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On L-functions of cyclotomic function fields

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Abstract

We study two criterions of cyclicity for divisor class groups of function fields, the first one involves Artin L-functions and the second one involves "affine" class groups. We show that, in general, these two criterions are not linked.

Let P be a prime of $\mathbb{F}_q[T]$ of degree d and let K_P be the Pth cyclotomic function field. In this paper we study the relation between the p-part of $\mathrm{Cl}^0(K_P)$ and the zeta function of K_P, where p is the characteristic of $\mathbb{F}_q$.

Let χ be an even character of the Galois group of $K_P/\mathbb{F}_q(T)$, $\chi \neq 1$. Let $g(X, \bar{\chi})$ be the "congruent to one modulo p" part of the L-function of $K_P/\mathbb{F}_q(T)$ associated to the character $\bar{\chi}$. We have two criterions of cyclicity [2, Chapter 8]: if $\deg_X g(X, \bar{\chi}) \leqslant 1$ then $\mathrm{Cl}^0(K_P)_p(\chi)$ is a cyclic $\mathbb{Z}_p[\mu_{q^d-1}]$-module, and if $\mathrm{Cl}(O_{K_P})_p(\chi) = \{0\}$ then $\mathrm{Cl}^0(K_P)_p(\chi)$ is a cyclic $\mathbb{Z}_p[\mu_{q^d-1}]$-module. Goss has obtained that if $\mathrm{Cl}(O_{K_P})_p(\chi)$ is trivial then $g(X, \bar{\chi})$ is of degree at most one [2, Theorem 8.21.2]. Unfortunately, there is a gap in the proof of this result. In fact, we show that in general $\mathrm{Cl}(O_{K_P})_p(\chi) = \{0\}$ does not imply $\deg_X g(X, \bar{\chi}) \leqslant 1$ (Proposition 3.6). In order to prove this result, we introduce a characteristic p Dirichlet series, $H(s)$, which continues to be an essentially algebraic entire function on Goss complex plane, and we give an arithmetic interpretation of its values at negative integers (Proposition 3.4). We also prove that if i is a

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q-magic number and if ω_P is the Teichmüller character at P, then $g(X, \omega_P^i)$ has simple roots when $i \equiv 0 \pmod{q-1}$ (Proposition 5.1).

Note the Goss conjectures that if i is a q-magic number then $\deg_X g(X, \omega_P^i) \leqslant 1$. This problem is still open and can be viewed as an analogue of Vandiver's Conjecture for function fields (see Section 5).

1. Notations

Let $\mathbb{F}_q$ be a finite field having q elements, $q = p^s$, where p is the characteristic of $\mathbb{F}_q$. Let T be an indeterminate over $\mathbb{F}_q$ and set $A = \mathbb{F}_q[T]$, $k = \mathbb{F}_q(T)$. We denote the set of monic elements of A by A^+. A prime of A is a monic irreducible polynomial in A. We fix $\bar{k}$ an algebraic closure of k. We denote the unique place of k which is a pole of T by ∞.

Let L/k be a finite geometric extension of k, $L \subset \bar{k}$. We set:

- O_L: the integral closure of A in L,
- O_L^*: the group of units of O_L,
- $S_\infty(L)$: the set of places of L above ∞,
- $\mathrm{Cl}^0(L)$: the group of divisors of degree zero of L modulo the group of principal divisors,
- $\mathrm{Cl}(O_L)$: the ideal class group of O_L,
- $R(L)$: the groupe of divisors of degree zero with supports in $S_\infty(L)$ modulo the group of principal divisors with supports in $S_\infty(L)$.

If d is the greatest common divisor of the degrees of the elements in $S_\infty(L)$, we have the following exact sequence:

$$0 \to R(L) \to \mathrm{Cl}^0(L) \to \mathrm{Cl}(O_L) \to \frac{\mathbb{Z}}{d\mathbb{Z}} \to 0.$$

Let P be a prime of A of degree d. We denote the Pth cyclotomic function field by K_P (see [2, Chapter 7; 5]). Recall that K_P/k is the maximal abelian extension of k contained in k such that:

- K_P/k is unramified outside of P, ∞,
- K_P/k is tamely ramified at P, ∞,
- for every place v of K_P above ∞, the completion of K_P at v is equal to $\mathbb{F}_q((\frac{1}{T}))(\sqrt[q-1]{-T})$.

We recall that $\mathrm{Gal}(K_P/k) \simeq (A/PA)^*$, and that the decomposition group of ∞ in K_P/k is equal to its inertia group and is isomorphic to $\mathbb{F}_q^*$.

Let $E/\mathbb{F}_q$ be a global function field and let F/E be a finite geometric abelian extension. Set $G = \mathrm{Gal}(F/E)$ and $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$.

Let $\chi \in \hat{G}$, $\chi \neq 1$, we set

$$L(X, \chi) = \prod_{v \text{ place of } E} (1 - \chi(v) X^{\deg v})^{-1},$$

where $\chi(v) = 0$ if v is ramified in $F^{\mathrm{Ker}(\chi)}/E$, and if v is unramified in $F^{\mathrm{Ker}(\chi)}/E$, $\chi(v) = \chi((v, F^{\mathrm{Ker}(\chi)}/E))$, where $(., F^{\mathrm{Ker}(\chi)}/E)$ is the global reciprocity map. If $\chi = 1$, we set $L(X,\chi) = L_E(X)$ where $L_E(X)$ is the numerator of the zeta function of E.

Therefore, if $L_F(X)$ is the numerator of the zeta function of F, we get

$$L_F(X) = \prod_{\chi \in \hat{G}} L(X,\chi).$$

Let Δ be a finite abelian group and let M be a Δ-module. Let ℓ be a prime number such that $|\Delta| \not\equiv 0 \pmod{\ell}$. We fix an embedding of $\bar{\mathbb{Q}}$ in $\overline{\mathbb{Q}_\ell}$. Let $W = \mathbb{Z}_\ell[\mu_{|\Delta|}]$. For $\chi \in \hat{\Delta}$, we set

$$e_\chi = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \chi(\delta)\delta^{-1} \in W[\Delta]$$

and

$$M_\ell(\chi) = e_\chi(M \otimes_{\mathbb{Z}} W).$$

Thus, we have

$$M \otimes_{\mathbb{Z}} W = \bigoplus_{\chi \in \hat{\Delta}} M_\ell(\chi).$$

Let N be a W-module, we denote the length of N by $\mathrm{Long}_W(N)$.

