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}\right) r(\mathcal{F}[I_\lambda(f)]) = r(\mathcal{F}[f]), \tag{44}$$

where $\mathcal{I}$ denotes the identity on $L_2[-\pi, \pi]$.

Suppose that $g \in L_2[-\pi, \pi]$, and let

$$\tilde{g}(t) = \begin{cases} g(t) & \text{if } t \in [-\pi, \pi]; \\ 0 & \text{if } t \in \mathbb{R} \setminus [-\pi, \pi]. \end{cases}$$

Then there is an $f \in PW_\pi$ such that $\mathcal{F}[f] = \tilde{g}$; in fact, we may take f to be the following:

$$f(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \mathrm{e}^{\mathrm{i}xt} \mathrm{d}t, \quad x \in \mathbb{R}.$$

Let $I_\lambda(f)$ denote the interpolant to f at the points (x_j), and define $L_\lambda(g) := r(\mathcal{F}[I_\lambda(f)])$; in other words, $L_\lambda = r \circ \mathcal{F} \circ I_\lambda \circ \mathcal{F}^{-1}$. As the Fourier transform $\mathcal{F}$ is a linear isomorphism on $L_2(\mathbb{R})$, and the maps I_λ and r are linear and continuous, the map $g \mapsto L_\lambda(g)$ is a continuous linear operator on $L_2[-\pi, \pi]$; moreover, Eq. (44) affirms that

$$\left(\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}\right) L_\lambda(g) = g \quad \text{a.e. on } [-\pi, \pi]. \tag{45}$$

This being true for every $g \in L_2[-\pi, \pi]$, we deduce that the map $\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}$ is surjective on $L_2[-\pi, \pi]$, and that L_λ is a right inverse of $\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}$. We now show that $\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}$ is, in fact, invertible on $L_2[-\pi, \pi]$.

Proposition 4.2. *The map $\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}$ is injective, hence invertible, on $L_2[-\pi, \pi]$. Moreover, there is a constant Δ, depending only on the sequence (x_j), such that*

$$\left\| \left(\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}\right)^{-1} \right\| \le \Delta, \quad 0 < \lambda \le 1.$$

Proof. Suppose that $\left(\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}\right) g = 0$ for some $g \in L_2[-\pi, \pi]$. Then

$$\begin{aligned} 0 &= \left\langle \left(\mathcal{I} + \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}\right)(g), \mathcal{M}_{1/(4\lambda)}(g) \right\rangle \\ &= \left\langle g, \mathcal{M}_{1/(4\lambda)}(g) \right\rangle + \left\langle \mathcal{T}_{1/(4\lambda)} \mathcal{M}_{1/(4\lambda)}(g), \mathcal{M}_{1/(4\lambda)}(g) \right\rangle \\ &\ge \left\langle g, \mathcal{M}_{1/(4\lambda)}(g) \right\rangle \ge 0, \end{aligned}$$

where we have used the positivity of the operators $\mathcal{T}_{1/(4\lambda)}$ and $\mathcal{M}_{1/(4\lambda)}$ to obtain the two inequalities above. It follows that $\mathcal{M}_{1/(4\lambda)}(g) = 0$, and, as $\mathcal{M}_{1/(4\lambda)}$ is strictly positive, g must be zero. Hence $\mathcal{I} + \mathcal{T}_{1/(4\lambda)}\mathcal{M}_{1/(4\lambda)}$ is injective, and therefore invertible. So (45) may now be stated as follows:

$$\left(\mathcal{I} + \mathcal{T}_{1/(4\lambda)}\mathcal{M}_{1/(4\lambda)}\right)^{-1} = L_\lambda.$$

Consequently, the uniform boundedness of $\left\|\left(\mathcal{I} + \mathcal{T}_{1/(4\lambda)}\mathcal{M}_{1/(4\lambda)}\right)^{-1}\right\|$ for $\lambda \in (0, 1]$ obtains from recalling the equation $L_\lambda = r \circ \mathcal{F} \circ I_\lambda \circ \mathcal{F}^{-1}$, along with Theorem 3.9. □

We are now ready for the first of the two main results of this section.

Theorem 4.3. *If $f \in PW_\pi$, then*

$$\lim_{\lambda \to 0^+} \|f - I_\lambda(f)\|_{L_2(\mathbb{R})} = 0.$$

Proof. In view of the first identity in (2), it is sufficient to show that

$$\lim_{\lambda \to 0^+} \|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|_{L_2(\mathbb{R})} = 0. \tag{46}$$

Assume that $0 < \lambda \le 1$, and let $\lambda' := 1/(4\lambda)$. As $\mathcal{F}[f]$ is zero almost everywhere outside $[-\pi, \pi]$, we see that

$$\begin{aligned}\|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|^2_{L_2(\mathbb{R})} &= \|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|^2_{L_2[-\pi,\pi]} \\ &\quad + \|\mathcal{F}[I_\lambda(f)]\|^2_{L_2(\mathbb{R}\setminus[-\pi,\pi])}.\end{aligned} \tag{47}$$

On the interval $[-\pi, \pi]$, we have, via (44), that

$$\begin{aligned}\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)] &= \left[\mathcal{I} - \left(\mathcal{I} + \mathcal{T}_{\lambda'}\mathcal{M}_{\lambda'}\right)^{-1}\right]\mathcal{F}[f] \\ &= \left(\mathcal{I} + \mathcal{T}_{\lambda'}\mathcal{M}_{\lambda'}\right)^{-1}\mathcal{T}_{\lambda'}\mathcal{M}_{\lambda'}(\mathcal{F}[f]),\end{aligned}$$

where the second step is a matter of direct verification. Consequently

$$\begin{aligned}&\|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|_{L_2[-\pi,\pi]} \\ &\quad \le \left\|\left(\mathcal{I} + \mathcal{T}_{\lambda'}\mathcal{M}_{\lambda'}\right)^{-1}\right\| \, \left\|\mathcal{T}_{\lambda'}\right\| \, \left\|\mathcal{M}_{\lambda'}(\mathcal{F}[f])\right\|_{L_2[-\pi,\pi]}.\end{aligned} \tag{48}$$

