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Nuclear Physics B 824 (2010) 273–295

NUCLEAR PHYSICS B

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A note on local GUT models in F-theory

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Received 3 April 2009; received in revised form 8 August 2009; accepted 4 September 2009

Available online 8 September 2009

Abstract

We construct non-minimal GUT local models in the F-theory configuration. The gauge group on the bulk G_S is one rank higher than the GUT gauge group. The line bundles on the curves are nontrivial to break G_S down to the GUT gauge groups. We demonstrate examples of $SU(5)$ GUT from $G_S = SU(6)$ and $G_S = SO(10)$, the flipped $SU(5)$ from $G_S = SO(10)$, and the $SO(10)$ GUT from $G_S = SO(12)$ and $G_S = E_6$. We obtain complete GUT matter spectra and couplings, with minimum exotic matter contents. GUT gauge group breaking to MSSM is achievable by instanton configurations.

1. Introduction

String theory is so far the most promising candidate of the unified theory as an extension of quantum field theory and a consistent quantum theory of gravity. It is expected to answer the fundamental questions in physics. Many of these questions can be explained by the extra dimensions or by the internal manifold from the string compactification point of view. On the other hand, one of the fundamental issues to be addressed from particle physics is the unification of gauge couplings. The natural solution to this question is the framework of the grand unified theory (GUT). There are two procedures to realize GUTs in the string theory compactification. The first is the top-down procedure in which the full compactification is consistent with the conditions of global geometry of extra dimensions and then the spectrum is close to GUT after breaking some symmetries [1]. In the bottom-up procedure, this gauge breaking can be understood in the decoupling

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0550-3213/$ – see front matter
doi:10.1016/j.nuclphysb.2009.09.008

limit of gravity [2,3], particularly in the framework that D-branes are introduced on the local regions within the extra dimensions in type IIB compactification [2–4]. In this case we can neglect the effects from the global geometry. In principle, the top-down procedure is the more satisfactory scenario theoretically than the bottom-up procedure. However, the later procedure is more efficient for model building than the former one.

There is no local model in type I and heterotic string compactifications since the matter fields live in the entire extra dimensions. It is possible to construct D-brane local and global models in type IIB compactification, however it is difficult to engineer the $\mathbf{10}\,\mathbf{10}\,\mathbf{5}_H$ coupling in a GUT model. This problem can be traced to the non-realization of the exceptional gauge groups in type IIB. In the perturbative type IIB theory, an $SU(N)$ and an $SO(2N)$ gauge group can be realized as N D-branes and N D-branes along O-planes, respectively [5]. The anti-symmetric representations of a GUT come from the intersection of a stack of D-branes and its image (as well as the orientifold), and it is not possible in this construction to find another such intersection to finish the Yukawa coupling without introducing exotic matter. Recently this problem is solved in the type IIB orientifold configuration with non-perturbative instantons corrections [6] based on [7]. On the other hand, the exceptional groups are believed to exist in the non-perturbative regime of type IIB theory. It is well known that the strong coupling version of type IIB theory can be realized as F-theory [8]. Actually, those gauge groups of ADE-type are naturally encoded in the geometry of the F-theory compactification [9,10]. Thus F-theory is a natural choice for local GUT model building.

F-theory is a non-perturbative 12-d theory built on the type IIB framework with an auxiliary two-torus ([8], see [11] for review). The ordinary string extra dimensions are regarded as a base B and the two-torus is equivalent to an elliptic curve as a fiber on this base manifold. The modulus of the elliptic curve is identified as axion-dilaton in type IIB theory. Due to the $SL(2,Z)$ monodromy of the modulus, F-theory is essentially non-perturbative in type IIB language. The locations of fiber degeneracies are defined by a codimension-one locus Δ within B, which also indicates the locations of seven-branes. The fiber degeneracies lead to singularities whose nature determines the worldvolume gauge groups of ADE-type on the seven-branes [9]. In the strong version of the local model, the gravity is decoupled from the gauge theory, so we can focus on the local properties by restricting the geometries on the submanifold S, which is a component of Δ and is wrapped by seven-branes. In order to achieve that, the volume of S is required to be contractible to zero size,[1] which is followed from the condition that the anti-canonical bundle K_S^{-1} of S is ample. It implies that S is a del Pezzo surface [12–14]. Given a Kähler surface S, the maximal supersymmetric Yang–Mills theory in 8-d admits a unique twist on $\mathbb{R}^{3,1}\times S$ which preserves $\mathcal{N}=1$ SUSY in $\mathbb{R}^{3,1}$ [12,13]. Matter comes from two sources, one is from the irreducible subgroups of the bulk gauge group by turning on nontrivial gauge bundles on S, and the other is from the intersection of two del Pezzo surfaces along a codimension-two Riemann surface Σ, which is the intersecting brane picture in type IIB theory [10]. Along this curve Σ the gauge group is enhanced and is able to be broken again by the nontrivial gauge bundles on it. The Yukawa couplings can be realized as couplings of either two fields from different curves intersecting at a point and a field from the bulk, or three fields from different curves intersecting at the same point, where the singularity is further enhanced [12,13]. The generation numbers of matter on the bulk and on the curve Σ are then determined by the dimensions of the bundle-

[1] There are two ways in which we could take $V_S\to 0$. The first one is requiring S to contract to a point, and the second is requiring S to contract to a curve of singularities. See [14] for the details.

valued cohomology groups on S and Σ, respectively [12,13]. One of the advantages of F-theory is that it naturally explains the unification of the gauge couplings.

Recently some local GUT models are built in this F-theory configuration [12,13,15–22], and some progresses in global models [23,24]. Supersymmetry breaking is discussed in [25–27], and the application to cosmology is studied in [28]. From [12,29], the upper bound on the rank of a candidate GUT group is six. In [12,15], the authors consider the minimal construction by using rank four gauge group $SU(5)$ to build $SU(5)$ GUT, and show some examples of exotic-free models. These models do not have the problems that a GUT model may have, such as proton decay, doublet–triplet splitting and so on. In this note, we shall consider non-minimal constructions of the GUT models, namely we consider rank five and six gauge groups to build local GUT models in F-theory.

In Section 2 of this paper, we briefly review F-theory and the construction in [12,13]. In Section 3, we shall consider $SU(5)$, flipped $SU(5)$ and $SO(10)$ GUT models from non-minimal gauge groups on S, and we conclude in Section 4. In Appendices A, B we collect some properties of del Pezzo surfaces and resolutions of triplet intersections for the Yukawa couplings.

2. F-Theory GUT models

The construction of local GUT models in F-theory has been analyzed in [12,13,15]. In this section we shall briefly review the essential ingredient of this construction, where the details can be found in [12,13,15]. Consider F-theory on an elliptically fibered Calabi–Yau four-fold X with base B. Generically, the fiber degenerates on the codimension-one reducible locus Δ within B. In local F-theory models, we focus on one component S of the locus Δ. S is a codimension one complex surface wrapped by seven-branes and supporting GUT models. The spirit of the bottom-up procedure leads to the choice of S being a del Pezzo surface [12,13,15]. To describe the spectrum of a local model, one has to study the gauge theory of the worldvolume on the seven-branes. As emphasized in [12,13], one can start from the maximal supersymmetric gauge theory on $\mathbb{R}^{3,1} \times \mathbb{C}^2$ and then replace $\mathbb{C}^2$ with the Kähler surface S. In order to make the low energy gauge theory preserve four supercharges, the maximal supersymmetric gauge theory on $\mathbb{R}^{3,1} \times \mathbb{C}^2$ should be twisted. It is shown that there exists a unique twist preserving $\mathcal{N} = 1$ supersymmetry in four dimensions and chiral matter can arise from the bulk S or the curve Σ [12,13,15].

Let us first discuss the spectrum of the bulk fields on S. The ADE-type singularity along S is corresponding to the gauge group G_S on S from seven-branes, and a nontrivial vector bundle over S with a structure group H_S leads to the unbroken gauge group Γ_S in four dimensions which is the commutant subgroup of H_S in G_S. After compactifying on S, the resulting theory is $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group Γ_S coupled to matter. The spectrum of the bulk theory on S transforms in the adjoint representation of G_S. The decomposition of $\operatorname{ad} G_S$ into representations of $\Gamma_S \times H_S$ is

$$\operatorname{ad} G_S = \bigoplus_k \rho_k \otimes \mathcal{R}_k, \tag{1}$$

where ρ_k and $\mathcal{R}_k$ are representations of Γ_S and H_S, respectively. The matter fields are determined by the zero modes of the Dirac operator on S. It is shown in [12,13] that the chiral and anti-chiral spectrum is determined by the bundle-valued cohomology groups

$$H^0_{\bar\partial}\big(S, R_k^\vee\big)^\vee \oplus H^1_{\bar\partial}(S, R_k) \oplus H^2_{\bar\partial}\big(S, R_k^\vee\big)^\vee \tag{2}$$

and

$$H^0_{\bar\partial}(S,R_k)\oplus H^1_{\bar\partial}\big(S,R_k^\vee\big)^\vee\oplus H^2_{\bar\partial}(S,R_k),\tag{3}$$

respectively, where $\vee$ stands for the dual bundle and R_k is the vector bundle on S whose sections transform in the representation $\mathcal{R}_k$ of the structure group H_S. Thus, the net number of the chiral field ρ_k and anti-chiral field ρ_k^* is given by

$$N_{\rho_k}-N_{\rho_k^*}=\chi\big(S,R_k^\vee\big)-\chi(S,R_k)=-\int_S c_1(R_k)c_1(S).\tag{4}$$

Moreover, by the vanishing theorem of del Pezzo surfaces [12] it shows that when $R_k\neq\mathcal{O}_S$, then $H^0_{\bar\partial}(S,R_k)=0$ and $H^2_{\bar\partial}(S,R_k)=0$. Thus the number of generations and anti-generations can be calculated by

$$N_{\rho_k}=-\chi(S,R_k)\tag{5}$$

and

$$N_{\rho_k^*}=-\chi\big(S,R_k^\vee\big),\tag{6}$$

respectively.