2. Cyclotomic function fields and Artin–Schreier extensions

Let Q be a prime of A of degree n, and write $Q(T) = T^n + \alpha T^{n-1} + \cdots$, $\alpha \in \mathbb{F}_q$. We set $i(Q) = Tr_{\mathbb{F}_q/\mathbb{F}_p}(\alpha)$. Let $a \in A$, $a \neq 0$, we set

$$i(a) = \sum_{Q \text{ prime of A}} v_Q(a) i(Q) \in \mathbb{F}_p,$$

where v_Q is the normalized Q-adic valuation on k. Note that the function $i : A\setminus\{0\} \to \mathbb{F}_p$ satisfies $i(ab) = i(a) + i(b)$. Observe that, if $a \in A^+$, $a = T^n + \alpha T^{n-1} + \cdots$, then $i(a) = Tr_{\mathbb{F}_q/\mathbb{F}_p}(\alpha)$.

Let $\theta \in \bar{k}$ such that $\theta^p - \theta = T$. Set $\tilde{A} = \mathbb{F}_q[\theta]$, $\tilde{k} = \mathbb{F}_q(\theta)$ and $G = \mathrm{Gal}(\tilde{k}/k)$. Note that $\tilde{k}/k$ is unramified outside ∞ and totally ramified at ∞. Let $\tilde{\infty}$ be the unique place of $\tilde{k}$ above ∞.

Lemma 2.1. *Let $(.,\tilde{k}/k)$ be the usual Artin symbol for ideals. For $a \in A\backslash\{0\}$:*

$$((a),\tilde{k}/k)(\theta)=\theta-i(a),$$

where (a) is the principal ideal generated by a.

Proof. By the classical properties of the Artin symbol, it is enough to prove the lemma when a is a prime of A. Thus, let P be a prime of A of degree d. We have

$$((P),\tilde{k}/k)(\theta)\equiv\theta^{q^d}\ (\mathrm{mod}\ P).$$

But, for $n\geqslant 0$, we have

$$\theta^{p^n}=\theta+T+T^p+\cdots+T^{p^{n-1}}.$$

Therefore

$$\theta^{q^d}\equiv\theta-i(P)\ (\mathrm{mod}\ P).$$

Thus

$$((P),\tilde{k}/k)(\theta)\equiv\theta-i(P)\ (\mathrm{mod}\ P).$$

But recall that there exists $\delta\in\mathbb{F}_p$ such that

$$((P),\tilde{k}/k)(\theta)\equiv\theta-\delta.$$

The lemma follows. □

Lemma 2.2. *Let P be a prime of A of degree d such that $i(P)\neq 0$. Then P is a prime of $\tilde{A}$ of degree pd. Let $\widetilde{K_P}$ be the Pth cyclotomic function field for the ring $\tilde{A}$, then $K_P\subset\widetilde{K_P}$.*

Proof. We have $-T=-\theta^p(1-\theta^{1-p})$. Note that

$$1-\theta^{1-p}\in\left(F_q\left(\left(\frac{1}{\theta}\right)\right)^*\right)^{q-1}.$$

Therefore

$$\sqrt[q-1]{-T}\in F_q\left(\left(\frac{1}{\theta}\right)\right)\left(\sqrt[q-1]{-\theta}\right).$$

Thus

- $\tilde{k}K_P/\tilde{k}$ is unramified outside P, $\tilde{\infty}$,
- $\tilde{k}K_P/\tilde{k}$ is tamely ramified at P, $\tilde{\infty}$,
- for every place w of $\tilde{k}K_P$ above $\tilde{\infty}$, the completion of $\tilde{k}K_P$ at w is contained in $F_q((\frac{1}{\theta}))(\sqrt[q-1]{-\theta})$.

The lemma follows by class field theory. □

We will need the following lemma:

Lemma 2.3. *Let P be a prime of A, $d = \deg_T(P)$. Let χ be a character of* $\mathrm{Gal}(K_P/k)$, *$\chi \neq 1$. Then $\deg_X L(X,\chi) = d-2$ if $\chi(\mathbb{F}_q^*) = \{1\}$, and $\deg_X L(X,\chi) = d-1$ otherwise.*

Proof. By Rosen [7, Proposition 4.3], we have

- $\deg_X L(X,\chi) \leqslant d-2$ if $\chi(\mathbb{F}_q^*) = \{1\}$,
- $\deg_X L(X,\chi) \leqslant d-1$ otherwise.

Let g be the genus of K_P, by Rosen [7, Proposition 16.7], we have

$$2g = \sum_{\chi \in \widehat{\mathrm{Gal}(K_P/k)}, \chi \neq 1} \deg_X L(X,\chi).$$

But, recall that K_P/k is tamely ramified and ramified at exactly P and ∞. Thus, by the Riemann–Hurwitz Theorem [7, Theorem 7.16], we get

$$2g = \left(\frac{q^d-1}{q-1} - 1\right)(d-2) + \frac{(q^d-1)(q-2)}{q-1}(d-1).$$

The lemma follows. □

Let P be a prime of A, $\deg_T P(T) = d$ and $i(P) \neq 0$. Let $L = \tilde{k}K_P \subset \widetilde{K_P}$. Let $\Delta = \mathrm{Gal}(K_P/k) \simeq \mathrm{Gal}(L/\tilde{k})$. We have an isomorphism compatible to class field theory: $\hat{\Delta} \to \widehat{\mathrm{Gal}(L/\tilde{k})}$, $\chi \to \tilde{\chi} = \chi \circ N_{\tilde{k}/k}$. We fix $\zeta_p \in \bar{\mathbb{Q}}$ a primitive pth root of unity.

Lemma 2.4. (1) *Let $\chi \in \hat{\Delta}$, $\chi \neq 1$. Let $L(X,\tilde{\chi})$ be the Artin L-function relative to $L/\tilde{k}$ and to the character $\tilde{\chi}$.* We have

$$L(X,\tilde{\chi}) = \prod_{\phi \in \hat{G}} L(X,\phi\chi),$$

where $L(X,\phi\chi)$ is the Artin L-function relative to L/k and the character $\phi\chi$.

(2) *Let $\chi \in \hat{\Delta}$, $\chi \neq 1$, χ even (i.e. $\chi(\mathbb{F}_q^*) = \{1\}$). Then*

$$\frac{L(X,\tilde{\chi})}{L(X,\chi)} \equiv (1-X)^{p-1} L(X,\chi)^{p-1} \pmod{(1-\zeta_p)}.$$

Proof. Assertion (1) is a consequence of the usual properties of Artin L-functions. Now, let $\phi \in \hat{G}$, $\phi \neq 1$. Since $\phi\chi$ is ramified at ∞, we get

$$L(X, \phi\chi) = \sum_{n \geqslant 0} \left(\sum_{a \in A^+, \deg(a) = n} \phi(a)\chi(a) \right) X^n.$$

Thus

$$L(X, \phi\chi) \equiv \sum_{n \geqslant 0} \left(\sum_{a \in A^+, \deg(a) = n} \chi(a) \right) X^n \pmod{(1 - \zeta_p)}.$$

But, since χ is even, we have $\chi(\infty) = 1$. Therefore

$$L(X, \phi\chi) \equiv (1 - X) L(X, \chi) \pmod{(1 - \zeta_p)}.$$

The lemma follows. □

Let $i \in \mathbb{F}_p$ and let $\sigma_i \in G$ such that $\sigma_i(\theta) = \theta - i$. Let $\psi \in \hat{G}$ given by $\psi(\sigma_i) = \zeta_p^i$.