Now Proposition 4.2 provides a positive constant Δ which bounds the first term on the right-hand side of (48) for every $\lambda \in (0, 1]$. As $\kappa(\pi^2/(4\lambda)) = O(\mathrm{e}^{-\pi^2/(4\lambda)})$ for $0 < \lambda \le 1$, Proposition 3.5 implies that

$$\left\|\mathcal{T}_{\lambda'}\right\| = O\left(\mathrm{e}^{\pi^2/(4\lambda)}\kappa(\pi^2/(4\lambda))\right) = O(1), \quad 0 < \lambda \le 1,$$

for some Big-O constant which is independent of λ. Using this pair of estimates in (48) provides

$$\|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|_{L_2[-\pi,\pi]} = O\left(\left\|\mathcal{M}_{\lambda'}(\mathcal{F}[f])\right\|_{L_2[-\pi,\pi]}\right), \quad 0 < \lambda \le 1. \tag{49}$$

Turning to the second term on the right-hand side of (47), we see from Proposition 3.8, (43), and (42) that

$$\|\mathcal{F}[I_\lambda(f)]\|^2_{L_2(\mathbb{R}\setminus[-\pi,\pi])} = O\left(\frac{\|\psi_\lambda\|^2_{L_2[-\pi,\pi]}\kappa(\pi^2/(2\lambda))}{\lambda}\right)$$

$$= O\left(\left\|e^{(\cdot)^2/(4\lambda)}\mathcal{F}[I_\lambda(f)]\right\|^2_{L_2[-\pi,\pi]}\kappa(\pi^2/(2\lambda))\right)$$
$$= O\left(e^{\pi^2/(2\lambda)}\left\|\mathcal{M}_{\lambda'}(\mathcal{F}[I_\lambda(f)])\right\|^2_{L_2[-\pi,\pi]}\kappa(\pi^2/(2\lambda))\right)$$
$$= O\left(\left\|\mathcal{M}_{\lambda'}(\mathcal{F}[I_\lambda(f)])\right\|^2_{L_2[-\pi,\pi]}\right), \qquad 0 < \lambda \le 1, \tag{50}$$

the final step resulting from the (oft cited) estimate $\kappa(\pi^2/(2\lambda)) = O(e^{-\pi^2/(2\lambda)})$, $0 < \lambda \le 1$. Now

$$\left\|\mathcal{M}_{\lambda'}(\mathcal{F}[I_\lambda(f)])\right\|^2_{L_2[-\pi,\pi]}$$
$$= O\left(\left\|\mathcal{M}_{\lambda'}(\mathcal{F}[I_\lambda(f)] - \mathcal{F}[f])\right\|^2_{L_2[-\pi,\pi]} + \left\|\mathcal{M}_{\lambda'}(\mathcal{F}[f])\right\|^2_{L_2[-\pi,\pi]}\right)$$
$$= O\left(\left\|\mathcal{M}_{\lambda'}(\mathcal{F}[f])\right\|^2_{L_2[-\pi,\pi]}\right), \tag{51}$$

because $\|\mathcal{M}_{\lambda'}\| \le 1$ (Proposition 4.1), and (49) holds. Combining (51) and (49) with (47), we find that

$$\|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|^2_{L_2(\mathbb{R})} = O\left(\left\|\mathcal{M}_{1/(4\lambda)}(\mathcal{F}[f])\right\|^2_{L_2[-\pi,\pi]}\right) = o(1), \quad \lambda \to 0^+, \tag{52}$$

the last assertion being a consequence of the Dominated Convergence Theorem. This establishes (46), and the proof is complete. □

The final theorem of the section deals with uniform convergence.

Theorem 4.4. *If $f \in PW_\pi$, then $\lim_{\lambda\to 0^+} I_\lambda(f)(x) = f(x)$, $x \in \mathbb{R}$, and the convergence is uniform on $\mathbb{R}$. In particular, the operators I_λ, $0 < \lambda \le 1$, are uniformly bounded as operators from PW_π to $C_0(\mathbb{R})$, via the Uniform Boundedness Principle.*

Proof. Assume that $0 < \lambda \le 1$, and let $x \in \mathbb{R}$. From (4), part (iv) of Theorem 2.7, and the fact that $\mathcal{F}[f] = 0$ almost everywhere on $\mathbb{R} \setminus [-\pi, \pi]$, we see that

$$f(x) - I_\lambda(f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\mathcal{F}[f](u) - \mathcal{F}[I_\lambda(f)](u))e^{ixu}du$$
$$- \frac{1}{2\pi}\int_{\mathbb{R}\setminus[-\pi,\pi]}\mathcal{F}[I_\lambda(f)](u)e^{ixu}du. \tag{53}$$

The modulus of the first term on the right-hand side of (53) is no larger than

$$\|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|_{L_1[-\pi,\pi]} = O\left(\|\mathcal{F}[f] - \mathcal{F}[I_\lambda(f)]\|_{L_2[-\pi,\pi]}\right)$$
$$= o(1), \quad \lambda \to 0^+, \tag{54}$$

via (2) and Theorem 4.3. The second term on the right-hand side of (53) is estimated in a familiar way:

$$\left|\int_{\mathbb{R}\setminus[-\pi,\pi]}\mathcal{F}[I_\lambda(f)](u)e^{ixu}du\right|$$
$$\le \int_{\mathbb{R}\setminus[-\pi,\pi]}|\mathcal{F}[I_\lambda(f)](u)|\,du$$

$$= \sqrt{\frac{\pi}{\lambda}} \sum_{l\in\mathbb{Z}\setminus\{0\}} \int_{(2l-1)\pi}^{(2l+1)\pi} e^{-u^2/(4\lambda)} |\Psi_\lambda(u)| du$$
$$\leq \sqrt{\frac{\pi}{\lambda}} \sum_{l\in\mathbb{Z}\setminus\{0\}} e^{-(2|l|-1)^2\pi^2/(4\lambda)} \|A_l(\psi_\lambda)\|_{L_1[-\pi,\pi]}$$
$$= O\left(\frac{1}{\sqrt{\lambda}} \sum_{l\in\mathbb{Z}\setminus\{0\}} e^{-(2|l|-1)^2\pi^2/(4\lambda)} \|A_l(\psi_\lambda)\|_{L_2[-\pi,\pi]}\right)$$
$$= O\left(\frac{\|\psi_\lambda\|_{L_2[-\pi,\pi]}\, \kappa(\pi^2/(4\lambda))}{\sqrt{\lambda}}\right). \tag{55}$$