In particular, when a gauge bundle is a line bundle L with structure group $U(1)$, according to Eq. (5), the chiral spectrum of ρ_r is determined by

$$N_{\rho_r}=-\chi\big(S,L^r\big)=-\left[1+\frac{1}{2}\left(\int_S c_1\big(L^r\big)c_1(S)+\int_S c_1\big(L^r\big)^2\right)\right],\tag{7}$$

where r corresponds to the $U(1)$ charges of the representations in the group theory decomposition. In order to preserve supersymmetry, the line bundle L has to obey the BPS equation [12,13]

$$J_S\wedge c_1(L)=0,\tag{8}$$

where J_S is the Kähler form on S and its expression can be found in Appendices A, B. According to Eq. (7), by switching on the suitable supersymmetric line bundle which satisfies the condition $c_1(L)c_1(S)=0$, the bulk fields ρ_r and $\bar\rho_{-r}$ form a vector-like pair or vanish, depending on the value of $c_1(L)^2$.

Another way to obtain chiral matter is from intersecting seven-branes along a curve, which is a Riemann surface. Let S and S' be two components of the discriminant locus Δ with gauge groups G_S and $G_{S'}$, respectively intersecting along a curve Σ. The gauge group on the curve Σ will be enhanced to G_Σ, where $G_\Sigma\supset G_S\times G_{S'}$. Therefore, chiral matter appears as the bi-fundamental representations in the decomposition of $\operatorname{ad}G_\Sigma$

$$\operatorname{ad}G_\Sigma=\operatorname{ad}G_S\oplus\operatorname{ad}G_{S'}\oplus_k\big(\mathcal{U}_k\otimes\mathcal{U}'_k\big).\tag{9}$$

As mentioned above, the presence of H_S and $H_{S'}$ will break $G_S\times G_{S'}$ to the commutant subgroup when nontrivial gauge bundles on S and S' with structure groups H_S and $H_{S'}$ are turned on. Let $\Gamma=\Gamma_S\times\Gamma_{S'}$ and $H=H_S\times H_{S'}$, the decomposition of $\mathcal{U}\otimes\mathcal{U}'$ into irreducible representation is

$$\mathcal{U}\otimes\mathcal{U}'=\bigoplus_k(v_k,\mathcal{V}_k),\tag{10}$$

where v_k and $\mathcal{V}_k$ are representations of Γ and H, respectively. The light chiral fermions in the representation v_k are determined by the zero modes of the Dirac operator on Σ. It is shown in [12,13] that the net number of the chiral field v_k and anti-chiral field v_k^* is given by

$$N_{v_k} - N_{v_k^*} = \chi\big(\Sigma, K_\Sigma^{1/2} \otimes V_k\big), \tag{11}$$

where V_k is the vector bundle whose sections transform in the representation $\mathcal{V}_k$ of the structure group H. In particular, if H_S and $H_{S'}$ are $U(1)$ gauge groups, the vector bundles over S and S' reduce into line bundles L and L', respectively, then the adjoint representation $\operatorname{ad} G_\Sigma$ will be decomposed into

$$\operatorname{ad} G_S \oplus \operatorname{ad} G_{S'} \oplus_j (\sigma_j, \sigma'_j)_{r_j, r'_j}, \tag{12}$$

where r_j and r'_j correspond to the $U(1)$ charges of the representations in the group theory decomposition. The bi-fundamental representation $(\sigma_j, \sigma'_j)_{r_j, r'_j}$ are localized on Σ [10,12,13]. As shown in [12,13], the generation number of the representation $(\sigma_j, \sigma'_j)_{r_j, r'_j}$ can be calculated by

$$N_{(\sigma_j,\sigma'_j)_{r_j,r'_j}} = h^0\big(\Sigma, K_\Sigma^{1/2} \otimes L_\Sigma^{r_j} \otimes L_\Sigma^{\prime r'_j}\big), \tag{13}$$

where the restrictions of line bundles to Σ are denoted by $L_\Sigma^{r_j} \equiv L^{r_j}|_\Sigma$ and $L_\Sigma^{\prime r'_j} \equiv L^{\prime r'_j}|_\Sigma$, respectively. It follows that the net chirality on Σ is given by

$$N_{(\sigma_j,\sigma'_j)_{r_j,r'_j}} - N_{\overline{(\sigma_j,\sigma'_j)_{r_j,r'_j}}} = c_1\big(L_\Sigma^{r_j} \otimes L_\Sigma^{\prime r'_j}\big). \tag{14}$$

In addition to the analysis of the spectrum, the pattern of Yukawa couplings is also studied [12,13,24]. By the vanishing theorem of del Pezzo surfaces [12,13], Yukawa couplings can form in two different ways. In the first type, the coupling comes from the interaction between two fields on the curves and one field on the bulk S. In the second type, all three fields are localized on the curves which intersect at a point where the gauge group G_p is further enhanced by two ranks. In the paper, we shall primarily focus on the couplings of the second case.

3. Model building

In this section we shall explore $SU(5)$, $SO(10)$ and flipped $SU(5)$ GUT models by taking G_S as higher rank groups. The $SU(5)$ models from $G_S = SU(5)$ and the $SO(10)$ models from $G_S = SO(10)$ have been discussed in [12,13,15]. In these models, the restriction of line bundles on the bulk to the matter curves are required to be trivial to maintain the GUT fermion spectrum, while they are nontrivial on the curves for Higgs fields to explain the phenomenology of doublet–triplet splitting when GUT breaks to the Minimum Supersymmetric Standard Model (MSSM). The curve self-intersection mechanism makes it possible to explain the rank three quark and lepton mass matrices from the Yukawa couplings. The bulk line bundle can be nontrivial on the matter curves, which is useful in discussing a flipped $SU(5)$ model [20], and a rich SM Yukawa mass structure [18].

We shall mainly focus on the cases that the gauge groups on S have higher ranks than the GUT gauge groups, so the bulk line bundles will be nontrivial on all the curves to obtain GUT spectra. There is no GUT adjoint representation on a del Pezzo surface, but it is still possible to break the GUT gauge groups to the Standard Model (SM) gauge group by introducing non-Abelian instanton configurations on the bulk [15]. For the maximum degrees of freedom of model building, the del Pezzo surfaces in the following models are all dP_8.

3.1. $SU(5)$ GUT

3.1.1. $G_S = SU(6)$

Consider seven-branes wrapping on a del Pezzo surface $S = dP_8$ with $G_S = SU(6)$. From Eq. (7), the bulk field ρ_r is determined by the bundle-valued Euler characteristic $\chi(S, L^r)$ where r is the $U(1)$ charge in the group theory decomposition. According to the property of the Chern class, $c_n(L^{-r}) = (-1)^n c_n(L^r)$, where L^{-r} is the dual bundle of L^r. In particular, when $n = 1$ we obtain $c_1(L^{-r}) = -c_1(L^r)$, and it turns out that $N_{\rho_r} - N_{\bar{\rho}_{-r}} = -r\int c_1(L)c_1(S)$. If $N_{\rho_r} \neq 0$, it implies that the bulk fields ρ_r and $\bar{\rho}_{-r}$ form a vector-like pair if

$$c_1(L)c_1(S) = 0, \tag{15}$$

for example, $L = \mathcal{O}_S(\sum_{m=1}^{2l}(-1)^{m+1}E_{i_m})$, $l \leqslant 4$, where all indices are distinct. It is easy to see that it solves Eq. (15) and the BPS equation (8) by choosing suitable polarization of J_S, for example, $J_S = AH - \sum_{i=1}^{8} E_i$, $A \gg 1$. If L is a line bundle satisfying $\chi(S, L^r) = \chi(S, L^{-r}) = 0$, then $N_{\rho_r} = N_{\bar{\rho}_{-r}} = 0$. In other words, no chiral field lives on the bulk. In this case, it is not difficult to find that $L = \mathcal{O}_S(E_i - E_j)^{1/r}$, $i \neq j$, which is a well-defined fractional line bundle[2] due to the fact that $c_1(L^r)$ is a integer class [12,13,15].