Lemma 2.5. *Let $\chi \in \hat{\Delta}$, χ even and non-trivial.*

(1) *Let $\phi \in \hat{G}$, $\phi \neq 1$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_P)/\mathbb{Q})$ such that $\phi = \psi^\sigma$. Then*

$$L(X, \phi\chi) = L(X, \psi\chi)^\sigma.$$

Furthermore $\deg_X L(X, \phi\chi) = d$.

(2) *We have*

$$L(1, \psi\chi) \equiv \left(\sum_{a \in A^+, \deg(a) \leqslant d} i(a)\chi(a) \right) (\zeta_p - 1) \pmod{(1 - \zeta_p)^2}.$$

Proof. Let $\mathbb{Q}(\chi)$ be the abelian extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ the values of χ. Let $\mathbb{Z}[\chi]$ be the ring of integers of $\mathbb{Q}(\chi)$. Note that p is unramified in $\mathbb{Q}(\chi)$ and

$$\mathrm{Gal}(\mathbb{Q}(\chi)(\zeta_p)/\mathbb{Q}(\chi)) \simeq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}).$$

Since $L(X, \phi\chi)$ is a polynomial in $\mathbb{Z}[\chi][\zeta_p][X]$, we have

$$L(X, \phi\chi) = L(X, \psi\chi)^\sigma.$$

Since χ and $\tilde{\chi}$ are non-trivial even characters, by Lemma 2.3, we have

$$\deg_X L(X, \tilde{\chi}) = pd - 2$$

and

$$\deg_X L(X, \chi) = d - 2.$$

Therefore $\deg_X L(X, \phi\chi) = d$.

Now, we have

$$L(X, \psi\chi) = \sum_{n=0}^{d} \left(\sum_{a \in A^+ \deg(a) = n} \zeta_p^{i(a)} \chi(a) \right) X^n.$$

But recall that

$$\zeta_p^{i(a)} \equiv 1 + i(a)(\zeta_p - 1) \pmod{(1 - \zeta_p)^2}.$$

Thus, since χ is even and non-trivial, we get

$$\begin{aligned} L(X, \psi\chi) \equiv\ & L(X, \chi)(1 - X) + (\zeta_p - 1) \\ & \times \left(\sum_{n=1}^{d} \left(\sum_{a \in A^+ \deg(a) = n} i(a)\chi(a) \right) X^n \right) \pmod{(1 - \zeta_p)^2}. \end{aligned}$$

The lemma follows. □

We are now ready to prove the main result of this section:

Proposition 2.6. *Let $\chi \in \hat{\Delta}$, $\chi \neq 1$, χ even. Let $W = \mathbb{Z}_p[\mu_{q^d - 1}]$. We have*

$$\mathrm{Long}_W \left(\frac{\mathrm{Cl}(O_L)_p(\tilde{X})}{\mathrm{Cl}(O_{K_P})_p(\chi)} \right) \geqslant 1 \Leftrightarrow \sum_{a \in A^+ \deg(a) \leqslant d} i(a)\bar{\chi}(a) \equiv 0 \pmod{p}.$$

Proof. Fix τ a generator of $G \simeq \mathrm{Gal}(L/K_P)$. Let $\varepsilon \in O_L^*$. Since L/K_P is totally ramified at any prime above ∞, there exists $\zeta \in \mathbb{F}_q^*$ such that $\tau(\varepsilon) = \zeta\varepsilon$. But $\tau^p(\varepsilon) = \zeta^p(\varepsilon) = \varepsilon$. Since we are in characteristic p, we deduce that $\varepsilon \in O_{K_P}^*$. Therefore

$$O_L^* = O_{K_P}^*.$$

Let I be an ideal of O_{K_P} such that $IO_L = \alpha O_L$ for some $\alpha \in O_L$. Then, there exists $\varepsilon \in O_L^*$ such that $\tau(\alpha) = \varepsilon\alpha$. Since $O_L^* = O_{K_P}^*$ and since τ is of order p, we deduce that $\alpha \in O_{K_P}$. This implies that

$$\mathrm{Cl}(O_{K_P}) \hookrightarrow \mathrm{Cl}(O_L).$$

One can also show that

$$\mathrm{Cl}^0(K_P) \hookrightarrow \mathrm{Cl}^0(L).$$

Set $\Delta^+ = \frac{\Delta}{\mathbb{F}_q^*}$. Let $\mathcal{I}$ be the augmentation ideal of $\mathbb{F}_p[\Delta^+]$. One sees that we have the following isomorphism of Δ-modules:

$$\frac{R(L)}{R(K_P)} \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq \mathcal{I}.$$

This implies that we have the following exact sequence of W-modules:

$$0 \to \frac{W}{pW} \to \frac{\mathrm{Cl}^0(L)_p(\tilde{\chi})}{\mathrm{Cl}^0(K_P)_p(\chi)} \to \frac{\mathrm{Cl}(O_L)_p(\tilde{\chi})}{\mathrm{Cl}(O_{K_P})_p(\chi)} \to 0.$$

Now, by the results of Goss and Sinnott [4]:

$$\mathrm{Long}_W \mathrm{Cl}^0(L)_p(\tilde{\chi}) = v_p(L(1, \bar{\tilde{\chi}}))$$

and

$$\mathrm{Long}_W \mathrm{Cl}^0(K_P)_p(\chi) = v_p(L(1, \bar{\chi})).$$

Thus by Lemma 2.4,

$$\mathrm{Long}_W \left(\frac{\mathrm{Cl}(O_L)_P(\tilde{\chi})}{\mathrm{Cl}(O_{K_P})_P(\chi)} \right) = (p-1)v_p(L(1, \psi\bar{\chi})) - 1.$$

It remains to apply Lemma 2.5. $\square$

3. Derivatives of L-functions

Let P be a prime of A of degree d. We fix an embedding of $\bar{\mathbb{Q}}$ in $\overline{\mathbb{Q}_p}$. Set $\Delta = \mathrm{Gal}(K_P/k)$ and $W = \mathbb{Z}_p[\mu_{q^d-1}]$. We fix an isomorphism $\Phi_P : A/PA \to W/pW$. Then Φ_P induces an isomorphism

$$\omega_P : \Delta \to \mu_{q^d-1} \subset W^*.$$

The morphism ω_P is called "the" Teichmüller character at P. Note that $\hat{\Delta}$ is a cyclic group and ω_P is a generator of this group.