Borrowing the argument which led up to (50), (51), and the final conclusion of (52), we deduce that $I_\lambda(f)(x)$ converges to $f(x)$ uniformly in $\mathbb{R}$. □

5. A multidimensional extension

We now consider the multidimensional Gaussian interpolation operator. Let $d \in \mathbb{N}$, and let $(x_j : j \in \mathbb{N}) \subset \mathbb{R}^d$. We say that $(x_j : j \in \mathbb{N}) = (x_{(j,1)}, x_{(j,2)}, \ldots, x_{(j,d)} : j \in \mathbb{N}) \subset \mathbb{R}^d$ is a *d-dimensional Riesz-basis sequence* if the sequence $(e^{(j)} : j \in \mathbb{N})$, with

$$e_j : [-\pi,\pi]^d \to \mathbb{C}, \quad e_j(t_1, t_2, \ldots, t_d) := e^{-i\langle x_j, t\rangle} = e^{-i\sum_{l=1}^{d} x_{(j,l)} t_l},$$

is a Riesz basis of $L_2[-\pi,\pi]^d$.

In general there is no ‘natural’ indexing of the elements of $\{x_j\}$ by $\mathbb{Z}$ or $\mathbb{Z}^d$ if $d \geq 2$; so we index generic d-dimensional Riesz-basis sequences by $\mathbb{N}$. Later we shall confine ourselves to *grids* in $\mathbb{R}^d$, *i.e.,* to Riesz bases in $\mathbb{R}^d$ which are products of one-dimensional Riesz-basis sequences. In that case the natural indexing set is $\mathbb{Z}^d$. We note that, as in the one-dimensional case, a Riesz basis-sequence in $\mathbb{R}^d$ also has to be separated [2].

The *d-dimensional Gaussian function with scaling parameter* $\lambda > 0$ is defined by

$$g_\lambda^{(d)}(x_1, x_2, \ldots, x_d) = e^{-\lambda\|x\|^2} = e^{-\lambda\sum_{j=1}^{d} x_j^2}, \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d.$$

The *Fourier transform* on $L_1(\mathbb{R}^d)$ and that in $L_2(\mathbb{R}^d)$ are defined as in the one-dimensional case, and we denote it by $\hat{g}$, if $g \in L_1(\mathbb{R}^d)$, and by $\mathcal{F}^{(d)}[g]$, if $g \in L_2(\mathbb{R}^d)$.

The Paley–Wiener functions on $\mathbb{R}^d$ are given by

$$PW_\pi^{(d)} := \{g \in L_2(\mathbb{R}^d) : \mathcal{F}^{(d)}[g] = 0 \text{ almost everywhere outside } [-\pi,\pi]^d\}.$$

The following multidimensional result can be shown in the same way as the corresponding result in the one-dimensional case (see Theorem 2.7). Its proof is omitted.

Theorem 5.1. *Let $d \in \mathbb{N}$, let λ be a fixed positive number, and let $(x_j : j \in \mathbb{N})$ be a Riesz-basis sequence in $\mathbb{R}^d$. For any $f \in PW_\pi^{(d)}$ there exists a unique square-summable sequence $(a(j,\lambda) : j \in \mathbb{N})$ such that*

$$\sum_{j\in\mathbb{N}} a(j,\lambda) g_\lambda^{(d)}(x_k - x_j) = f(x_k), \quad k \in \mathbb{N}. \tag{56}$$

The Gaussian Interpolation Operator $I_\lambda^{(d)} : PW_\pi^{(d)} \to L_2[-\pi, \pi]^d$, *defined by*

$$I_\lambda^{(d)}(f)(\cdot) = \sum_{j\in\mathbb{N}} a(j,\lambda) g_\lambda^{(d)}(\cdot - x_j),$$

where $(a(j,\lambda) : j \in \mathbb{N})$ *satisfies* (56), *is a well-defined, bounded linear operator from* $PW_\pi^{(d)}$ *to* $L_2[-\pi,\pi]^d$. *Moreover,* $I_\lambda^{(d)}(f) \in C_0(\mathbb{R}^d)$.

One can use tensor products to extend Theorems 4.3 and 4.4 to the multidimensional case, assuming that the underlying Riesz-basis sequence is a grid, *i.e.,* if the $\{x_j : j \in \mathbb{N}\}$ is a Cartesian product of one-dimensional Riesz-basis sequences.

Specifically, the following can be proved. The proof is withheld at the referees' behest.

Theorem 5.2. *Assume that the Riesz-basis sequence* (x_j) *is a grid. Suppose that* $F \in PW_\pi^{(d)}$. *Then* $\lim_{\lambda\to 0^+} \|I_\lambda^{(d)}(F) - F\|_{L_2(\mathbb{R}^d)} = 0$, *and* $\lim_{\lambda\to 0} I_\lambda^{(2)}(F)(z) = F(z)$ *uniformly for* $z \in \mathbb{R}^d$.

The general multidimensional case eludes us:

Problem 5.3. Let $d \in \mathbb{N}$, let $(x_n : n \in \mathbb{N}) \subset \mathbb{R}^d$ be a (general) d-dimensional Riesz-basis sequence, and let $f \in PW_\pi^{(d)}$. Is it true that

$$f = \lim_{\lambda\to 0+} I_\lambda^{(d)}(f), \quad \text{in } L_2(\mathbb{R}^d) \text{ and uniformly on } \mathbb{R}^d?$$

6. Further results on univariate Gaussian interpolation

In this final section we return to univariate interpolation, in order to discuss extensions of some results obtained in Section 2. We begin with a general result concerning bi-infinite matrices which appears to be folkloric. We have seen two articles which cite a well-known treatise for it, but our search of the latter came up emptyhanded. A proof of the said result is indicated in [11], but for the sake of completeness and record, we include a fairly self-contained and expanded rendition of this argument here.