In this model where $G_S = SU(6)$, the possible breaking patterns on the local curve by $U(1)$ line bundle from S' and by $U(1)_S$ line bundle on the bulk are [30]:

$$\begin{aligned}
&SU(7) \to SU(6)_S \times U(1) \to SU(5) \times U(1) \times U(1)_S,\\
&48 \quad\ \to 35_0 + 1_0 \qquad\quad \to 24_{0,0} + 1_{0,0} + 5_{0,6} + \bar{5}_{0,-6} + 1_{0,0}\\
&\qquad\quad + 6_{-7} + \bar{6}_7 \qquad\quad + 5_{-7,1} + 1_{-7,-5} + \bar{5}_{7,-1} + 1_{7,5},
\end{aligned} \tag{16}$$

$$\begin{aligned}
&SO(12) \to SU(6)_S \times U(1) \to SU(5) \times U(1) \times U(1)_S,\\
&66 \quad\ \to 35_0 + 1_0 \qquad\quad \to 24_{0,0} + 1_{0,0} + 5_{0,6} + \bar{5}_{0,-6} + 1_{0,0}\\
&\qquad\quad + 15_2 + \overline{15}_{-2} \qquad + 10_{2,2} + 5_{2,-4} + \overline{10}_{-2,-2} + \bar{5}_{-2,4},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&E_6 \to SU(6)_S \times U(1) \to SU(5) \times U(1) \times U(1)_S,\\
&78 \to 35_0 + 1_0 + \mathit{1}_{\pm 2} \to 24_{0,0} + 2 \times 1_{0,0} + 5_{0,6} + \bar{5}_{0,-6} + \mathit{1}_{\pm 2,0}\\
&\qquad + 20_1 + \mathit{20}_{-1} \qquad + 10_{1,-3} + \overline{10}_{1,3} + \mathit{10}_{-1,-3} + \overline{\mathit{10}}_{-1,3}.
\end{aligned} \tag{18}$$

We shall consider the supersymmetric line bundle $L = \mathcal{O}_S(E_1 - E_2)^{1/6}$ so that there is no chiral field on the bulk, *i.e.* $N_{\mathbf{5}_6} = N_{\bar{\mathbf{5}}_{-6}} = 0$. Therefore, there is no Yukawa coupling of $\Sigma\Sigma S$-type, such as $\mathbf{10}_{-1,-3}\mathbf{10}_{-1,-3}\mathbf{5}_{0,6}$, $\mathbf{10}_{2,2}\bar{\mathbf{5}}_{-2,4}\bar{\mathbf{5}}_{0,-6}$ and their complex conjugates. The first $U(1)$ charge of each representation is from S' and the second is from the bulk. Since the bulk line bundle is not trivial in our discussion, the $U(1)_S$ charges should be conserved in each Yukawa coupling.

[2] It is not the only solution, for example, it could be $L = \mathcal{O}_S(\sum_{m=1}^{8}(-1)^{m+1}E_m)^{1/2r}$. However, $L = \mathcal{O}_S(E_i - E_j)^{1/r}$, $i \neq j$, is the only solution that $c_1(L^r) \in H_2(S, \mathbb{Z})$.

Now let us turn to the chiral spectra from the curves. The same representation can come from alternate breaking patterns giving varied charges. The difference from the cases in [12,15] is that the restriction of bulk fluxes to the matter curves are nontrivial here. Therefore we have to choose proper representations from the curves that intersect at a double enhanced point forming the corresponding Yukawa coupling. One possible choice of such $SU(5)$ model from $G_S = SU(6)$ in terms of the matter representations on the curves is:

$$\mathcal{W} \supset \mathbf{10}_{2,2}\mathbf{10}_{2,2}\mathbf{5}_{2,-4} + \mathbf{10}_{2,2}\bar{\mathbf{5}}_{7,-1}\bar{\mathbf{5}}_{7,-1} + \cdots. \tag{19}$$

The corresponding Yukawa coupling patterns on the double enhanced points of $\mathbf{10}\,\mathbf{10}\,\mathbf{5}$ and $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$ can be found in Eqs. (B.8) and (B.4), respectively.

In what follows, we engineer the minimal spectrum by introducing suitable supersymmetric line bundles. Let L and L' be the line bundles over S and S' respectively, and consider Σ to be a curve of genus zero. Let $L_\Sigma = \mathcal{O}_\Sigma(a_\Sigma)$ and $L'_\Sigma = \mathcal{O}_\Sigma(b_\Sigma)$ be the line bundles restricted to the curve Σ. The parameters a_Σ and b_Σ from the line bundles L and L' need to be fixed by the constraints from the matter spectrum, and there could be more than two conditions from these constraints resulting in the existence of exotic matter.

According to [24], it is not necessary to use the self-intersecting mechanism in [12,15] to obtain the codimension three Yukawa coupling $\mathbf{10}\,\mathbf{10}\,\mathbf{5}_H$, and one can instead simply engineer two intersecting curves supporting $\mathbf{10}_{2,2}$ and $\mathbf{5}_{2,-4}$ to get a rank one coupling. We will follow the latter to construct the Yukawa coupling.

The three generations are from the curve Σ_M^1 with the enhanced group $G_{\Sigma_M^1} = SO(12)$. Let the line bundles on this curve be $L_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(a_M^1)$ and $L'_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(b_M^1)$. It is required to obtain the desired field content that

$$\begin{aligned}
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(2a_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(2b_M^1\big)\big) = 3,\\
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(-2a_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(-2b_M^1\big)\big) = 0,\\
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(-4a_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(2b_M^1\big)\big) = 0,\\
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(4a_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(-2b_M^1\big)\big) = 0.
\end{aligned}$$

It is easy to find that the unique solution is $a_M^1 = \frac{1}{2}$ and $b_M^1 = 1$, so there exist

$$3 \times \mathbf{10}_{2,2}$$

localized on the curve Σ_M^1.

Let the matter multiple $\bar{\mathbf{5}}$ be from the curve Σ_M^2. We choose this curve to be genus zero with the enhanced group $G_{\Sigma_M^2} = SU(7)$ and the line bundles on Σ_M^2 to be $L_{\Sigma_M^2} = \mathcal{O}_{\Sigma_M^1}(a_M^2)$ and $L'_{\Sigma_M^2} = \mathcal{O}_{\Sigma_M^1}(b_M^2)$. In this case, we obtain the unique solution $a_M^2 = -\frac{1}{2}$ and $b_M^2 = \frac{5}{14}$. The resulting field content is

$$3 \times \bar{\mathbf{5}}_{7,-1}.$$

Let the up-type Higgs multiplet be from the curve Σ_H^1. Then we choose it also a genus zero curve with the enhanced group $G_{\Sigma_H^1} = SO(12)$ and the line bundles on Σ_H^1 as $L_{\Sigma_H^1} = \mathcal{O}_{\Sigma_H^1}(a_H^1)$ and $L'_{\Sigma_H^1} = \mathcal{O}_{\Sigma_H^1}(b_H^1)$. The unique solution is $a_H^1 = -\frac{1}{6}$ and $b_H^1 = \frac{1}{6}$, so the field content is

Table 1
An $SU(5)$ GUT model from $G_S = SU(6)$, where $L = \mathcal{O}_S(E_1 - E_2)^{1/6}$.

Multiplet	Curve	Class	g_Σ	L_Σ	L'_Σ
$3 \times \mathbf{10}_{2,2}$	Σ_M^1	$4H + 2E_2 - E_1$	0	$\mathcal{O}_{\Sigma_M^1}(1)^{1/2}$	$\mathcal{O}_{\Sigma_M^1}(1)$
$3 \times \bar{\mathbf{5}}_{7,-1}$	Σ_M^2	$5H + 3E_1 - E_6$	0	$\mathcal{O}_{\Sigma_M^2}(-1)^{1/2}$	$\mathcal{O}_{\Sigma_M^2}(1)^{5/14}$
$1 \times \mathbf{5}_{2,-4}$	Σ_H^1	$3H + E_1 - E_3$	0	$\mathcal{O}_{\Sigma_H^1}(-1)^{1/6}$	$\mathcal{O}_{\Sigma_H^1}(1)^{1/6}$
$1 \times \bar{\mathbf{5}}_{7,-1}$	Σ_H^2	$H - E_2 - E_3$	0	$\mathcal{O}_{\Sigma_H^2}(-1)^{1/6}$	$\mathcal{O}_{\Sigma_H^2}(1)^{5/42}$

$$1 \times \mathbf{5}_{2,-4}.$$

Similarly, for the down-type Higgs multiplet on Σ_H^2, we again take it as a genus zero curve with the enhanced group $G_{\Sigma_H^2} = SU(7)$ and the line bundles on Σ_H^2 are $L_{\Sigma_H^2} = \mathcal{O}_{\Sigma_H^2}(a_H^2)$ and $L'_{\Sigma_H^2} = \mathcal{O}_{\Sigma_H^2}(b_H^2)$. In this case, we obtain the unique solution $a_H^2 = -\frac{1}{6}$ and $b_H^2 = \frac{5}{42}$ and the field content is

$$1 \times \bar{\mathbf{5}}_{7,-1}.$$

After determining the line bundles, we look for the suitable curves to support these bundles. In our construction we require all curves effective and genus zero. Of course it is possible to choose the curves with higher genus, such as a genus one curve with non-effective divisors. However, there will exist vector-like Higgs fields on these curves, which may result in the problem of rapid proton decay [15]. Therefore, we only consider curves of genus zero and separate up-type and down-type Higgs fields on different curves.

We summarize the spectrum and the homology classes of the curves of this model in Table 1.

3.1.2. $G_S = SO(10)$

Consider a $G_S = SO(10)$ model with nontrivial line bundles on all the curves, so $SO(10)$ is broken down to $SU(5) \times U(1)_S$ on the bulk. Like the previous case, we choose a supersymmetric line bundle $L = \mathcal{O}_S(E_1 - E_2)^{1/4}$ on S such that the chiral matter fields on the bulk vanish, *i.e.* $N_{\mathbf{10}_4} = N_{\overline{\mathbf{10}}_{-4}} = 0$. The Yukawa couplings of $\Sigma\Sigma S$-type such as $\mathbf{10}_{0,4}\bar{\mathbf{5}}_{2,-2}\bar{\mathbf{5}}_{-2,-2}$ and $\mathbf{10}_{0,4}\mathbf{10}_{-3,-1}\mathbf{5}_{3,-3}$ and their complex conjugates are vanishing. We shall only consider the Yukawa couplings of $\Sigma\Sigma\Sigma$-type where chiral fields are from local curves Σs in the following example.