Let $i \in \mathbb{N}$, set:

- $\beta(0) = 1$,
- $\beta(i) = \sum_{a \in A^+} a^i$ if $i \geqslant 1$, $i \not\equiv 0 \pmod{q-1}$,
- $\beta(i) = -\sum_{a \in A^+} \deg(a) a^i$ if $i \geqslant 1$, $i \equiv 0 \pmod{q-1}$.

One can prove that for all $i \in \mathbb{N}$, $\beta(i) \in A$. We also see that

$$\forall i \in \mathbb{N},\ 0 \leqslant i \leqslant q^d - 2, \quad \Phi_P(\beta(i)) \equiv L(1, \omega_P^i) \pmod{p}.$$

Therefore, if $1 \leqslant i \leqslant q^d - 2$, by the results of Goss and Sinnott [4], we have

$$\mathrm{Long}_W \mathrm{Cl}^0(K_P)_p(\omega_P^{-i}) \geqslant 1 \Leftrightarrow \beta(i) \equiv 0 \pmod{P}.$$

The numbers $\beta(i)$ are called the Bernoulli–Goss polynomials.

Recall that we have a surjective morphism of Δ-modules

$$W[\Delta^+] \to R(K_P) \otimes_{\mathbb{Z}} W,$$

where $\Delta^+ = \Delta/\mathbb{F}_q^*$. Thus for $\chi \in \hat{\Delta}$, χ even, $R(K_P)_p(\chi)$ is a cyclic W-module. But, for such a character, we have the exact sequence of W-modules:

$$0 \to R(K_P)_p(\chi) \to \mathrm{Cl}^0(K_P)_p(\chi) \to \mathrm{Cl}(O_{K_P})_p(\chi) \to 0.$$

This implies that, if $\mathrm{Cl}(O_{K_P})_p(\chi) = \{0\}$, $\mathrm{Cl}^0(K_P)_p(\chi)$ is a cyclic W-module.

Goss has shown [2, Corollary 8.16.2] that for χ is even, $\chi \neq 1$, if $L'(1, \bar{\chi}) \not\equiv 0 \pmod{p}$ (here $L'(1, \bar{\chi})$ is the derivative of $L(X, \bar{\chi})$ taken at $X = 1$), then $\mathrm{Cl}^0(K_P)_p(\chi)$ is a cyclic W-module.

Therefore a natural question arises. Let $\chi \in \hat{\Delta}$, $\chi \neq 1$, χ even. Assume that $L(1, \bar{\chi}) \equiv 0 \pmod{p}$. Do we have

$$\mathrm{Cl}(O_{K_P})_p(\chi) = \{0\} \Rightarrow L'(1, \bar{\chi}) \not\equiv 0 \pmod{p}?$$

Our aim in this section is to show that in general the answer is no.

Lemma 3.1. *Let j be an integer, $1 \leqslant j \leqslant q^d - 2$, for some integer $d \geqslant 1$. Let P be a prime of A of degree d such that $i(P) \neq 0$. Then for all integer h, $h \geqslant d + 1$, we have*

$$\sum_{a \in A^+, \deg(a) = h} i(a) a^j \equiv 0 \pmod{P}.$$

Proof. We keep the notations of Section 2. Write $\chi = \omega_P^j$. Recall that (in fact we proved such a fact for even characters, but one can also show that this is true for any non-trivial character):

$$\deg_X L(X, \psi\chi) = d.$$

Thus

$$\sum_{a \in A^+, \deg(a)=h} \psi(a)\chi(a) = 0.$$

But, since $h \geqslant d + 1$ and $j \not\equiv 0 \pmod{q^d - 1}$, we have

$$\sum_{a \in A^+, \deg(a)=h} \chi(a) = 0.$$

Therefore

$$\sum_{a \in A^+, \deg(a)=h} (\psi(a) - 1)\chi(a) = 0.$$

Thus

$$\sum_{a \in A^+, \deg(a)=h} i(a)\chi(a) \equiv 0 \pmod{p}.$$

But

$$\Phi_P \left(\sum_{a \in A^+, \deg(a)=h} i(a)a^j \right) \equiv \sum_{a \in A^+, \deg(a)=h} i(a)\chi(a) \pmod{p}.$$

The lemma follows. □

Lemma 3.2. *Let j be an integer, $j \geqslant 1$. Then, for $h >> 0$, we have*

$$\sum_{a \in A^+, \deg(a)=h} i(a)a^j = 0.$$

Proof. Let n_0 be the smallest integer such that $q^{n_0} - 2 \geqslant j$. Let $h \geqslant n_0 + 1$. Let S be the set of primes in A of degree d, $n_0 \leqslant d \leqslant h - 1$, such that $i(P) \neq 0$. Then, by Lemma 3.1, we have

$$\forall P \in S, \qquad \sum_{a \in A^+, \deg(a)=h} i(a)a^j \equiv 0 \pmod{P}.$$

There are exactly $\frac{q^d}{p}$ elements $x \in \mathbb{F}_{q^d}$ such that $Tr_{\mathbb{F}_{q^d}/\mathbb{F}_p}(x) = 0$. Therefore, there are at most $\frac{q^d}{p^d}$ primes P in A of degree d such that $i(P) = 0$. But, recall that there are at least $\frac{q^d}{d} - \frac{q}{(q-1)d} q^{d/2}$ primes in A of degree d. Thus, there are at least $(1 - \frac{1}{p})\frac{q^d}{d} - \frac{q}{(q-1)d} q^{d/2}$ primes of degree d such that $i(P) \neq 0$. Therefore

$$\deg_T\left(\prod_{P\in S} P\right) \geqslant \left(1 - \frac{1}{p}\right)\frac{q^h - q^{n_0}}{q-1} - \frac{q}{q-1}\,\frac{q^{h/2} - q^{n_0/2}}{q-1}.$$

Thus, if $\sum_{a\in A^+,\deg(a)=h} i(a)a^j \neq 0$, we get

$$\deg_T\left(\sum_{a\in A^+,\deg(a)=h} i(a)a^j\right) \geqslant \left(1 - \frac{1}{p}\right)\frac{q^h - q^{n_0}}{q-1} - \frac{q}{q-1}\,\frac{q^{h/2} - q^{n_0/2}}{q-1}.$$

But

$$\deg_T\left(\sum_{a\in A^+,\deg(a)=h} i(a)a^j\right) \leqslant hj.$$

Thus, if $\sum_{a\in A^+,\deg(a)=h} i(a)a^j \neq 0$, we have

$$hj \geqslant \left(1 - \frac{1}{p}\right)\frac{q^h - q^{n_0}}{q-1} - \frac{q}{q-1}\,\frac{q^{h/2} - q^{n_0/2}}{q-1}.$$

The lemma follows. □

Let $j \geqslant 1$ be an integer and set

$$\gamma(j) = \sum_{\ell\geqslant 0}\sum_{a\in A^+,\deg(a)=\ell} i(a)a^j.$$

By Lemma 3.2, $\gamma(j) \in A$.