Theorem 6.1. *Suppose that* $(A(j,k))_{j,k\in\mathbb{Z}}$ *is a bi-infinite matrix which, as an operator on* $\ell^2(\mathbb{Z})$, *is self-adjoint, positive and invertible. Assume further that there exist positive constants* τ *and* γ *such that* $|A(j,k)| \le \tau e^{-\gamma|j-k|}$ *for every pair of integers* j *and* k. *Then there exist constants* $\tilde\tau$ *and* $\tilde\gamma$ *such that* $|A^{-1}(s,t)| \le \tilde\tau e^{-\tilde\gamma|s-t|}$ *for every* $s,t \in \mathbb{Z}$.

For the proof of Theorem 6.1 we shall need the following pair of lemmata.

Lemma 6.2. *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a Hilbert space, and let* $A : H \to H$ *be a bounded linear, self-adjoint, positive and invertible operator. Let* $R := I - \frac{A}{\|A\|}$, *where* I *denotes the identity. Then* $R = R^*$, $\langle x, Rx\rangle \ge 0$ *for every* $x \in H$, *and* $\|R\| < 1$.

Proof. The symmetry of R is evident. If $\|x\| = 1$, then

$$\langle x, Rx\rangle = \|x\|^2 - \left\langle x, \frac{A}{\|A\|}x\right\rangle.$$

By the assumptions on A and the Bunyakovskii–Cauchy–Schwarz inequality we see that the term on the right-hand side of the preceding equation is between 0 and 1. Therefore $\|R\| = \sup\{\langle x, Rx\rangle : \|x\| = 1\} \le 1$. If $\|R\| = 1$, then there is a sequence $(x_n : n \in \mathbb{N})$ such that $\|x_n\| = 1$ for every n, and

$$1 = \lim_{n\to\infty} \langle x_n, Rx_n \rangle = \lim_{n\to\infty} \left(1 - \left\langle x_n, \frac{A}{\|A\|} x_n \right\rangle \right).$$

which contradicts the invertibility of A. $\square$

Lemma 6.3. *Suppose that $(R(s,t))_{s,t\in\mathbb{Z}}$ is a bi-infinite matrix satisfying the following condition: there exist positive constants C and γ such that $|R(s,t)| \le C\mathrm{e}^{-\gamma|s-t|}$ for every pair of integers s and t. Given $0 < \gamma' < \gamma$, there is a constant $C(\gamma,\gamma')$, depending on γ and γ', such that $|R^n(s,t)| \le C^n C(\gamma,\gamma')^{n-1} \mathrm{e}^{-\gamma'|s-t|}$ for every $s, t \in \mathbb{Z}$.*

Proof. Suppose firstly that $s \neq t \in \mathbb{Z}$, and assume without loss that $s < t$. Note that

$$\begin{aligned}\sum_{u=-\infty}^{\infty} \mathrm{e}^{-\gamma|s-u|}\mathrm{e}^{-\gamma'|t-u|} &= \sum_{u=s}^{t} \mathrm{e}^{-\gamma(u-s)}\mathrm{e}^{-\gamma'(t-u)} \\ &\quad + \sum_{u=-\infty}^{s-1} \mathrm{e}^{-\gamma(s-u)}\mathrm{e}^{-\gamma'(t-u)} + \sum_{u=t+1}^{\infty} \mathrm{e}^{-\gamma(u-s)}\mathrm{e}^{-\gamma'(u-t)} \\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3. \end{aligned} \tag{57}$$

Now

$$\begin{aligned}\Sigma_1 &= \mathrm{e}^{-\gamma'(t-s)}\mathrm{e}^{(\gamma-\gamma')s} \sum_{u=s}^{t} \mathrm{e}^{-u(\gamma-\gamma')} \\ &= \mathrm{e}^{-\gamma'(t-s)}\mathrm{e}^{(\gamma-\gamma')s} \sum_{v=0}^{t-s} \mathrm{e}^{-s(\gamma-\gamma')-v(\gamma-\gamma')} \\ &= \mathrm{e}^{-\gamma'(t-s)} \sum_{v=0}^{t-s} \mathrm{e}^{-v(\gamma-\gamma')} \le \frac{\mathrm{e}^{-\gamma'(t-s)}}{1-\mathrm{e}^{-(\gamma-\gamma')}}. \end{aligned} \tag{58}$$

Moreover,

$$\Sigma_2 = \sum_{v=1}^{\infty} \mathrm{e}^{-\gamma v}\mathrm{e}^{-\gamma'(t-s+v)} = \mathrm{e}^{-\gamma'(t-s)} \sum_{v=1}^{\infty} \mathrm{e}^{-v(\gamma+\gamma')} \le \frac{\mathrm{e}^{-\gamma'(t-s)}}{1-\mathrm{e}^{-(\gamma+\gamma')}}, \tag{59}$$

whereas

$$\Sigma_3 = \sum_{v=1}^{\infty} \mathrm{e}^{-\gamma' v}\mathrm{e}^{-\gamma(v+t-s)} = \mathrm{e}^{-\gamma(t-s)} \sum_{v=1}^{\infty} \mathrm{e}^{-(\gamma+\gamma')v} \le \frac{\mathrm{e}^{-\gamma'(t-s)}}{1-\mathrm{e}^{-(\gamma+\gamma')}}. \tag{60}$$

If $s = t$, then

$$\sum_{u=-\infty}^{\infty} \mathrm{e}^{-\gamma|s-u|}\mathrm{e}^{-\gamma'|t-u|} = \sum_{u=-\infty}^{\infty} \mathrm{e}^{-(\gamma+\gamma')|s-u|} \le \frac{2}{1-\mathrm{e}^{-(\gamma+\gamma')}}. \tag{61}$$

From (57)–(61) we conclude that

$$|R^2(s,t)| \le C^2 \left[\frac{1}{1-\mathrm{e}^{-(\gamma-\gamma')}} + \frac{2}{1-\mathrm{e}^{-(\gamma+\gamma')}} \right] =: C^2 C(\gamma,\gamma').$$