The breaking chains and matter content from the enhanced adjoints of the curves are

$$\begin{aligned} &SO(12) \to SO(10)_S \times U(1) \to SU(5) \times U(1) \times U(1)_S, \\ &66 \quad \to 45_0 + 1_0 \qquad \to 24_{0,0} + 1_{0,0} + 10_{0,4} + \overline{10}_{0,-4} + 1_{0,0} \\ &\qquad\quad + 10_2 + \overline{10}_{-2} \qquad + 5_{2,2} + \bar{5}_{2,-2} + \bar{5}_{-2,-2} + 5_{-2,2}, \end{aligned} \tag{20}$$

$$\begin{aligned} &E_6 \to SO(10)_S \times U(1) \to SU(5) \times U(1) \times U(1)_S, \\ &78 \to 45_0 + 1_0 \qquad \to 24_{0,0} + 1_{0,0} + 10_{0,4} + \overline{10}_{0,-4} + 1_{0,0} \\ &\qquad + 16_{-3} + \overline{16}_3 \qquad + (10_{-3,-1} + \bar{5}_{-3,3} + 1_{-3,-5} + c.c.). \end{aligned} \tag{21}$$

Let us turn to the spectrum from the curves. Again, since the bulk line bundle is nontrivial in our discussion, the $U(1)_S$ charges of the fields localized on the curves should be conserved in each Yukawa coupling. The superpotential is:

$$\mathcal{W} \supset \mathbf{10}_{-3,-1}\mathbf{10}_{-3,-1}\mathbf{5}_{-2,2} + \mathbf{10}_{-3,-1}\bar{\mathbf{5}}_{-3,3}\bar{\mathbf{5}}_{2,-2} + \cdots. \tag{22}$$

The corresponding Yukawa coupling patterns on the double enhanced points of $\mathbf{10}\,\mathbf{10}\,\mathbf{5}$ and $\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}$ can be found in Eqs. (B.5) and (B.2), respectively.

To obtain the spectrum, first we choose the genus zero curve Σ_M^1 with $G_{\Sigma_M^1} = E_6$ and let $L_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(d_M^1)$ and $L'_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(e_M^1)$. In order to get the desired field content, it is required that

$$h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(-d_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(-3e_M^1\big)\big) = 3,$$
$$h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(d_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(3e_M^1\big)\big) = 0,$$
$$h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(3d_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(-3e_M^1\big)\big) = 0,$$
$$h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(-3d_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(3e_M^1\big)\big) = 0,$$
$$h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(-5d_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(-3e_M^1\big)\big) = 0,$$
$$h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(5d_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(3e_M^1\big)\big) = 0.$$

It is easy to see no solution satisfies all conditions, which means that there exists exotic matter. We choose $d_M^1 = -\frac{3}{4}$ and $e_M^1 = -\frac{3}{4}$, then the field content includes exotic singlets:

$$3 \times \mathbf{10}_{-3,-1}, \quad 6 \times \mathbf{1}_{-3,-5}.$$

For Σ_M^2, we take it as a genus zero curve with $G_\Sigma = E_6$ and let the line bundles be $L_{\Sigma_M^2} = \mathcal{O}_{\Sigma_M^2}(d_M^2)$ and $L'_{\Sigma_M^2} = \mathcal{O}_{\Sigma_M^2}(e_M^2)$. Again, no solution satisfies all the conditions, which means that there exists exotic matter. We choose $d_M^2 = \frac{3}{4}$ and $e_M^2 = -\frac{1}{4}$ so then the field content is

$$3 \times \bar{\mathbf{5}}_{-3,3}, \quad 3 \times \mathbf{1}_{3,5}.$$

We choose Σ_H^1 to be a genus zero curve with $G_{\Sigma_H^1} = SO(12)$ and let the line bundles be $L_{\Sigma_H^1} = \mathcal{O}_{\Sigma_H^1}(d_H^1)$ and $L'_{\Sigma_H^1} = \mathcal{O}_{\Sigma_H^1}(e_H^1)$. The unique solution is $d_H^1 = \frac{1}{4}$ and $e_H^1 = -\frac{1}{4}$. The resulting field content is

$$1 \times \mathbf{5}_{-2,2}.$$

We choose Σ_H^2 to be a genus zero curve with $G_{\Sigma_H^2} = SO(12)$ and let the line bundles be $L_{\Sigma_H^2} = \mathcal{O}_{\Sigma_H^2}(d_H^2)$ and $L'_{\Sigma_H^2} = \mathcal{O}_{\Sigma_H^2}(e_H^2)$. The solution is $d_H^2 = -\frac{1}{4}$ and $e_H^2 = \frac{1}{4}$, thus the resulting field content is

$$1 \times \bar{\mathbf{5}}_{2,-2}.$$

We summarize the result in Table 2.

In the first example with $G_S = SU(6)$, the flux is nontrivial in order to break the bulk gauge group into the desired $SU(5)$ gauge group. We choose the case that all matter fields come from the curves without exotic fields. We avoid the possibilities of up-type and down-type Higgs fields coming from the bulk or from the same curve that will cause rapid proton decay by the induced quartic terms in the superpotential. The $U(1)_S$ charges are consistent in the fermion mass Yukawa couplings.

Table 2
An $SU(5)$ GUT model from $G_S = SO(10)$, where $L = \mathcal{O}_S(E_1 - E_2)^{1/4}$.

Multiplet	Curve	Class	g_Σ	L_Σ	L'_Σ
$3 \times \mathbf{10}_{-3,-1}$	Σ_M^1	$4H + 2E_1 - E_2$	0	$\mathcal{O}_{\Sigma_M^1}(-1)^{3/4}$	$\mathcal{O}_{\Sigma_M^1}(-1)^{3/4}$
$3 \times \bar{\mathbf{5}}_{-3,3}$	Σ_M^2	$5H + 3E_2 - E_5$	0	$\mathcal{O}_{\Sigma_M^2}(1)^{3/4}$	$\mathcal{O}_{\Sigma_M^2}(-1)^{1/4}$
$1 \times \mathbf{5}_{-2,2}$	Σ_H^1	$3H + E_3 - E_1$	0	$\mathcal{O}_{\Sigma_h^1}(1)^{1/4}$	$\mathcal{O}_{\Sigma_h^1}(-1)^{1/4}$
$1 \times \bar{\mathbf{5}}_{2,-2}$	Σ_H^2	$H - E_2 - E_3$	0	$\mathcal{O}_{\Sigma_h^2}(-1)^{1/4}$	$\mathcal{O}_{\Sigma_h^2}(1)^{1/4}$

In the second example with $G_S = SO(10)$, the flux is nontrivial as well in order to break the bulk gauge group into the desired $SU(5)$ gauge group. All matter fields are from the curves without exotic fields on the bulk. The $U(1)_S$ charges are consistent in the Yukawa couplings and it explains that an $SU(5)$ GUT is descended from the $SO(10)$ unified gauge group.

3.1.3. Split gauge bundle

The Standard Model (SM) gauge group is two ranks lower than G_S, therefore in principle, if we want to break G_S to $SU(3) \times SU(2) \times U(1)_Y$ it is possible to introduce an instanton configuration to break G_S [15]. This instanton can be a $SU(2)$ or $U(1) \times U(1)$ gauge group. In the models discussed above, the $U(1)_S$ is a substructure of $U(1) \times U(1)$, and the additional $U(1)_{\tilde{S}}$ can be utilized on the bulk to break the $SU(5)$ GUT to SM. $U(1)_Y$ which can be the linear combination of these $U(1)$s. In this case, the $U(1)_{\tilde{S}}$ charges are consistent with the $U(1)_Y$ charges. There is also a possibility to solve the doublet–triplet problem from controlling the Higgs multiplets by this $U(1)_{\tilde{S}}$ gauge group. In what follows we demonstrate an example that how this Abelian gauge bundle breaks the $SU(5)$ GUT group on the bulk.

Consider V to be a split vector bundle of rank two over S. Write $V = L_1 \oplus L_2$, where L_i, $i = 1, 2$, are nontrivial line bundles. In order to solve the BPS equation (8), the line bundles are required to be supersymmetric, in other words, $J_S \wedge c_1(L_1) = J_S \wedge c_1(L_2) = 0$. To be more concrete, let $V = \mathcal{O}_S(E_i - E_j) \oplus \mathcal{O}_S(E_j - E_i)^{1/6}$, $i \neq j$, it is easy to check that it solves BPS equation. In this case, the structure group is $U(1)_{\tilde{S}} \times U(1)_S$. Therefore, by switching on the gauge bundle V, $G_S = SU(6)$ can be broken into $SU(3) \times SU(2) \times U(1)_{\tilde{S}} \times U(1)_S$. The breaking pattern is as follows

$$
\begin{aligned}
& SU(6) \to SU(3) \times SU(2) \times U(1)_{\tilde{S}} \times U(1)_S, \\
& 35 \quad \to (8,1)_{0,0} + (1,3)_{0,0} + (3,2)_{-5,0} + (\bar{3},2)_{5,0} + (1,1)_{0,0} \\
& \qquad + (1,1)_{0,0} + (1,2)_{3,6} + (3,1)_{-2,6} + (1,\bar{2})_{-3,-6} + (\bar{3},1)_{2,-6}.
\end{aligned} \tag{23}
$$

It turns out that in this case, all fields on the bulk form vector-like pairs. The spectrum on the bulk is then given by

$$
\begin{cases}
N_{(3,2)_{-5,0}} = N_{(\bar{3},2)_{5,0}} = 24, \\
N_{(1,2)_{3,6}} = N_{(1,\bar{2})_{-3,-6}} = 3, \\
N_{(\bar{3},1)_{2,-6}} = N_{(3,1)_{-2,6}} = 8.
\end{cases} \tag{24}
$$

Of course this is not the only choice for the split gauge bundle of rank two over S. The detailed configuration and the spectrum of the chiral fields from curves will be presented elsewhere [36].