Let S_∞ be Goss complex plane [2, Chapter 8, paragraph 8.1]. Consider the Dirichlet series

$$H(s) = \sum_{a\in A^+} i(a)a^{-s}.$$

By the proof of Lemma 3.2, $H(s)$ is in the motivic class $\mathfrak{M}$ (see [3]). Thus, by the main theorem of [3], $H(s)$ continues to an essentially algebraic entire function on S_∞. Observe that

$$\forall j \geqslant 1, \quad H(-j) = \gamma(j).$$

Lemma 3.3. *Let* $\tau \in \mathrm{Gal}(\mathbb{F}_q(T)/\mathbb{F}_q(T^p - T))$ *such that* $\tau(T) = T + 1$. *Let* $j \in \{1, \ldots, q^d - 2\}$, $j \equiv 0 \pmod{q-1}$. *Recall that* $q = p^s$. *We have*

$$\tau(\gamma(j)) = \gamma(j) + s\beta(j).$$

Proof. By Goss [2, Remark 8.12.1.1] if $j < q^h - 1$, we have

$$\sum_{a \in A^+, \deg(a) = h} a^j = 0.$$

Therefore, by Lemma 3.2, we can select an integer h such that

$$\beta(j) = \sum_{a \in A^+, \deg(a) \leqslant h} \deg(a) a^j$$

and

$$\gamma(j) = \sum_{a \in A^+, \deg(a) \leqslant h} i(a) a^j.$$

Let Q be a prime of A of degree n. Write $Q = T^n + \alpha T^{n-1} + \cdots$, where $\alpha \in \mathbb{F}_q$. Then $\tau(Q) = T^n + (\alpha + n)T^{n-1} + \cdots$. Therefore $i(\tau(Q)) = i(Q) + s\deg(Q)$. This implies that

$$\forall a \in A \backslash \{0\}, \quad i(\tau(a)) = i(a) + s\deg(a).$$

Now

$$\tau(\gamma(j)) = \sum_{a \in A^+, \deg(a) \leqslant h} i(a) \tau(a)^j.$$

Therefore

$$\tau(\gamma(j)) = \sum_{a \in A^+, \deg(a) \leqslant h} (i(\tau(a)) - s\deg(a)) \tau(a)^j.$$

Thus

$$\tau(\gamma(j)) = \sum_{a\in A^+,\deg(a)\leqslant h} i(\tau(a))\tau(a)^j - s \sum_{a\in A^+,\deg(a)\leqslant h} \deg(\tau(a))\tau(a)^j.$$

Observe that $\sum_{a\in A^+,\deg(a)\leqslant h} i(\tau(a))\tau(a)^j = \gamma(j)$ and $-\sum_{a\in A^+,\deg(a)\leqslant h} \deg(\tau(a))\tau(a)^j = \beta(j)$. $\square$

Proposition 3.4. *Let P be a prime of A of degree d such that $i(P)\neq 0$. Set $Q(T) = P(T^p - T)$. Then Q is a prime of A of degree pd. Let i be an integer such that $1\leqslant i\leqslant q^d-2$, $i\equiv 0 \pmod{q-1}$ and $\mathrm{Cl}(O_{K_P})_p(\omega_P^{-i}) = \{0\}$. Then*

$$\mathrm{Long}_W \mathrm{Cl}(O_{K_Q})_p(\omega_Q^{-i(q^{pd}-1)/(q^d-1)}) \geqslant 1 \Leftrightarrow \gamma(i) \equiv 0 \pmod{P}.$$

Proof. By Lemmas 3.1 and 3.2, for $j\in\{1,\dots,q^d-2\}$, we have

$$\Phi_P(\gamma(i)) \equiv \sum_{a\in A^+,\deg(a)\leqslant d} i(a)\omega_P^j(a) \pmod{p}.$$

It remains to apply Proposition 2.6. $\square$

Lemma 3.5. *Assume $p\neq 2$. Let $d\geqslant 1$ be an integer. There exists a prime P in A, $\deg(P)=d$, such that $i(P(T))i(P(T+1))\neq 0$.*

Proof. Let Q be a prime of A of degree d such that $i(Q)\neq 0$. Such a prime exists by the normal basis Theorem. Fix $\overline{\mathbb{F}_q}$ an algebraic closure of $\mathbb{F}_q$. We assume that $i(Q(T+1)) = 0$. Write $Q = T^d + \alpha T^{d-1} + \cdots$. Then $Tr_{\mathbb{F}_q/\mathbb{F}_p}(\alpha) = -sd$. Therefore $sd \not\equiv 0 \pmod{p}$. Let $\theta\in\overline{\mathbb{F}_q}$ such that $Q(\theta)=0$. We observe that

$$\forall\zeta\in\mathbb{F}_p,\quad Tr_{\mathbb{F}_{q^d}/\mathbb{F}_p}(\zeta\theta) = -\zeta sd.$$

Since $p\geqslant 3$, we can find $\zeta\in\mathbb{F}_p^*$ such that $-\zeta sd\neq -sd$. Set $P(T) = \mathrm{Irr}(\zeta\theta,\mathbb{F}_q;T)$. Then P is a prime of degree d such that $i(P)i(\tau(P))\neq 0$. $\square$

Proposition 3.6. *Assume that $p\neq 2$ and $s\not\equiv 0 \pmod{p}$. Led d be an integer, $d\geqslant 2$, and let P be a prime of degree d such that $i(P(T))i(P(T+1))\neq 0$. Set $Q(T) = P(T^p - T)$. Then*

- $L(1,\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}) \equiv 0 \pmod{p}$,
- $L'(1,\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}) \equiv 0 \pmod{p}$,
- $\mathrm{Cl}(O_{K_Q})_p(\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}) = \{0\}$.

