Distance to $C^k$ Hypersurfaces

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It is shown by elementary means that a $C^k$ hypersurface $M$ of positive reach in $\mathbb{R}^{n+1}$ has the property that the signed distance function to it is $C^k$, $k \geq 1$. This extends and complements work of Federer, Gilbarg and Trudinger, and Serrin.

INTRODUCTION

In the study of various problems of analysis, for example, the Dirichlet problem for quasilinear differential equations, the function $\delta_M = \delta_M(x)$, defined as the distance from $x$ to an $n$-dimensional manifold $M$ in $\mathbb{R}^{n+1}$, is a useful tool. One must ensure that $\delta_M$ be sufficiently differentiable for one's purposes. The most natural result to assert regarding the relationship between the differentiability properties of $M$ and those of $\delta_M$ is apparently that if $M$ is a $C^k$ submanifold, with $k \geq 2$, then, near $M$, $\delta_M$ is of class $C^{k-1}$. This is what is asserted in Lemma 3.1 of Chapter I of Serrin's paper [3], with $k = 3$, for example.

In order to set one's theorems and constructions in the most natural, or most general, context, it is essential to know that the above result can be strengthened. In fact, in the Appendix to their book, Gilbarg and Trudinger [2] state and prove that if $M$ is a $C^k$ manifold, with $k \geq 2$, then, near $M$, $\delta_M$ is of class $C^k$. This extra smoothness of $\delta_M$ is surprising and remains little known.

The purpose of this note is to draw attention to the lemma of Gilbarg and Trudinger and to give an alternate proof which, it is felt, exposes what is geometrically significant in the relationship between $M$ and $\delta_M$. The key element in the analysis of $\delta_M$ is the fact that if $M$ is $C^2$, then $M$ is of positive reach; this result is due to Federer (see [1]). Given that $M$ is of positive reach, the differentiability properties of $\delta_M$ follow from an elementary, but elegant and surprising, computation. As an additional benefit of our more explicit proof, we are able to extend the result of Gilbarg and Trudinger to the $C^1$ case. The problem we address here was left open in [1] and its

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resolution completes the analysis, initiated by Federer, of the distance function.

It follows from [1; 4.8 (3)] that, in the absence of the positive reach assumption, $\delta_M$ need not even be differentiable near $M$. We give an example in Section 4 of a compact $C^2$ curve in $\mathbb{R}^2$ which does not have positive reach.

1. NOTATION AND DEFINITIONS. Let $M \subset \mathbb{R}^{n+1}$ be a compact $C^k$ ($k \geq 1$) submanifold of dimension $n$ and suppose $M = \partial \Omega$ for some open $\Omega \subset \mathbb{R}^{n+1}$; set

$$\delta(x) = \begin{cases} \text{dist}(x, M) & \text{if } x \in \Omega, \\ \text{dis}(x, M) & \text{if } x \in \mathbb{R}^{n+1} \sim \Omega. \end{cases}$$

If $x \in \mathbb{R}^m$, $r > 0$, then $U^m(x, r)$ denotes the euclidean $m$-ball with center $x$ and radius $r$. Let $F \subset \mathbb{R}^{n+1}$, $\delta > 0$. Define $U_{\delta}(F) = \{x \in \mathbb{R}^{n+1}: \text{dist}(x, F) < \delta\}$. The set $F$ is said to be of positive reach if there is a $\delta > 0$ so that each $x \in U_{\delta}(F)$ has a unique nearest point in $F$. Define reach$(F)$ to be the greatest such $\delta$ (possibly $+\infty$).

2. THEOREM. If $k = 1$ and reach$(M) > 0$ hold, then there exists an open neighborhood $U$ of $M$ such that $\delta_M|U$ is a $C^1$ function.

Proof. Fix a point $P \in M$ and choose coordinates so that $P$ is at the origin and so that a neighborhood of $P$ in $M$ is given by $\{(t_1, \ldots, t_n, f(t_1, \ldots, t_n)) : (t_1, \ldots, t_n) \in U^m(0, r)\}$, where $r > 0$ and $f$ is $C^1$ with $f(0) = 0$ and $\left(\frac{\partial f}{\partial t_j}\right)(0) = 0$ for $j = 1, \ldots, n$. We need only show that there is an open ball containing $P$ on which $\delta_M$ is $C^1$.

We can choose $\mu$ with $0 < \mu < \text{reach}(M)$ such that if we set $W = U^{m+1}(0, \mu)$, then for $(x, y) = (x_1, \ldots, x_n, y) \in W$ the unique nearest point of $M$, denoted by $(t_1, \ldots, t_n, f(t_1, \ldots, t_n))$, where $t_j = t_j(x, y)$ for $j = 1, \ldots, n$, satisfies

$$\left(\sum_{j=1}^{n} t_j^2\right)^{1/2} < r.$$

Now, $(t_1(x, y), \ldots, t_n(x, y))$ is an extreme point for the function

$$\sum_{j=1}^{n} (t_j - x_j) + (f(t_1, \ldots, t_n) - y)^2.$$
of \((t_1, \ldots, t_n)\). Thus we have

\[
0 = \frac{\partial}{\partial t_i} \left[ \sum_{j=1}^{n} (t_j - x_j)^2 + (f(t) - y)^2 \right]
\]

\[
= 2(t_i - x_i) + 2(f(t) - y) \frac{\partial f}{\partial t_i}.
\]  

(1)

By \([1, 4.8(4)]\), \(t_j(x, y), j = 1, \ldots, n\), is a continuous function of \((x, y) \in W\).