The self-intersection mechanism of the **10** curve in the **10 10 5** coupling is not the only way to obtain higher rank Yukawa mass matrices. It has been shown in [18] that a generalization of the conditions on the $U(1)_{B-L}$ flux with the chiral fermions from two different curves can take the work. With the introduction of this additional $U(1)$, the generation numbers of MSSM fields in the **10** and **5** representations of $SU(5)$ can be controlled to achieve a richer structure of the fermion mass matrices.

3.2. Flipped $SU(5)$ GUT

In a flipped $SU(5)\times U(1)_X$ [31–33] unified model, the electric charge generator is only partially embedded in $SU(5)$. In other words, the photon is shared between $SU(5)$ and $U(1)_X$. The SM fermions plus the right-handed neutrino states reside within the representations $\bar{\mathbf{5}}$, **10**, and **1** of $SU(5)$, which are collectively equivalent to a spinor **16** of $SO(10)$. The quark and lepton assignments are flipped by $u_L^c \leftrightarrow d_L^c$ and $\mu_L^c \leftrightarrow e_L^c$ relative to a conventional $SU(5)$ GUT embedding. Since **10** contains a neutral component ν_L^c, we can spontaneously break the GUT gauge symmetry by using a pair of $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ of superheavy Higgs where the neutral components receive a large VEV. The spontaneous breaking of electroweak gauge symmetry is generated by the Higgs doublets embedded in the Higgs pentaplet $\mathbf{5}_h$. It then has a natural solution to the doublet–triplet splitting problem through the trilinear coupling of the Higgs fields $\mathbf{10}_H\mathbf{10}_H\mathbf{5}_h$. The generic superpotential $\mathcal{W}$ is

$$\mathcal{W}\supset \mathbf{10}\,\mathbf{10}\,\mathbf{5}_h+\mathbf{10}\,\bar{\mathbf{5}}\,\bar{\mathbf{5}}_h+\bar{\mathbf{5}}\,\mathbf{1}\,\mathbf{5}_h+\mathbf{10}\overline{\mathbf{10}}_H\mathbf{1}_\phi+\mathbf{10}_H\mathbf{10}_H\mathbf{5}_h+\overline{\mathbf{10}}_H\overline{\mathbf{10}}_H\bar{\mathbf{5}}_h+\cdots. \tag{25}$$

3.2.1. $G_S=SU(6)$

Since the flipped $SU(5)$ model has a similar fermion spectrum as the $SU(5)$ model, and there are limited options for the matter from the curves, we may make the $SU(5)\times U(1)_X$ model from $G_S=SU(6)$ based on the setup of the previous Section 3.1.1 with additional fields such as the singlet $\mathbf{1}_M$ and the GUT Higgs $\mathbf{10}_H$, $\overline{\mathbf{10}}_H$. One possible choice for the Yukawa couplings is:

$$\mathcal{W}\supset \mathbf{10}_{2,2}\mathbf{10}_{2,2}\mathbf{5}_{2,-4}+\mathbf{10}_{2,2}\bar{\mathbf{5}}_{7,-1}\bar{\mathbf{5}}_{7,-1}+\bar{\mathbf{5}}_{7,-1}\mathbf{5}_{2,-4}\mathbf{1}_{7,5}+\cdots. \tag{26}$$

The construction is similar to the $SU(5)$ model from $G_S=SU(6)$ in the previous section, and we need the additional matter singlet and the superheavy Higgs pairs. We choose Σ_M^3 to be a genus zero curve with $G_{\Sigma_M^3}=SU(7)$ and let the line bundles be $L_{\Sigma_M^3}=\mathcal{O}_{\Sigma_M^3}(\tilde{a}_M^3)$ and $L'_{\Sigma_M^3}=\mathcal{O}_{\Sigma_M^3}(\tilde{b}_M^3)$. The unique solution is $\tilde{a}_M^3=\frac{1}{2}$ and $\tilde{b}_M^3=\frac{1}{14}$ and the resulting field content is

$$3\times\mathbf{1}_{7,5}.$$

We choose Σ_H^1 to be a genus zero curve with $G_{\Sigma_H^1}=SO(12)$. Let $L_{\Sigma_H^1}=\mathcal{O}_{\Sigma_H^1}(\tilde{a}_H^1)$ and $L'_{\Sigma_H^1}=\mathcal{O}_{\Sigma_H^1}(\tilde{b}_H^1)$, the unique solution is $\tilde{a}_H^1=\frac{1}{6}$ and $\tilde{b}_H^1=\frac{1}{3}$ and the resulting field content is

$$1\times\mathbf{10}_{2,2}.$$

Similarly, for Σ_H^2, we make it genus zero. The resulting field content is

$$1\times\overline{\mathbf{10}}_{-2,-2}.$$

We summarize the pinched model in Table 3.

From the spectrum the matter fields **10** and $\bar{\mathbf{5}}$ are from the curves that have different enhanced gauge groups, which implies they are not unified in the same representation of a higher rank gauge group, such as the **15** of $SU(6)$. Furthermore, we are not able to obtain the corresponding

Table 3
An $SU(5)\times U(1)_X$ model from $G_S=SU(6)$, where $L=\mathcal{O}_S(E_1-E_2)^{1/6}$.

Multiplet	Curve	Class	g_Σ	L_Σ	L'_Σ
$3\times\mathbf{10}_{2,2}$	Σ_M^1	$4H+2E_2-E_1$	0	$\mathcal{O}_{\Sigma_M^1}(1)^{1/2}$	$\mathcal{O}_{\Sigma_M^1}(1)$
$3\times\bar{\mathbf{5}}_{7,-1}$	Σ_M^2	$5H+3E_1-E_6$	0	$\mathcal{O}_{\Sigma_M^2}(-1)^{1/2}$	$\mathcal{O}_{\Sigma_M^2}(1)^{5/14}$
$3\times\mathbf{1}_{7,5}$	Σ_M^3	$6H+3E_2-3E_3-2E_5$	0	$\mathcal{O}_{\Sigma_M^3}(1)^{1/2}$	$\mathcal{O}_{\Sigma_M^3}(1)^{1/14}$
$1\times\mathbf{10}_{2,2}$	Σ_H^1	$2H-E_1-E_3-E_5$	0	$\mathcal{O}_{\Sigma_H^1}(1)^{1/6}$	$\mathcal{O}_{\Sigma_H^1}(1)^{1/3}$
$1\times\overline{\mathbf{10}}_{-2,-2}$	Σ_H^2	$2H-E_2-E_3-E_5$	0	$\mathcal{O}_{\Sigma_H^2}(-1)^{1/6}$	$\mathcal{O}_{\Sigma_H^2}(-1)^{1/3}$
$1\times\mathbf{5}_{2,-4}$	Σ_h^3	$3H+E_1-E_3$	0	$\mathcal{O}_{\Sigma_h^3}(-1)^{1/6}$	$\mathcal{O}_{\Sigma_h^3}(1)^{1/6}$
$1\times\bar{\mathbf{5}}_{7,-1}$	Σ_h^4	$H-E_2-E_3$	0	$\mathcal{O}_{\Sigma_h^4}(-1)^{1/6}$	$\mathcal{O}_{\Sigma_h^4}(1)^{5/42}$

$U(1)_X$ charges of the matter after rotating the two charges of each representation. These imply that a flipped $SU(5)$ gauge group is not naturally embedded in $SU(6)$. The approach to build an $SU(5)\times U(1)_X$ from $G_S=SU(6)$ is not a success.

3.2.2. $G_S=SO(10)$

In this section we shall build the flipped $SU(5)$ model from the bulk $G_S=SO(10)$. Again, we achieve this by extending the spectrum of the $SU(5)$ model constructed from $G_S=SO(10)$ in Section 3.1.2. The $U(1)_S$ charges of the fields on the curves should be conserved in the Yukawa couplings due to the nontrivial bulk flux. The Yukawa couplings in the superpotential are

$$\mathcal{W}\supset\mathbf{10}_{-3,-1}\mathbf{10}_{-3,-1}\mathbf{5}_{-2,2}+\mathbf{10}_{-3,-1}\bar{\mathbf{5}}_{-3,3}\bar{\mathbf{5}}_{2,-2}+\bar{\mathbf{5}}_{-3,3}\mathbf{5}_{-2,2}\mathbf{1}_{-3,-5}+\cdots. \tag{27}$$

The matter singlet has 6 copies and is from the same curve Σ_{E_6} as the $\mathbf{10}_M$. The additional GUT Higgs multiplets $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ can be engineered by the following calculation.