Proof. Set $R = P(T+1)$ and $Z = R(T^p - T)$. We observe that we have an isomorphism

$$\mathrm{Cl}(O_{K_Q})_P\big(\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}\big) \simeq \mathrm{Cl}(O_{K_Z})_p\big(\omega_Z^{-(q-1)(q^{pd}-1)/(q^d-1)}\big).$$

Note also that $\beta(q-1) = 1$. Thus

$$\mathrm{Cl}(O_{K_P})_p\big(\omega_P^{-(q-1)}\big) = \mathrm{Cl}(O_{K_R})_p\big(\omega_R^{-(q-1)}\big) = \{0\}.$$

We have

$$L\big(1, \omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}\big) \equiv L\big(1, \omega_Z^{-(q-1)(q^{pd}-1)/(q^d-1)}\big) \equiv 0 \pmod p.$$

And, by Lemma 2.4, since $p \geqslant 3$:

$$L'\big(1, \omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}\big) \equiv L'\big(1, \omega_Z^{-(q-1)(q^{pd}-1)/(q^d-1)}\big) \equiv 0 \pmod p.$$

Suppose that we have $\mathrm{Cl}(O_{K_Q})_p\big(\omega_Q^{-(q-1)(q^{pd}-1)/(q^d-1)}\big) \neq \{0\}$. Then by Proposition 3.4

$$\gamma(q-1) \equiv 0 \pmod P,$$

and also

$$\gamma(q-1) \equiv 0 \pmod R.$$

Thus

$$\tau(\gamma(q-1)) \equiv 0 \pmod{\tau(P)}.$$

Now, by Lemma 3.3, and the fact that $\tau(P) = R$, we get

$$\gamma(q-1) + s\beta(q-1) \equiv 0 \pmod R.$$

Therefore we get $s \equiv 0 \pmod p$ which is a contradiction. The proposition follows. □

4. Cyclicity of class groups and L-functions

Let $E/\mathbb{F}_q$ be a global function field and let F/E be a finite geometric abelian extension. Set $\Delta = \mathrm{Gal}(F/E)$. Let ℓ be a prime number. Let us recall some well-known facts about L-functions.

Set $T_\ell = \mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, J)$, where J is the inductive limit of the $\mathrm{Cl}^0(\mathbb{F}_{q^n}F)$, $n \geqslant 1$. We fix an embedding of $\bar{\mathbb{Q}}$ in $\overline{\mathbb{Q}_\ell}$. Let γ be the Frobenius of $\mathbb{F}_q$. Then γ and Δ act on T_ℓ.

If $\ell \neq p$, we have (see [7, Chapter 15]):

$$\mathrm{Det}(1-\gamma X|_{T_\ell}) = L_F(X),$$

where $L_F(X)$ is the numerator of the zeta function of F.

If $\ell = p$, write $L_F(X) = \prod_i (1-\alpha_i X)$ and set $L_F^{nr}(X) = \prod_{v_p(\alpha_i)=0}(1-\alpha_i X)$. Then (see [1] and also [4]):

$$\mathrm{Det}(1-\gamma X|_{T_p}) = L_F^{nr}(X).$$

Now assume that ℓ does not divide the cardinal of Δ, then the above results are also valid character by character. More precisely, if $\ell \neq p$, we have

$$\forall \chi \in \hat{\Delta}, \quad \mathrm{Det}(1-\gamma X|_{T_\ell(\chi)}) = L(X, \bar{\chi}).$$

If $\ell = p$, for $\chi \in \hat{\Delta}$, write $L(X,\chi) = \prod_i (1-\alpha_i(\chi)X)$ and set $L^{nr}(X,\chi) = \prod_{v_p(\alpha_i(\chi))=0} (1-\alpha_i(\chi)X)$. Then

$$\forall \chi \in \hat{\Delta}, \quad \mathrm{Det}(1-\gamma X|_{T_p(\chi)}) = L^{nr}(X, \bar{\chi}).$$

Now, let $\chi \in \hat{\Delta}$, write

$$L(X,\chi) = \prod_i (1-\alpha_i(\chi)X),$$

and set

$$g(X,\chi) = \prod_{v_\ell(\alpha_i(\chi)-1)>0} (1-\alpha_i(\chi)X).$$

Set

$$g(X) = \prod_{\chi \in \hat{\Delta}} g(X,\chi).$$

We also set:

$$\forall \chi \in \hat{\Delta}, \quad H(X,\chi) = (1+X)^{\deg_X g(X,\chi)} g\big((1+X)^{-1}, \chi\big),$$

and

$$H(X)=\prod_{\chi\in\hat{\Delta}} H(X,\chi).$$

For $n\geqslant 0$, set $F_n=\mathbb{F}_{q^{\ell^n}}F$, and let A_n be the ℓ-Sylow subgroup of $\mathrm{Cl}^0(F_n)$. Let $F_\infty=\bigcup_{n\geqslant 0}F_n$ and let A_∞ be the inductive limit of the A_n, $n\geqslant 0$. We set

$$Y=\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A_\infty).$$

Set $\Gamma=\mathrm{Gal}(F_\infty/F)$, then γ is a topological generator of $\Gamma\simeq\mathbb{Z}_\ell$.

Lemma 4.1. (1) *For all $n\geqslant 0$, we have an isomorphism of Δ-modules*

$$\frac{Y}{(\gamma^{\ell^n}-1)Y}\simeq A_n.$$

(2) *Assume $|\Delta|\not\equiv 0 \pmod{\ell}$. Then, $\forall\chi\in\hat{\Delta}$, $\forall n\geqslant 0$, we have*

$$\frac{Y(\chi)}{(\gamma^{\ell^n}-1)Y(\chi)}\simeq A_n(\chi).$$

Proof. We prove assertion (1), and note that (2) is a consequence of (1). Recall that A_∞ is a divisible group (see [7, Proposition 11.16]). We start with the following exact sequence:

$$0\to A_n\to A_\infty\to A_\infty\to 0,$$

where the middle map is the multiplication by $\gamma^{\ell^n}-1$. We apply $\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell,.)$ to this sequence, we get

$$0\to Y\to Y\to \mathrm{Ext}^1(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A_n)\to 0.$$

We also have the following exact sequence:

$$0\to\mathbb{Z}_\ell\to\mathbb{Q}_\ell\to\frac{\mathbb{Q}_\ell}{\mathbb{Z}_\ell}\to 0.$$

We apply $\mathrm{Hom}(., A_n)$ to this last sequence, using the fact that

$$\mathrm{Ext}^1(\mathbb{Q}_\ell, A_n)=\{0\},$$

we get

$$\mathrm{Hom}(\mathbb{Z}_\ell, A_n) \simeq \mathrm{Ext}^1(\mathbb{Q}_\ell/\mathbb{Z}_\ell, A_n).$$

The lemma follows. $\square$

Proposition 4.2. (1) *Let $\Lambda = \mathbb{Z}_\ell[[X]]$ be the Iwasawa algebra of Γ over $\mathbb{Z}_\ell$ where X acts like $\gamma - 1$. Then Y is a finitely generated Λ-module and a torsion Λ-module. The characteristic polynomial of the Λ-module Y is equal to $H(X)$.*