The general result follows from this via induction. $\square$

Proof of Theorem 6.1. Let $R = I - \frac{A}{\|A\|}$ be the matrix given in that lemma. As

$$R(j,k) = \frac{A(j,k)}{\|A\|} \quad \text{if } j \neq k, \quad \text{and} \quad R(k,k) = \frac{\|A\| - A(k,k)}{\|A\|},$$

there is some constant C such that $|R(s,t)| \leq C\mathrm{e}^{-\gamma|s-t|}$ for every pair of integers s and t. As $A = \|A\|\,(I - R)$, and $r := \|R\| < 1$ (Lemma 6.2), the standard Neumann series expansion yields the relations

$$\begin{aligned} A^{-1} &= \|A\|^{-1} \sum_{n=0}^{\infty} R^n \\ &= \|A\|^{-1} \sum_{n=0}^{N-1} R^n + \|A\|^{-1} R^N \sum_{n=0}^{\infty} R^n = \|A\|^{-1} \sum_{n=0}^{N-1} R^n + R^N A^{-1}, \end{aligned} \tag{62}$$

for any positive integer N. As $R^0(s,t) = I(s,t) = 0$ if $s \neq t$, $s, t \in \mathbb{Z}$, we see from (62) that

$$A^{-1}(s,t) = \|A\|^{-1} \sum_{n=1}^{N-1} R^n(s,t) + [R^N A^{-1}](s,t), \quad s \neq t. \tag{63}$$

Choose and fix a positive number $\gamma' < \gamma$, and recall from Lemma 6.3 that there is a constant $C(\gamma, \gamma')$ such that $|R^n(s,t)| \leq C^n C(\gamma,\gamma')^{n-1} \mathrm{e}^{-\gamma'|s-t|}$ for every positive integer n, and every pair of integers s and t. So we may assume that there is some constant $D := D(\gamma,\gamma') > 1$ such that $|R^n(s,t)| \leq D^n \mathrm{e}^{-\gamma'|s-t|}$ for every positive integer n, and every pair of integers s and t. Using this bound in (63) provides the following estimate for every $s \neq t$:

$$\begin{aligned} |A^{-1}(s,t)| &\leq \|A\|^{-1} \mathrm{e}^{-\gamma'|s-t|} \sum_{n=1}^{N-1} D^n + \|A^{-1}\| r^N \\ &\leq \|A\|^{-1} \mathrm{e}^{-\gamma'|s-t|} \frac{D^N}{D-1} + \|A^{-1}\| r^N. \end{aligned} \tag{64}$$

Let m be a positive integer such that $\mathrm{e}^{-\gamma'} D^{1/m} < 1$, and let $s, t \in \mathbb{Z}$ with $|s-t| \geq m$. Writing $|s-t| = Nm + k$, $0 \leq k \leq m-1$, we find that

$$\mathrm{e}^{-\gamma'|s-t|} D^N = \left[\mathrm{e}^{-\gamma'} D^{\frac{1}{m+(k/N)}}\right]^{|s-t|} \leq [\mathrm{e}^{-\gamma'} D^{1/m}]^{|s-t|}. \tag{65}$$

Further,

$$r^N = \left[r^{\frac{1}{m+(k/N)}}\right]^{|s-t|} \leq [r^{1/2m}]^{|s-t|}, \tag{66}$$

and combining (65) and (66) with (64) leads to the following bounds for every $|s-t| \geq m$ and an appropriately chosen $\tilde{\gamma} > 0$:

$$|A^{-1}(s,t)| \leq \frac{\|A\|^{-1}}{D-1} [\mathrm{e}^{-\gamma'} D^{1/m}]^{|s-t|} + \|A^{-1}\| [r^{1/2m}]^{|s-t|} = O(\mathrm{e}^{-\tilde{\gamma}|s-t|}). \tag{67}$$

On the other hand, if $|s-t| < m$, we obtain

$$|A^{-1}(s,t)| \leq \|A^{-1}\| \leq (\|A^{-1}\| \mathrm{e}^{m\tilde{\gamma}}) \mathrm{e}^{-\tilde{\gamma}|s-t|}, \tag{68}$$

and combining (67) with (68) finishes the proof. □

A direct consequence of the previous theorem is the following:

Corollary 6.4. *Let $\lambda > 0$, and let $(x_j : j \in \mathbb{Z})$ be a sequence of real numbers satisfying (8) for some $q > 0$. Let $A := A_\lambda$ be a bi-infinite matrix whose entries are given by $A(j,k) := \mathrm{e}^{-\lambda(x_j - x_k)^2}$, $j, k \in \mathbb{Z}$. Then there exist positive constants β_1 and γ_1, depending on λ and q, such that $|A^{-1}(s,t)| \le \beta_1 \mathrm{e}^{-\gamma_1 |s-t|}$, $s, t \in \mathbb{Z}$.*

Proof. The hypothesis on the x_j's implies that $|x_j - x_k| \ge |j-k|q$, for $j, k \in \mathbb{Z}$. □

Remark 6.5. The foregoing result implies, in particular, that A^{-1} is a bounded operator on every $\ell_p(\mathbb{Z})$, $1 \le p \le \infty$.

We now turn to the second half of this section, in which we introduce the fundamental functions for Gaussian interpolation (at scattered data sites), and set out some of their basic properties.

Theorem 6.6. *Let $\lambda > 0$ be fixed, and let $(x_j : j \in \mathbb{Z})$ be a sequence satisfying (8) for some $q > 0$. Let $A = A_\lambda$ be the bi-infinite matrix whose entries are given by $A(j,k) = \mathrm{e}^{-\lambda(x_j - x_k)^2}$, $j, k \in \mathbb{Z}$. Given $l \in \mathbb{Z}$, let the l-th* fundamental function *be defined as follows:*

$$L_{l,\lambda}(x) := L_l(x) := \sum_{k \in \mathbb{Z}} A^{-1}(k,l) \mathrm{e}^{-\lambda(x - x_k)^2}, \quad x \in \mathbb{R}.$$

The following hold:

(i) *The function L_l is continuous throughout $\mathbb{R}$.*
(ii) *Each L_l obeys the fundamental interpolatory conditions $L_l(x_m) = \delta_{lm}$, $m \in \mathbb{Z}$.*
(iii) *If in addition, there is a positive number Q such that (9) holds, then there exist positive constants β_2 and ρ, depending on λ, q, and Q such that $|L_l(x)| \le \beta_2 \mathrm{e}^{-\rho|x - x_l|}$ for every $x \in \mathbb{R}$ and every $l \in \mathbb{Z}$.*
(iv) *Assume that $(x_j : j \in \mathbb{Z})$ satisfies the condition stipulated in* (iii). *Let $(b_l : l \in \mathbb{Z})$ be a sequence satisfying the following condition: there exists a positive number K and a positive integer P such that $|b_l| \le K|l|^P$ for every integer l. Then the function $\mathbb{R} \ni x \mapsto \sum_{l \in \mathbb{Z}} b_l L_l(x)$ is continuous on $\mathbb{R}$.*

Proof. (i) As A^{-1} is a bounded operator on $\ell_\infty(\mathbb{Z})$, the sequence $(A^{-1}(k,l) : k \in \mathbb{Z})$ is bounded. Hence the continuity of L_l follows from Proposition 2.1.

(ii) Given $m \in \mathbb{Z}$, we have

$$L_l(x_m) = \sum_{k \in \mathbb{Z}} A^{-1}(k,l) \mathrm{e}^{-\lambda(x_m - x_k)^2} = \sum_{k \in \mathbb{Z}} A^{-1}(k,l) A(m,k) = I(m,l) = \delta_{lm}.$$

(iii) The assumption $x_{j+1} - x_j \le Q$ for every integer j implies that $|x_k - x_l| \le |k-l|Q$ for every pair of integers k and l. Therefore Corollary 6.4 leads to the bound $|A^{-1}(k,l)| \le \beta_1 \mathrm{e}^{-\gamma_2 |x_k - x_l|}$, $k, l \in \mathbb{Z}$, where $\gamma_2 := \gamma_1 / Q$. Consequently,

$$\begin{aligned} |L_l(x)| &\le \beta_1 \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\gamma_2 |x_k - x_l|} \mathrm{e}^{-\lambda(x - x_k)^2} \\ &\le \beta_1 \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\rho |x_k - x_l|} \mathrm{e}^{-\rho(x - x_k)^2}, \quad x \in \mathbb{R}, \end{aligned} \tag{69}$$

where $\rho := \min\{\lambda, \gamma_2\}$. Therefore

$$e^{\rho|x-x_l|}|L_l(x)| \le \beta_1 \sum_{k\in\mathbb{Z}} e^{\rho[|x-x_l|-|x_k-x_l|]} e^{-\rho(x-x_k)^2}$$

$$\le \beta_1 \sum_{k\in\mathbb{Z}} e^{\rho|x-x_k|} e^{-\rho(x-x_k)^2}, \quad x \in \mathbb{R}. \tag{70}$$

Fix $x \in \mathbb{R}$, and let s be the integer such that $x_s \le x < x_{s+1}$. From (70) we obtain

$$e^{\rho|x-x_l|}|L_l(x)| \le \beta_1 \sum_{k=s}^{s+1} e^{\rho|x-x_k|} e^{-\rho(x-x_k)^2} + \beta_1 \sum_{k\in\mathbb{Z}\setminus\{s,s+1\}} e^{\rho|x-x_k|} e^{-\rho(x-x_k)^2}$$

$$\le 2\beta_1 e^{\rho Q} + \beta_1 \sum_{k\in\mathbb{Z}\setminus\{s,s+1\}} e^{\rho|x-x_k|} e^{-\rho(x-x_k)^2}. \tag{71}$$

As $mq \le |x - x_{s-m}| \le (m+1)Q$ for every positive integer m, and $(m-1)q \le |x - x_{s+m}| \le mQ$ for every positive integer $m \ge 2$, the final sum on the right-hand side of (71) is no larger than

$$\beta_1 \sum_{m=1}^{\infty} e^{\rho(m+1)q} e^{-\rho m^2 q^2} + \beta_1 \sum_{m=2}^{\infty} e^{\rho m q} e^{-\rho(m-1)^2 q^2} =: \beta_2.$$

Combining this with (71) and (70) finishes the proof.

(iv) Each summand is continuous by assertion (i). Let $x \in \mathbb{R}$, and let $x_s \le x < x_{s+1}$. Assertion (iii) implies that

$$\left|\sum_{l\in\mathbb{Z}} b_l L_l(x)\right| \le K\beta_2 \left[\sum_{l=s}^{s+1} |l|^P + \sum_{l\in\mathbb{Z}\setminus\{s,s+1\}} |l|^P e^{-\rho|x-x_l|}\right]$$

$$\le K\beta_2 \left[\sum_{l=s}^{s+1} |l|^P + \sum_{m=1}^{\infty} |l|^P e^{-\rho m q} + \sum_{m=2}^{\infty} |l|^P e^{-\rho(m-1)q}\right].$$

It follows that the series $\sum_{l\in\mathbb{Z}} b_l L_l(x)$ is locally uniformly convergent, whence the stated result follows. □

The counterpart of Part (iii) of the foregoing theorem for Gaussian cardinal interpolation was obtained in [7], and its analogue for spline interpolation was proved in [12].

Earlier in this paper we have discussed Gaussian interpolation operators associated to functions, specifically to bandlimited functions. Here our perspective changes slightly, as we begin to think of these interpolation operators acting on sequence spaces.

Theorem 6.7. *Let $\lambda > 0$ be fixed. Suppose that $(x_j : j \in \mathbb{Z})$ is a real sequence satisfying (8) and (9) for some positive numbers q and Q. Let $A = A_\lambda$ be the bi-infinite matrix whose entries are given by $A(j,k) = e^{-\lambda(x_j - x_k)^2}$, $j, k \in \mathbb{Z}$. Given $p \in [1, \infty]$, and $\overline{y} := (y_l : l \in \mathbb{Z}) \in \ell_p(\mathbb{Z})$, define*

$$I_\lambda(\overline{y}, x) := \sum_{k\in\mathbb{Z}} (A^{-1}\overline{y})_k e^{-\lambda(x-x_k)^2}, \quad x \in \mathbb{R},$$

where $(A^{-1}\overline{y})_k$ denotes the kth component of the sequence $A^{-1}\overline{y}$. The following hold:

(i) *The function* $\mathbb{R} \ni x \mapsto I_\lambda(\overline{y}, x)$ *is continuous on* $\mathbb{R}$.