$\mathbf{10}_H$ has the same charge as the $\mathbf{10}_M$ does, so we also choose the enhanced gauge group of curve Σ_H^1 to be $G_{\Sigma_H^1}=E_6$. Let $L_{\Sigma_H^1}=\mathcal{O}_{\Sigma_H^1}(f_H^1)$ and $L'_{\Sigma_H^1}=\mathcal{O}_{\Sigma_H^1}(g_H^1)$. In order to obtain the desired field content, it is required that

$$\begin{aligned}
&h^0\big(\Sigma_H^1,K_{\Sigma_H^1}^{1/2}\otimes\mathcal{O}_{\Sigma_H^1}\big(-f_H^1\big)\otimes\mathcal{O}_{\Sigma_H^1}\big(-3g_H^1\big)\big)=1,\\
&h^0\big(\Sigma_H^1,K_{\Sigma_H^1}^{1/2}\otimes\mathcal{O}_{\Sigma_H^1}\big(f_H^1\big)\otimes\mathcal{O}_{\Sigma_H^1}\big(3g_H^1\big)\big)=0,\\
&h^0\big(\Sigma_H^1,K_{\Sigma_H^1}^{1/2}\otimes\mathcal{O}_{\Sigma_H^1}\big(3f_H^1\big)\otimes\mathcal{O}_{\Sigma_H^1}\big(-3g_H^1\big)\big)=0,\\
&h^0\big(\Sigma_H^1,K_{\Sigma_H^1}^{1/2}\otimes\mathcal{O}_{\Sigma_H^1}\big(-3f_H^1\big)\otimes\mathcal{O}_{\Sigma_H^1}\big(3g_H^1\big)\big)=0,\\
&h^0\big(\Sigma_H^1,K_{\Sigma_H^1}^{1/2}\otimes\mathcal{O}_{\Sigma_H^1}\big(-5f_H^1\big)\otimes\mathcal{O}_{\Sigma_H^1}\big(-3g_H^1\big)\big)=0,\\
&h^0\big(\Sigma_H^1,K_{\Sigma_H^1}^{1/2}\otimes\mathcal{O}_{\Sigma_H^1}\big(5f_H^1\big)\otimes\mathcal{O}_{\Sigma_H^1}\big(3g_H^1\big)\big)=0.
\end{aligned}$$

It is easy to see no solution satisfies all the conditions, which means there exists exotic matter. We choose $f_H^1=-\frac{1}{4}$ and $g_H^1=-\frac{1}{4}$ and the field content is

$$1\times\mathbf{10}_{-3,-1},\quad 2\times\mathbf{1}_{-3,-5}.$$

Similarly, we take Σ_H^2 as a genus zero curve with $G_{\Sigma_H^2}=E_6$ and let the line bundles be $L_{\Sigma_H^2}=\mathcal{O}_{\Sigma_H^2}(f_H^2)$ and $L'_{\Sigma_H^2}=\mathcal{O}_{\Sigma_H^2}(g_H^2)$. Following the same process, we find that there is no solution

Table 4
An $SU(5)\times U(1)_X$ model from $G_S = SO(10)$, where $L=\mathcal{O}_S(E_1-E_2)^{1/4}$.

Multiplet	Curve	Class	g_Σ	L_Σ	L'_Σ
$3\times \mathbf{10}_{-3,-1}$	Σ_M^1	$4H+2E_1-E_2$	0	$\mathcal{O}_{\Sigma_M^1}(-1)^{3/4}$	$\mathcal{O}_{\Sigma_M^1}(-1)^{3/4}$
$3\times \bar{\mathbf{5}}_{-3,3}$	Σ_M^2	$5H+3E_2-E_5$	0	$\mathcal{O}_{\Sigma_M^2}(1)^{3/4}$	$\mathcal{O}_{\Sigma_M^2}(-1)^{1/4}$
$3\times \mathbf{1}_{-3,-5}$	Σ_M^1	$6H+3E_1-3E_4-2E_5$	0	$\mathcal{O}_{\Sigma_M^1}(-1)^{3/4}$	$\mathcal{O}_{\Sigma_M^1}(-1)^{3/4}$
$1\times \mathbf{10}_{-3,-1}$	Σ_H^1	$2H-E_2-E_4-E_5$	0	$\mathcal{O}_{\Sigma_H^1}(-1)^{1/4}$	$\mathcal{O}_{\Sigma_H^1}(-1)^{1/4}$
$1\times \overline{\mathbf{10}}_{3,1}$	Σ_H^2	$2H-E_1-E_4-E_5$	0	$\mathcal{O}_{\Sigma_H^2}(1)^{1/4}$	$\mathcal{O}_{\Sigma_H^2}(1)^{1/4}$
$1\times \mathbf{5}_{-2,2}$	Σ_h^3	$H-E_1-E_5$	0	$\mathcal{O}_{\Sigma_h^3}(1)^{1/4}$	$\mathcal{O}_{\Sigma_h^3}(-1)^{1/4}$
$1\times \bar{\mathbf{5}}_{2,-2}$	Σ_h^4	$H-E_2-E_5$	0	$\mathcal{O}_{\Sigma_h^4}(-1)^{1/4}$	$\mathcal{O}_{\Sigma_h^4}(1)^{1/4}$

for all the conditions. So we set $f_H^2=\frac{1}{4}$ and $g_H^2=\frac{1}{4}$ for a minimum content. The resulting field content is

$$1\times \overline{\mathbf{10}}_{3,1},\quad 2\times \mathbf{1}_{3,5}.$$

We summarize the result in Table 4.

The $U(1)_S$ charges in the spectrum are consistent with the $U(1)_X$ charges, which is natural since $SU(5)\times U(1)_X$ is embedded in $SO(10)$. However, to make $U(1)_X$ massless we have to rotate the $U(1)$ gauge groups to satisfy the constraints from the Green–Schwarz mechanism in a global picture. In addition, we are not able to avoid a few copies of exotic singlets. This model includes all the terms of the generic superpotential $\mathcal{W}$ of $SU(5)\times U(1)_X$ stated in Eq. (25).

From the first case, we find the generic structure of $G_S=SU(6)$ cannot produce a flipped $SU(5)$ model due to the inconsistent charges of the fermion and Higgs fields. It is difficult to construct a flipped $SU(5)$ model unless we are able to turn on a line bundle to break G_Σ to an $SO(10)$ gauge group.

In the second case, the $SU(5)\times U(1)_X$ model from $SO(10)$ is similar to the constructions in [15] and [20]. In our model, the curves in the spectrum have alternate classes. The nontrivial bulk fluxes on the curves are turned on so we can study the substructure of $\mathbf{16}$ from $SO(10)$. $\bar{\mathbf{5}}$ and $\mathbf{10}$ are not on the same curve, while $\mathbf{10}$ still forms a $\mathbf{10}\,\mathbf{10}\,\mathbf{5}$ coupling but $\bar{\mathbf{5}}$ gets rid of the coupling $\bar{\mathbf{5}}\,\bar{\mathbf{5}}\,\mathbf{5}_h$. The $U(1)_S$ charges are consistent with the $U(1)_X$ charges. This implies the bulk $SO(10)$ is corresponding to the $SO(10)$ GUT which is the higher unification of the flipped $SU(5)$.

Again the self-intersecting geometry can be introduced to obtain a rank three Yukawa mass structure, and we can also construct a flipped $SU(5)$ model by splitting chiral fermions on two different matter curves [18].

3.3. $SO(10)$ GUT

In this section we shall discuss the $SO(10)$ GUT from the breaking of a higher rank bulk gauge group. There are two possible choices, $G_S=SO(12)$ and $G_S=E_6$.

3.3.1. $G_S=SO(12)$

Consider seven-branes wrapping on S where $G_S=SO(12)$. There exist the following breaking patterns from the enhanced adjoints of the curves:

Table 5
An $SO(10)$ GUT model from $G_S = SO(12)$, where $L = \mathcal{O}_S(E_1 - E_2 - E_3)^{1/2}$ and Higgs **10** is from the bulk.

Multiplet	Curve	Class	g_Σ	L_Σ	L'_Σ
$3 \times \mathbf{16}_{1,-1}$	Σ_M^1	$3H + E_1 - E_2 - E_3$	0	$\mathcal{O}_{\Sigma_M^1}(-1)^{3/2}$	$\mathcal{O}_{\Sigma_M^1}(1)^{3/2}$

$$\begin{aligned}
&SO(14) \rightarrow SO(12)_S \times U(1) \rightarrow SO(10) \times U(1) \times U(1)_S,\\
&91 \quad\ \ \rightarrow 66_0 + 1_0 \qquad\quad \rightarrow 45_{0,0} + 1_{0,0} + 10_{0,2} + \overline{10}_{0,-2} + 1_{0,0}\\
&\qquad\quad + 12_2 + \overline{12}_{-2} \qquad + (10_{2,0} + 1_{2,2} + 1_{2,-2} + c.c.),
\end{aligned} \tag{28}$$

$$\begin{aligned}
&E_7 \ \rightarrow SO(12)_S \times U(1) \rightarrow SO(10) \times U(1) \times U(1)_S,\\
&133 \rightarrow 66_0 + 1_0 + \mathit{1}_{\pm 2} \ \rightarrow 45_{0,0} + 2 \times 1_{0,0} + 10_{0,2} + \overline{10}_{0,-2} + \mathit{1}_{\pm 2,0}\\
&\qquad + 32'_1 + \mathit{32}'_{-1} \qquad + 16_{1,-1} + \overline{16}_{1,1} + \mathit{16}_{-1,-1} + \overline{\mathit{16}}_{-1,1}.
\end{aligned} \tag{29}$$

To obtain a **16 16 10** coupling, the **10** can only be from the bulk due to the conservation of the $U(1)_S$ charges, and it implies that the coupling is a $\Sigma\Sigma S$-type instead of a $\Sigma\Sigma\Sigma$-type. From the above breaking patterns, the possible choices are $\mathbf{16}_{1,-1}\mathbf{16}_{1,-1}\mathbf{10}_{0,2}$, $\mathbf{16}_{-1,-1}\mathbf{16}_{-1,-1}\mathbf{10}_{0,2}$, and $\mathbf{16}_{1,-1}\mathbf{16}_{-1,-1}\mathbf{10}_{0,2}$, and we take the first as an example whose superpotential is:

$$\mathcal{W} \supset \mathbf{16}_{1,-1}\mathbf{16}_{1,-1}\mathbf{10}_{0,2} + \cdots. \tag{30}$$

The corresponding Yukawa coupling pattern on the double enhanced point can be found in Eq. (B.11).