(2) *Assume that ℓ does not divide the cardinal of Δ. Let $\Lambda = W[[X]]$ be the Iwasawa algebra of Γ over $W = \mathbb{Z}_\ell[\mu_{|\Delta|}]$ where X acts like $\gamma - 1$. Then, for $\chi \in \hat{\Delta}$, $Y(\chi)$ is a finitely generated Λ-module and a torsion Λ-module. The characteristic polynomial of the Λ-module Y is equal to $H(X, \bar{\chi})$.*

Proof. We prove (1), the proof of (2) is essentially similar. For all $n \geqslant 0$, we set $\omega_n(X) = (1 + X)^{\ell^n} - 1$. By Lemma 4.1, we have

$$\forall n \geqslant 0, \quad \frac{Y}{\omega_n Y} \simeq A_n.$$

Therefore Y is a finitely generated Λ-module and a torsion Λ-module. Let $r \in \mathbb{N}$ such that we have an isomorphism of groups

$$Y \simeq \mathbb{Z}_\ell^r.$$

Then, there exists a constant $v \in \mathbb{Z}$, such that, for all n sufficiently large:

$$\left| \frac{Y}{\omega_n Y} \right| = \ell^{rn+v}.$$

But, for all $n \geqslant 0$, we have

$$|A_n| = \ell^{v_\ell(L_{F_n}(1))}.$$

Therefore, there exists a constant $v' \in \mathbb{Z}$ such that, for all n sufficiently large:

$$|A_n| = \ell^{\deg_X H(X) n + v'}.$$

Thus $r = \deg_X H(X)$. But let $V(X)$ be the characteristic polynomial of the Λ-module Y. We know that $r = \deg_X V(X)$, and we also know that $V(X)$ divides $(1 + X)^{\deg L_F(X)} L_F((1 + X)^{-1})$. But $V(X)$ is a distinguished polynomial, thus $V(X)$ divides $H(X)$. The proposition follows. $\square$

Proposition 4.3. (1) *If A_0 is a cyclic $\mathbb{Z}_\ell$-module then $g(X)$ has simple roots.*

(2) *Assume that $|\Delta| \not\equiv 0 \pmod{\ell}$. Let $\chi \in \hat{\Delta}$. If $A_0(\chi)$ is a cyclic W-module then $g(X, \bar{\chi})$ has simple roots.*

Proof. We prove (1). By Nakayama's Lemma, Y is pseudo-isomorphic to $\Lambda/H(X)\Lambda$. But, by a result of Tate [9], we know that the action of γ on Y is semi-simple. This implies that $H(X)$ has simple roots. $\square$

Let us give an application of this last proposition.

Proposition 4.4. *We assume that $q \geqslant 5$. Let $E/\mathbb{F}_q(T)$ be a real quadratic field, i.e. $[E : \mathbb{F}_q(T)] = 2$ and ∞ splits completely in E. If O_E is a principal ideal domain then $L_E(X)$ has simple roots.*

Proof. Let g be the genus of E and write

$$L_E(X) = \prod_{i=1}^{2g} (1 - \alpha_i X).$$

Let $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_{2g})$. Let $\alpha \in \{\alpha_1, \ldots, \alpha_{2g}\}$. Then

$$(1 - \alpha)(1 - \bar{\alpha}) \geqslant q + 1 - 2\sqrt{q} > 1.$$

Therefore

$$N_{K/\mathbb{Q}}(1 - \alpha) > 1.$$

Thus $1 - \alpha$ is not a unit of K. Therefore, for $\alpha \in \{\alpha_1, \ldots, \alpha_{2g}\}$, there exist a prime number ℓ and a prime $\mathcal{L}$ of K above ℓ such that $\alpha \equiv 1 \pmod{\mathcal{L}}$.

Let ∞_1 and ∞_2 be the places of E above ∞. Then $R(E)$ is a quotient of $\mathbb{Z}(\infty_1 - \infty_2)$ and we have an exact sequence

$$0 \to R(E) \to \mathrm{Cl}^0(E) \to \mathrm{Cl}(O_E) \to 0.$$

Therefore, if O_E is a principal ideal domain then $\mathrm{Cl}^0(E)$ is a cyclic group. It remains to apply Proposition 4.3 for the prime numbers that divide $L_E(1)$. $\square$

It is conjectured that there exists infinitely many real quadratic function fields $E/\mathbb{F}_q(T)$ such that O_E is a principal ideal domain. In view of this conjecture, it will be interesting to prove that there exists infinitely many real quadratic function fields $E/\mathbb{F}_q(T)$ such that $L_E(X)$ has simple roots.

5. A conjecture of Goss

Set $D_0 = 1$ and for $i \geqslant 1$, $D_i = (T^{q^i} - T)D_{i-1}^q$. The Carlitz exponential is defined by

$$\mathrm{Exp}(X) = \sum_{i \geqslant 0} \frac{X^{q^i}}{D_i} \in k[[X]].$$

Let $n \in \mathbb{N}$, write $n = a_0 + a_1 q + \cdots + a_r q^r$, where $a_0, \ldots, a_r \in \{0, \ldots, q-1\}$. We set

$$\Gamma_n = \prod_{i=0}^{r} D_i^{a_i}.$$

The ith Bernoulli–Carlitz number, $B(i) \in k$, is defined by

$$\frac{X}{\mathrm{Exp}(X)} = \sum_{i \geqslant 0} \frac{B(i)}{\Gamma_i} X^i.$$

Let P be a prime of A of degree d and let $i \in \{1, \ldots, q^d - 2\}$, $i \equiv 0 \pmod{q-1}$. We have the following result [6]:

$$\mathrm{Cl}(O_{K_P})_p(\omega_P^i) \neq \{0\} \Rightarrow B(i) \equiv 0 \pmod{P}.$$

We fix an embedding of $\bar{\mathbb{Q}}$ in $\overline{\mathbb{Q}_p}$. Let $i \in \{1, \ldots, q^d - 2\}$. Write

$$L(X, \omega_P^i) = \prod_j (1 - \alpha_j(i)X),$$

and set

$$g(X, \omega_P^i) = \prod_{v_p(\alpha_j(i)-1)>0} (1 - \alpha_j(i)X).$$

Let $i \in \mathbb{N}$. We say that i is a q-magic number if there exist $c \in \{0, \ldots, q-2\}$ and an integer $n \in \mathbb{N}$ such that $i = cq^n + q^n - 1$.