(ii) *If* x *is any real number, then*

$$I_\lambda(\overline{y}, x) = \sum_{l\in\mathbb{Z}} y_l L_l(x),$$

where $(L_l : l \in \mathbb{Z})$ *is the sequence of fundamental functions introduced in the preceding theorem.*

(iii) *There is a constant* β_3*, depending on* λ*,* q*,* Q*, and* p*, such that*

$$\|I_\lambda(\overline{y}, \cdot)\|_{L_p(\mathbb{R})} \le \beta_3 \|\overline{y}\|_{\ell_p(\mathbb{Z})},$$

for every $\overline{y} \in \ell_p(\mathbb{Z})$.

Proof. (i) As A^{-1} is a bounded operator on $\ell_p(\mathbb{Z})$, the sequence $A^{-1}\overline{y}$ is bounded. Hence the continuity of $I_\lambda(\overline{y}, \cdot)$ follows from Proposition 2.1.

(ii) Let $x \in \mathbb{R}$. Then

$$I_\lambda(\overline{y}, x) = \sum_{k\in\mathbb{Z}} (A^{-1}\overline{y})_k \mathrm{e}^{-\lambda(x-x_k)^2} = \sum_{k\in\mathbb{Z}} \left[\sum_{l\in\mathbb{Z}} y_l A^{-1}(k,l)\right] \mathrm{e}^{-\lambda(x-x_k)^2}.$$

The required result is obtained by interchanging the order of summation, which is justified by the following series of estimates, the first of which is consequent upon Corollary 6.4, and the last of which follows from Proposition 2.1.

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \left[\sum_{l\in\mathbb{Z}} |y_l A^{-1}(k,l)|\right] \mathrm{e}^{-\lambda(x-x_k)^2} &= O\left(\sum_{k\in\mathbb{Z}} \left[\sum_{l\in\mathbb{Z}} \mathrm{e}^{-\gamma_1|k-l|}\right] \mathrm{e}^{-\lambda(x-x_k)^2}\right) \\ &= O\left(\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\lambda(x-x_k)^2}\right) = O(1).\end{aligned}$$

(iii) We begin with $p = \infty$. If $x_s \le x < x_{s+1}$, assertion (ii), the triangle inequality, and a now familiar argument lead to the following:

$$\begin{aligned}|I_\lambda(\overline{y}, x)| &= \left|\sum_{l\in\mathbb{Z}} y_l L_l(x)\right| \\ &\le \beta_2 \left[|y_s| + |y_{s+1}| + \sum_{m=1}^{\infty} |y_{s-m}|\mathrm{e}^{-\rho m q} + \sum_{m=2}^{\infty} |y_{s+m}|\mathrm{e}^{-\rho(m-1)q}\right] \\ &\le \frac{2\beta_2}{1-\mathrm{e}^{-\rho q}} \|\overline{y}\|_{\ell_\infty(\mathbb{Z})} =: \beta_3 \|\overline{y}\|_{\ell_\infty(\mathbb{Z})}. \qquad (72)\end{aligned}$$

It is immediate that $\|I_\lambda(\overline{y}, \cdot)\|_{L_\infty(\mathbb{R})} \le \beta_3 \|\overline{y}\|_{\ell_\infty(\mathbb{Z})}$.

Now suppose that $1 \le p < \infty$. Let $J_s := [x_s, x_{s+1})$, $s \in \mathbb{Z}$, and recall from (72) that, if $x \in J_s$, then

$$|I_\lambda(\overline{y}, x)| \le \beta_2 \left[|y_s| + |y_{s+1}| + \sum_{m=1}^{\infty} |y_{s-m}|\mathrm{e}^{-\rho m q} + \sum_{m=2}^{\infty} |y_{s+m}|\mathrm{e}^{-\rho(m-1)q}\right].$$

Therefore, as $|x_{s+1} - x_s| \le Q$, we have

$$\int_{J_s} |I_\lambda(\overline{y}, x)|^p \mathrm{d}x \le Q\beta_2^p \left[|y_s| + |y_{s+1}| + \sum_{m=1}^{\infty} |y_{s-m}| \mathrm{e}^{-\rho m q} + \sum_{m=2}^{\infty} |y_{s+m}| \mathrm{e}^{-\rho(m-1)q} \right]^p. \tag{73}$$

Let $\overline{u} = (u_k : k \in \mathbb{Z})$ and $\overline{v} = (v_k : k \in \mathbb{Z})$ be a pair of sequences defined by $u_k = |y_k|$, $k \in \mathbb{Z}$, and

$$v_k = \begin{cases} 1 & \text{if } k \in \{-1, 0\}; \\ \mathrm{e}^{-\rho k q} & \text{if } k \ge 1; \\ \mathrm{e}^{\rho(k+1)q} & \text{if } k \le -2. \end{cases}$$

Then (73) may be recast as follows:

$$\int_{J_s} |I_\lambda(\overline{y}, x)|^p \mathrm{d}x \le Q\beta_2^p \left| \sum_{m\in\mathbb{Z}} u_{s-m} v_m \right|^p,$$

so that

$$\|I_\lambda(\overline{y}, \cdot)\|_{L_p(\mathbb{R})} = \left[\sum_{s\in\mathbb{Z}} \int_{J_s} |I_\lambda(\overline{y}, x)|^p \mathrm{d}x \right]^{1/p} \le Q^{1/p}\beta_2 \left[\sum_{s\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} u_{s-m} v_m \right|^p \right]^{1/p}. \tag{74}$$