We choose a genus zero curve Σ_M^1 with $G_{\Sigma_M^1} = E_7$ and let $L_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(h_M^1)$ and $L'_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(k_M^1)$. In order to get the desired field content, it is required that

$$\begin{aligned}
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(-h_M^1\big)\mathcal{O}_{\Sigma_M^1}\big(k_M^1\big)\big) = 3,\\
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(h_M^1\big)\mathcal{O}_{\Sigma_M^1}\big(-k_M^1\big)\big) = 0,\\
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(-h_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(-k_M^1\big)\big) = 0,\\
&h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2} \otimes \mathcal{O}_{\Sigma_M^1}\big(h_M^1\big) \otimes \mathcal{O}_{\Sigma_M^1}\big(k_M^1\big)\big) = 0.
\end{aligned}$$

The unique solution is $h_M^1 = -\frac{3}{2}$ and $k_M^1 = \frac{3}{2}$, so the resulting field content is

$$3 \times \mathbf{16}_{1,-1}.$$

The Higgs multiplet $\mathbf{10}_{0,2}$ is from the bulk. By Eq. (7), we obtain

$$N_{\mathbf{10}_2} = 1, \quad N_{\overline{\mathbf{10}}_{-2}} = 0$$

where $L = \mathcal{O}_S(E_1 - E_2 - E_3)^{1/2}$ has been used. Note that in this case, we change the polarization to be $J_S = AH - 2E_1 - \sum_{i=2}^{8} E_i$ so that BPS equation (8) still holds. The spectrum is shown in Table 5.

3.3.2. $G_S = E_6$

In the case of $G_S = E_6$, E_6 is broken into $SO(10)\times U(1)_S$ by nontrivial fluxes on the bulk. In order to avoid chiral matter on the bulk, we choose a supersymmetric line bundle $L = \mathcal{O}_S(E_1 - E_2)^{1/3}$ over S. By doing so, all chiral matter on the bulk disappears, *i.e.* $N_{\mathbf{16}_{0,-3}} = N_{\overline{\mathbf{16}}_{0,3}} = 0$, which means that all the chiral fields are localized on the curves. The possible breaking chain and the matter content from the enhanced adjoint of the curve is

$$\begin{aligned} E_7 &\to E_6\times U(1) && \to SO(10)\times U(1)\times U(1)_S,\\ 133 &\to 78_0 + 1_0 && \to 45_{0,0} + 1_{0,0} + 1_{0,0} + 16_{0,-3} + \overline{16}_{0,3}\\ &\quad + 27_2 + \overline{27}_{-2} && \quad + (16_{2,1} + 10_{2,-2} + 1_{2,4} + c.c.). \end{aligned} \tag{31}$$

From the breaking pattern we find the Yukawa coupling in the superpotential is $\Sigma\Sigma\Sigma$-type instead of $\Sigma\Sigma S$-type:

$$\mathcal{W} \supset \mathbf{16}_{2,1}\mathbf{16}_{2,1}\mathbf{10}_{2,-2} + \cdots. \tag{32}$$

The corresponding Yukawa coupling pattern on the double enhanced point can be found in Eq. (B.10).

Consider Σ_M^1 a pinched curve of genus zero with $G_{\Sigma_M^1} = E_7$ and let $L_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(\tilde{h}_M^1)$ and $L'_{\Sigma_M^1} = \mathcal{O}_{\Sigma_M^1}(\tilde{k}_M^1)$. In order to get the desired field content, it is required that

$$\begin{aligned} &h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2}\otimes\mathcal{O}_{\Sigma_M^1}\big(\tilde{h}_M^1\big)\otimes\mathcal{O}_{\Sigma_M^1}\big(2\tilde{k}_M^1\big)\big) = 3,\\ &h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2}\otimes\mathcal{O}_{\Sigma_M^1}\big(-\tilde{h}_M^1\big)\otimes\mathcal{O}_{\Sigma_M^1}\big(-2\tilde{k}_M^1\big)\big) = 0,\\ &h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2}\otimes\mathcal{O}_{\Sigma_M^1}\big(-2\tilde{h}_M^1\big)\otimes\mathcal{O}_{\Sigma_M^1}\big(2\tilde{k}_M^1\big)\big) = 0,\\ &h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2}\otimes\mathcal{O}_{\Sigma_M^1}\big(2\tilde{h}_M^1\big)\otimes\mathcal{O}_{\Sigma_M^1}\big(2\tilde{k}_M^1\big)\big) = 0,\\ &h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2}\otimes\mathcal{O}_{\Sigma_M^1}\big(4\tilde{h}_M^1\big)\otimes\mathcal{O}_{\Sigma_M^1}\big(2\tilde{k}_M^1\big)\big) = 0,\\ &h^0\big(\Sigma_M^1, K_{\Sigma_M^1}^{1/2}\otimes\mathcal{O}_{\Sigma_M^1}\big(-4\tilde{h}_M^1\big)\otimes\mathcal{O}_{\Sigma_M^1}\big(-2\tilde{k}_M^1\big)\big) = 0. \end{aligned}$$

Since there is no solution for all the conditions, it implies there exists exotic matter. We choose $\tilde{h}_M^1 = 1$ and $\tilde{k}_M^1 = 1$, so the resulting field content is

$$3\times\mathbf{16}_{2,1},\quad 6\times\mathbf{1}_{2,4}.$$

We choose Σ_H^1 to be a genus zero curve with $G_{\Sigma_H^1} = E_7$. Let the line bundles on Σ_H^1 be $L_{\Sigma_H^1} = \mathcal{O}_{\Sigma_H^1}(\tilde{h}_H^1)$ and $L'_{\Sigma_H^1} = \mathcal{O}_{\Sigma_H^1}(\tilde{k}_H^1)$. Again, there is no solution for all the conditions. We then choose $\tilde{h}_H^1 = -\frac{1}{3}$ and $\tilde{k}_H^1 = \frac{1}{6}$, so the resulting field content is

$$1\times\mathbf{10}_{2,-2},\quad 1\times\mathbf{1}_{-2,-4}.$$

We summarize the result in Table 6.

In these models, the fluxes are nontrivial on all the curves in order to break the gauge group into $SO(10)$. In the first example, the fields come from both the bulk and the curve, while in the second the fields are from the curves.

To solve the doublet–triplet problem, we may consider the Dimopoulos–Wilczek mechanism [34]. There are several choices of Higgs fields to break the $SO(10)$ gauge group, but they are

Table 6
An $SO(10)$ GUT model from $G_S = E_6$, where $L = \mathcal{O}_S(E_1 - E_2)^{1/3}$.

Multiplet	Curve	Class	g_Σ	L_Σ	L'_Σ
$3 \times \mathbf{16}_{2,1}$	Σ_M^1	$4H + 2E_2 - E_1$	0	$\mathcal{O}_{\Sigma_M^1}(1)$	$\mathcal{O}_{\Sigma_M^1}(1)$
$1 \times \mathbf{10}_{2,-2}$	Σ_H^1	$H - E_2 - E_3$	0	$\mathcal{O}_{\Sigma_H^2}(-1)^{1/3}$	$\mathcal{O}_{\Sigma_H^1}(1)^{1/6}$

absent in these models. For example, we do not have **210**, **210**, and $\mathbf{126} + \overline{\mathbf{126}}$ to break the gauge group to the $SU(5)$ GUT or MSSM-like model [35]. However, the configurations of the non-Abelian instanton broken into a product of $U(1)$s may take the work [15]. The possible breaking pattern is

$$SO(10) \times U(1)_{S_a} \to SU(5) \times U(1)_{S_b} \times U(1)_{S_a} \to SU(3) \times SU(2) \times U(1)_S^3,$$

or

$$\begin{aligned} SO(10) \times U(1)_{S_a} &\to SU(2) \times SU(2) \times SU(4) \times U(1)_{S_a} \\ &\to SU(2) \times SU(2) \times SU(3) \times U(1)_S^2. \end{aligned} \tag{33}$$

4. Conclusion

In this paper we construct examples of $SU(5)$, $SU(5) \times U(1)_X$, and $SO(10)$ GUT local models from G_S which is one rank higher than these GUT gauge groups in the F-theory configuration. The bulk flux is nontrivial on all the curves to break G_S down to the GUT gauge group. We can study the unification of the GUT gauge groups to higher rank gauge groups in string theory. There is no GUT adjoint representation on a del Pezzo surface, but it is still possible to break the GUT gauge groups to the SM gauge group by introducing Abelian instanton configurations on the bulk [15].