Consider a point \((x, y) \in W\). Using Eq. (1), we compute

\[
(\delta_m)^2 (x, y) = \sum_{j=1}^{n} (t_j - x_j)^2 + (f(t) - y)^2
\]

\[
= \sum_{j=1}^{n} \left( (f(t) - y) \frac{\partial f}{\partial t_j} \right)^2 + (f(t) - y)^2
\]

\[
= (f(t) - y)^2 \left( 1 + \sum_{j=1}^{n} \left( \frac{\partial f}{\partial t_j} \right)^2 \right).
\]

Thus we may suppose, without loss of generality, that

\[
\delta_m(x, y) = (f(t) - y) \left( 1 + \sum_{j=1}^{n} \left( \frac{\partial f}{\partial t_j} \right)^2 \right)^{-1/2}
\]  

(2)

holds.

For \((x, y) \in W \sim M\) we have, by \([1; 4.8(3, 5)]\) and Eqs. (1) and (2),

\[
\frac{\partial}{\partial x_i} (\delta_m) = (\delta_m)^{-1} (x_i - t_i)
\]

\[
= \frac{(x_i - t_i)}{(f(t) - y)} \left( 1 + \sum_{j=1}^{n} \left( \frac{\partial f}{\partial t_j} \right)^2 \right)^{-1/2}
\]

\[
= \frac{\partial f}{\partial t_i} \cdot \left( 1 + \sum_{j=1}^{n} \left( \frac{\partial f}{\partial t_j} \right)^2 \right)^{-1/2}
\]  

(3)

and

\[
\frac{\partial}{\partial y} (\delta_m) = (\delta_m)^{-1} (y - f(t))
\]

\[
= - \left( 1 + \sum_{j=1}^{n} \left( \frac{\partial f}{\partial t_j} \right)^2 \right)^{-1/2}.
\]  

(4)
Now, the right-hand sides of Eqs. (3) and (4) are continuous functions of \((x, y)\) on all of \(\mathcal{W}\), so, by [1; 4.7], (3) and (4) hold for all \((x, y) \in \mathcal{W}\).

3. **Theorem.** If \(k \geq 2\) holds, then there is an open neighborhood \(U\) of \(M\) such that \(\delta_{\mathcal{W}}| U\) is a \(C^k\) function.

**Proof.** By [1; 4.12] we have \(\text{reach}(M) > 0\). Proceeding as in the proof of Theorem 2 we obtain Eq. (1). Choosing a smaller ball, \(W\), about \(P\) if necessary, we see by the implicit function theorem that \(t_j(x, y), j = 1, \ldots, n,\) is a \(C^{k-1}\) function on \(W\). As before we obtain Eqs. (3) and (4) from which we conclude that \((\partial/\partial x_i)(\delta_{\mathcal{W}}), i = 1, \ldots, n,\) and \((\partial/\partial y)(\delta_{\mathcal{W}})\) are \(C^{k-1}\) functions.

4. **Example.** There is a compact \(C^{2-\epsilon}\) curve \(\gamma\) in \(\mathbb{R}^2\) with \(\text{reach}(\gamma) = 0\). Moreover, \(\delta_{\gamma}|(U - \gamma)\) is not even differentiable for any neighborhood \(U\) of \(\gamma\).

Set 
\[
\gamma = \{(t, |t|^{2-\epsilon}) : -1 \leq t \leq 1\} \cup \gamma',
\]
where \(\gamma'\) is any arc in \(\{(x, y) \in \mathbb{R}^2 : y \geq 1\}\) connecting \((+1, 1)\) to \((-1, 1)\) which makes \(\gamma\) a compact \(C^{2-\epsilon}\) curve. Set 
\[
A = (0, a), \quad a > 0.
\]
We will show that for all sufficiently small choices of \(a\), the nearest point of \(\gamma\) to \(A\) is not the origin. Since \(\{(t, |t|^{2-\epsilon}) : -1 \leq t \leq 1\}\) is symmetric about the \(y\) axis, it then follows that \(A\) has at least two nearest points on \(\gamma\). Now if 
\[
\text{dist}[(t, |t|^{2-\epsilon}), A] \geq \text{dist}[(0, 0), A],
\]
then 
\[
t^2 + (|t|^{2-\epsilon} - a)^2 \geq 0,
\]
or, if \(t > 0\),
\[
t^{2-\epsilon}(t^2 + t^{2-\epsilon} - 2a) \geq 0
\]
or
\[
t^2 + t^{2-\epsilon} - 2a \geq 0,
\]
which is false if \(t\) is small enough. The second conclusion follows from [1; 4.8(3)].
Note added in proof: Theorem 3 remains true with $k = \infty$ and with $k = \omega$. This follows from Eq. (2), since, in this case, $f(x, y)$ is $C^k$ by the implicit function theorem (cf. J. Dieudonné, "Foundations of Modern Analysis," Academic Press, New York/London, 1960, Sect. 10.2).

It has been shown that, in case $k = 1$, $M$ is of positive reach if and only if the unit normal vector to $M$ satisfies a lipschitz condition with exponent 1 (cf. K. Lucas, "Submanifolds of Dimension $n - 1$ in $\mathbb{R}^n$ With Normals Satisfying a Lipschitz Condition, Studies in Eigenvalue Problems," Technical Report 18, Department of Mathematics, University of Kansas, 1957, Sect. 2).

**References**