Proposition 5.1. *Let P be a prime of A of degree d. Let i be a q-magic number, $1 \leqslant i \leqslant q^d - 2$, $i \equiv 0 \pmod{q-1}$. Then $g(X, \omega_P^i)$ has simple roots.*

Proof. We have $i = q^n - 1$ for some integer n, $1 \leqslant n \leqslant d-1$. By a result of Carlitz [2, Lemma 8.22.4]:

$$B(q^d - 1 - i) = \frac{(-1)^{d-n}}{L_{d-n}^{q^n}},$$

where $L_0 = 1$ and for $j \geqslant 1$, $L_j = (T^{q^j} - T)L_{j-1}$. Thus

$$B(q^d - 1 - i) \not\equiv 0 \pmod P.$$

Therefore,

$$\mathrm{Cl}(O_{K_P})_p(\omega^{-i}) = \{0\}.$$

It remains to apply Proposition 4.3. $\square$

In [2], Goss makes the following conjecture:
Let P be a prime of degree d and let i be a q-magic number, $1 \leqslant i \leqslant q^d - 2$. Then $\deg_X g(X, \omega_P^i) \leqslant 1$.

Note that the results of Section 3 do not give any counter-example to Goss conjecture. It is natural to ask if there exist primes P and q-magic numbers i, $1 \leqslant i \leqslant q^{\deg P} - 2$, such that $\deg_X g(X, \omega_P^i) \geqslant 1$. This is the case.

Proposition 5.2. *Let $c \in \{0, \ldots, q-2\}$. There exist infinitely many primes P such that*

$$\prod_{n=1}^{\deg P - 1} \beta(cq^n + q^n - 1) \equiv 0 \pmod P.$$

Proof. We prove this proposition for $c \neq 0$. This proof for $c = 0$ is very similar.

Let us recall some results from the work of Sheats [8]. Let $m \geqslant 1$ be an integer. Let $X = (X_1, \ldots, X_m) \in \mathbb{N}^m$, we set

$$pd(X) = X_1 + 2X_2 + \cdots + mX_m.$$

Let $V \subset \mathbb{N}^m$ be a finite set, an element $O \in V$ is called optimal if $\forall X \in V$, $pd(X) \leqslant pd(O)$.

Let $k \geqslant 0$ and $i \geqslant 1$ be two integers. Let $U_{k+1}(i)$ be the set of elements $r = (r_0, r_1, \ldots, r_k) \in \mathbb{N}^{k+1}$ such that

- $r_0 + r_1 + \cdots + r_k = i$,
- in the sum $r_0 + r_1 + \cdots + r_k$ there is no carry over p-adic digits,
- for $0 \leqslant j \leqslant k-1$, $r_j \geqslant 1$ and $r_j \equiv 0 \pmod{q-1}$

for $j \geqslant 0$ and $i \geqslant 1$, set

$$S_j(i) = \sum_{a \in A^+, \deg(a)=j} a^i.$$

Then:

- $S_j(i) \neq 0$ if and only if $U_{j+1}(i) \neq \emptyset$,
- if $S_j(i) \neq 0$, there exists an unique optimal element G in $U_{j+1}(i)$ and $\deg_T(S_j(i)) = pd(G) - i$.

Let us apply the above results. Since $c \in \{1, \dots, q-2\}$, we get

- $S_j(cq^n + q^n - 1) = 0$ if and only if $j \geqslant n+1$,
- for $j \leqslant n$, the optimal element of $U_{j+1}(cq^n + q^n - 1)$ is $G_j = (q-1, q(q-1), \dots, q^{j-1}(q-1), q^j(q-1) + \dots + q^{n-1}(q-1) + cq^n)$.

If $t_j = pd(G_j)$, we observe that for $1 \leqslant j \leqslant n-1$, $t_j < t_{j+1}$. Therefore

$$\begin{aligned}\deg_T \beta(cq^n + q^n - 1) &= (n+1)cq^n + nq^{n-1}(q-1) + \cdots \\ &\quad + 2q(q-1) + (q-1) - cq^n - q^n + 1,\end{aligned}$$

thus

$$\deg_T \beta(cq^n + q^n - 1) = n(c+1)q^n - \frac{q^{n+1} - q}{q-1}.$$

Let S be the set of primes P in A such that

$$\prod_{i=1}^{\deg P - 1} \beta(cq^n + q^n - 1) \equiv 0 \pmod{P}.$$

Let's assume that S is a finite set. We set

$$D = \prod_{P \in S} \deg P$$

and $D = 1$ if $S = \emptyset$. Note that

$$\forall P \in S, \quad q^D \equiv 1 \pmod{q^{\deg P} - 1}.$$

Therefore, since $\beta(c) = 1$, we have

$$\forall P \in S, \quad \beta(cq^D + q^D - 1) \equiv 1 \pmod{P}.$$

But $\deg_T \beta(cq^D + q^D - 1) \geqslant 1$, thus we can select a prime Q of A such that $\beta(cq^D + q^D - 1) \equiv 0 \pmod{Q}$. Note that $Q \notin S$. Set $d = \deg Q$. Since d does not divide D, there exists an integer r, $1 \leqslant r \leqslant d-1$, such that $D \equiv r \pmod{d}$. Therefore

$$\beta(cq^D + q^D - 1) \equiv \beta(cq^r + q^r - 1) \equiv 0 \pmod{Q}.$$

But this implies that $Q \in S$, which is a contradiction. $\square$

Let P be a prime of A of degree d. Let J be the jacobian of K_P, i.e. J is the inductive limit of the $\mathrm{Cl}^0(\mathbb{F}_{q^n} K_P)$, $n \geqslant 1$. Set $\mathbb{F}_{q^{p^\infty}} = \bigcup_{n \geqslant 0} \mathbb{F}_{q^{p^n}} \subset \overline{\mathbb{F}_q}$, where $\overline{\mathbb{F}_q}$ is the algebraic closure of $\mathbb{F}_q$ in $\bar{k}$. We consider the $\Delta = \mathrm{Gal}(K_P/k)$-module:

$$\mathcal{A}_P = \frac{J[p]^{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_{q^{p^\infty}})}}{\mathrm{Cl}^0(K_P)[p]}.$$

As a consequence of the results in Section 4, we get:

Proposition 5.3. *Let $W = \mathbb{Z}_p[\mu_{q^d-1}]$ and let $\chi \in \hat{\Delta}$. We have*

$$\dim_{\frac{W}{pW}} \mathcal{A}_P(\chi) = \deg_X g(X, \bar{\chi}) - \dim_{\frac{W}{pW}} \mathrm{Cl}^0(K_P)_p(\chi).$$

Note that in general, by Proposition 3.6, we do not have $\mathcal{A}_P = \{0\}$. But Goss conjecture implies the following:
Let P be a prime of A of degree d and let i be a q-magic number, $1 \leqslant i \leqslant q^d - 2$, then $\mathcal{A}_P(\omega_P^{-i}) = \{0\}$.

It would be interesting to prove (or find a counter-example) to this weak form of Goss conjecture.

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