Now the Generalized Minkowski Inequality (cf. [13, p. 123]) implies that

$$\left[\sum_{s\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} u_{s-m} v_m \right|^p \right]^{1/p} \le \sum_{m\in\mathbb{Z}} |v_m| \left[\sum_{s\in\mathbb{Z}} |u_{s-m}|^p \right]^{1/p} \le \frac{2}{1 - \mathrm{e}^{-\rho q}} \|\overline{y}\|_{\ell_p(\mathbb{Z})}, \tag{75}$$

and a combination of (75) and (74) completes the proof. □

We conclude the paper with a few remarks. Suppose that f is a bandlimited function, and let $d_k = f(x_k)$, $k \in \mathbb{Z}$. We have seen in Proposition 2.6 that this sequence $\overline{d} = (d_k : k \in \mathbb{Z})$ is square summable. Furthermore, as observed in the course of the proof of Theorem 2.7(v), the Gaussian interpolant $I_\lambda(f)(\cdot)$ studied earlier coincides with the function $I_\lambda(\overline{d}, \cdot)$ introduced in this section. Thus the final part of the previous theorem presents a twofold generalization of the estimate (24): to values of $p \in [1, \infty]$ other than 2, whilst requiring only that the underlying set of sampling points satisfies conditions (8) and (9). In particular we do *not* assume in Theorem 6.7(iii) that $(x_j : j \in \mathbb{Z})$ is a Riesz-basis sequence. However, it is not without interest to note that the main convergence theorems obtained in Section 4 do not hold for data sites which merely satisfy the quasi-uniformity conditions (8) and (9). For example, let $X := \mathbb{Z} \setminus \{0\}$, and let f be the bandlimited function given by $f(x) := \sin(\pi x)/(\pi x)$, $x \in \mathbb{R}$. As $f(x_j) = 0$ for every $x_j \in X$, $I_\lambda(f)$ is identically zero, so it is manifest that $I_\lambda(f)$ does not converge to f.

Counterparts of the result obtained in part (iii) of the preceding theorem, for the case when $x_j = j$, may be found in [14,7]. However, those estimates are much more precise in nature.

Suppose that $(x_j : j \in \mathbb{Z})$ is a strictly increasing sequence of real numbers satisfying the following condition:

$$|x_j - j| \le D < 1/4, \quad j \in \mathbb{Z}. \tag{76}$$

Then (x_j) is a Riesz-basis sequence [15]. Let $(L_{l,\lambda} : l \in \mathbb{Z})$ be the associated sequence of fundamental functions defined in Theorem 6.6. Define

$$G(x) := (x - x_0) \prod_{j=1}^{\infty} \left(1 - \frac{x}{x_j}\right)\left(1 - \frac{x}{x_{-j}}\right), \quad x \in \mathbb{R},$$

and let

$$G_l(x) := \frac{G(x)}{(x - x_l)G'(x_l)}, \quad x \in \mathbb{R},\ l \in \mathbb{Z}.$$

It is shown in [16] that each G_l is a bandlimited function satisfying the interpolatory conditions $G_l(x_m) = \delta_{lm}$, $m \in \mathbb{Z}$. Thus we find that $I_\lambda(G_l) = L_{l,\lambda}$, and conclude from Theorems 4.3 and 4.4 that $\lim_{\lambda \to 0^+} L_{l,\lambda} = G_l$ in $L_2(\mathbb{R})$ and uniformly on $\mathbb{R}$. As a particular example, we learn from [17] that, if $x_0 = 0$ and $x_j = x_{-j} = j + c^2/j$, $j \ge 1$, $0 < |c| < 1/2$, then (x_j) fulfills (76), and that the corresponding function G is given in closed form:

$$G(x) = x[\cos(\pi(x^2 - 4c^2)^{1/2}) - \cos(\pi x)]/2\sinh(\pi c), \quad x \in \mathbb{R}.$$

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References

[1] S.D. Riemenschneider, N. Sivakumar, Cardinal interpolation by Gaussian functions: A survey, J. Anal. 8 (2000) 157–178.

[2] Yu. Lyubarskii, W.R. Madych, The recovery of irregularly sampled band limited functions via tempered splines, J. Funct. Anal. 125 (1994) 201–222.

[3] R.R. Goldberg, Fourier Transforms, Cambridge University Press, 1961.

[4] K. Chandrasekharan, Classical Fourier Transforms, Springer Verlag, 1989.

[5] B.J.C. Baxter, N. Sivakumar, On shifted cardinal interpolation by Gaussians and multiquadrics, J. Approx. Theory 87 (1996) 36–59.

[6] S.D. Riemenschneider, Multivariate cardinal interpolation, in: C.K. Chui, L.L. Schumaker, J.D. Ward (Eds.), Approximation Theory VI, Vol. II, Academic Press, Boston, MA, 1989, pp. 561–580.

[7] S.D. Riemenschneider, N. Sivakumar, On cardinal interpolation by Gaussian radial-basis functions: Properties of fundamental functions and estimates for Lebesgue constants, J. Anal. Math. 79 (1999) 33–61.

[8] F.J. Narcowich, N. Sivakumar, J.D. Ward, On condition numbers associated with radial-function interpolation, J. Math. Anal. Appl. 186 (1994) 457–485.

[9] F.J. Narcowich, J.D. Ward, Norm estimates for the inverses of a general class of scattered-data radial-function interpolation matrices, J. Approx. Theory 69 (1992) 84–109.

[10] R.M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, 1980.

[11] S. Jaffard, Propriétés des matrices “bien localisées” près de leur diagonale et quelques applications, Ann. Inst. H. Poincaré 7 (1990) 461–476.

[12] C. de Boor, Odd-degree spline interpolation at a bi-infinite knot sequence, in: R. Schaback, K. Scherer (Eds.), Approximation Theory, in: Lecture Notes in Mathematics, vol. 556, Springer-Verlag, Berlin, 1976, pp. 30–53.

[13] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, 1934.

[14] S.D. Riemenschneider, N. Sivakumar, Gaussian radial-basis functions: Cardinal interpolation of ℓ^p and power-growth data, Adv. Comput. Math. 11 (1999) 229–251.

[15] M.I. Kadec, The exact value of the Paley–Wiener constant, Dokl. Akad. Nauk SSSR 155 (1964) 1243–1254.

[16] N. Levinson, On non-harmonic Fourier series, Ann. of Math. 37 (1936) 919–936.

[17] J.R. Higgins, A sampling theorem for irregularly spaced sample points, IEEE Trans. Inform. Theory (1976) 621–622.