We demonstrate how to obtain a model of $SU(5)$ Georgi–Glashow from $G_S = SU(6)$. In this model we are able to obtain three copies of quarks and leptons in the **10** and $\bar{\mathbf{5}}$ representations and one copy of the Higgs fields $\mathbf{5}_H$ and $\bar{\mathbf{5}}_H$. Due to the $U(1)_S$ charge structure when breaking $SU(6)$ to $SU(5)$, the up-type Higgs and down-type Higgs are not charge conjugates. To obtain the μ term a mixture state for the up-type Higgs from two curves may be considered and further studied. In these models $SU(5)$ descends from an $SU(6)$ unification. In the example of $SU(5)$ from $G_S = SO(10)$, the $U(1)_S$ charges are consistent in each term of superpotential, and we can see it is natural to embed $SU(5)$ into $SO(10)$. In our examples the matter **10** is either from a curve or two independent curves from which it is possible to use the left–right mechanism to generate rank three mass matrices elegantly as shown in [18]. In these $SU(5)$ models we can avoid rapid proton decay by separating the up- and down-type Higgs from vector-like pairs, and the generic doublet–triplet splitting problem may be controlled when GUT breaks down to MSSM by the additional $U(1)$ from the instanton.

We also try to construct a flipped $SU(5)$ model from $G_S = SU(6)$ and $G_S = SO(10)$. However we are not able to find a consistent set of $U(1)_X$ charges for the matter content in the model with $G_S = SU(6)$. This implies it is not natural to embed an $SU(5) \times U(1)_X$ GUT into an $SU(6)$ gauge group. In the example of $G_S = SO(10)$ the fermion spectrum is similar to what we obtained in the case of $SU(5)$ Georgi–Glashow, with an additional pair of $\mathbf{10}_H$ and $\overline{\mathbf{10}}_H$ Higgs fields. The $U(1)_S$ charges are consistent with the $U(1)_X$ charges which implies $SO(10)$ is a more natural unification from $SU(5) \times U(1)_X$. For a massless $U(1)_X$ one may have to refer to the global picture. The model construction is similar to that studied in [20], but $\bar{\mathbf{5}}$ fermion is from a different curve from $\mathbf{10}$. One advantage of the model is that we can avoid the $\bar{\mathbf{5}}\,\bar{\mathbf{5}}\,\mathbf{5}$ coupling in the superpotential.

In addition, we demonstrate how to obtain models of an $SO(10)$ GUT from $G_S = SO(12)$ and $G_S = E_6$. In the case of $G_S = SO(12)$, the $\mathbf{10}_H$ field is from the bulk so the matter Yukawa coupling is a $\Sigma\Sigma S$-type, while in the $G_S = E_6$ case, all the matter fields are from bi-fundamental representations. There is no $SO(10)$ adjoint $\Phi_{\mathbf{45}}$ for a coupling such as $\Phi_{\mathbf{45}}\,\mathbf{16}_H\,\overline{\mathbf{16}}_H$, however one may consider introducing the instanton configuration to break the GUT gauge symmetry.

The singularity types on the fibers are corresponding to the gauge groups on the seven-branes in F-theory. The introduction of fluxes can be regarded as resolutions of the singularities, and then we are able to analyze the fluxes via Cartan subalgebra [10]. There then arises an interesting question that whether the enhanced gauge group on the curve breaks to a gauge group different from the original bulk gauge group when the line bundle is turned on. It may result in interesting gauge group configurations on the curves.

F-theory has captured attention recently for its non-perturbative configuration and elegant way of constructing the matter spectrum of a local model. The next step is probably to find out the global constraints for building realistic models. Other topics, like supersymmetry breaking, non-Abelian gauge fluxes for gauge group breaking to MSSM, and explicit examples of del Pezzo surfaces for GUT models are interesting and worthy of study in the future.

Acknowledgements

We would like to thank K. Becker, C. Bertinato, B. Dutta, J. Heckman, and J. Marsano for valuable communications and discussions. The work of C.M.C. is supported in part by the Mitchell-Heep Chair in High Energy Physics. The work of Y.C.C. is supported in part by the NSF grant PHY-0555575 and Texas A&M University.

Appendix A. Del Pezzo surfaces

In this section we shall briefly review the geometric properties of del Pezzo surfaces. Del Pezzo surface dP_k, $k \leqslant 8$, is defined by blowing up k generic points of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. The divisors on dP_k can be generated by H and E_i, where H is a hyperplane divisor, and E_i is an exceptional divisor from blowing-up and is isomorphic to $\mathbb{P}^1$. The intersecting numbers are

$$H \cdot H = 1, \quad E_i \cdot E_j = -\delta_{ij}, \quad H \cdot E_i = 0.$$

The canonical divisor on dP_k is given by

$$K_{dP_k} = -c_1(dP_k) = -3H + \sum_{i=1}^{k} E_i. \tag{A.1}$$

The genus of the curve $\mathcal{C}$ within dP_k can be calculated by the formula

$$\mathcal{C} \cdot (K_{dP_k} + \mathcal{C}) = 2g - 2.$$

For a large volume limit, given a line bundle L on dP_k and

$$c_1(L) = \sum_{i=1}^{k} a_i E_i, \tag{A.2}$$

where $a_i a_j < 0$ for some $i \neq j$, there exits a parametric family of Kähler classes J_{dP_k} over dP_k constructed as [12]

$$J_{dP_k} = AH - \sum_{i=1}^{k} b_i E_i, \tag{A.3}$$

where $\sum_k a_k b_k = 0$ and $A \gg b_i > 0$. By the construction, it is easy to see that the line bundle L solves the BPS equation $J_{dP_k} \wedge c_1(L) = 0$.

Appendix B. Resolutions of the triplet intersections

B.1. $SU(5)$ GUT model

For $SU(5)$ GUT model, we consider G_S and G_p to be of rank five and seven, respectively. In general, we have $G_p = SU(8)$, $SO(14)$ or E_7. Here we only consider the group theory decompositions of ADE-type. It is straightforward to get the following resolutions [30]:

$G_p = SU(8)$:

$$\begin{aligned}
&SU(8) \to SU(7)\times U(1) \to SU(6)_S \times U(1)^2 &&\to SU(5)\times U(1)^2 \times U(1)_S,\\
&63 \quad\; \to 48_0 + 1_0 \qquad\; \to 35_{0,0} + 1_{0,0} + 1_{0,0} &&\to 24_{0,0,0} + 3\times 1_{0,0,0} + 5_{0,0,6}\\
&\qquad\qquad\qquad\qquad\qquad\quad + 6_{0,-7} + \bar{6}_{0,7} &&\quad + \bar{5}_{0,0,-6} + (5_{0,-7,1} + 1_{0,-7,-5}\\
&&&\quad + c.c.)\\
&\qquad\quad + 7_8 + \bar{7}_{-8} \qquad\quad + (6_{8,-1} + 1_{8,6} + c.c.) &&\quad + (5_{8,-1,1} + 1_{8,-1,-5} + 1_{8,6,0}\\
&&&\quad + c.c.).
\end{aligned} \tag{B.1}$$

$G_p = SO(14)$:

$$\begin{aligned}
&SO(14) \to SO(12)\times U(1) \to SO(10) \times U(1)^2 &&\to SU(5)\times U(1)^2 \times U(1)_S,\\
&91 \qquad \to 66_0 + 1_0 \qquad\quad \to 45_{0,0} + 1_{0,0} + 1_{0,0} &&\to 24_{0,0,0} + 3\times 1_{0,0,0} + 10_{0,0,4}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 10_{0,2} + \overline{10}_{0,-2} &&\quad + \overline{10}_{0,0,-4} + (5_{0,2,2} + \bar{5}_{0,2,-2}\\
&&&\quad + c.c.)\\
&\qquad\qquad + 12_2 + \overline{12}_{-2} \qquad + (10_{2,0} + 1_{2,2} &&\quad + (5_{2,0,2} + \bar{5}_{2,0,-2} + 1_{2,2,0}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 1_{2,-2} + c.c.) &&\quad + 1_{2,-2,0} + c.c.).
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&SO(14) \to SO(12)\times U(1) \to SU(6) \times U(1)^2 &&\to SU(5)\times U(1)^2 \times U(1)_S,\\
&91 \qquad \to 66_0 + 1_0 \qquad\quad \to 35_{0,0} + 1_{0,0} + 1_{0,0} &&\to 24_{0,0,0} + 3\times 1_{0,0,0} + 5_{0,0,6}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 15_{0,2} + \overline{15}_{0,-2} &&\quad + \bar{5}_{0,0,-6} + (10_{0,2,2} + 5_{0,2,-4}\\
&&&\quad + c.c.)\\
&\qquad\qquad + 12_2 + \overline{12}_{-2} \qquad + (6_{2,1} + \bar{6}_{2,-1} &&\quad + (5_{2,1,1} + 1_{2,1,-5} + \bar{5}_{2,-1,-1}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + c.c.) &&\quad + 1_{2,-1,5} + c.c.).
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
&SO(14) \to SU(7)\times U(1) \to SU(6)\times U(1)^2 \to SU(5)\times U(1)^2\times U(1)_S,\\
&91 \to 48_0 + 1_0 \to 35_{0,0} + 1_{0,0} + 1_{0,0} \to 24_{0,0,0} + 3\times 1_{0,0,0} + 5_{0,0,6}\\
&\qquad + 6_{0,-7} + \bar{6}_{0,7} \qquad + \bar{5}_{0,0,-6} + (5_{0,-7,1} + 1_{0,-7,-5} + c.c.)\\
&\quad + 21_4 + \overline{21}_{-4} \quad + (15_{4,-2} + 6_{4,5} + c.c.) \quad + (10_{4,-2,2} + 5_{4,-2,-4} + 5_{4,5,1} + 1_{4,5,-5} + c.c.).
\end{aligned} \tag{B.4}$$

$G_p = E_7$:

$$\begin{aligned}
&E_7 \to E_6\times U(1) \to SO(10)\times U(1)^2 \to SU(5)\times U(1)^2\times U(1)_S,\\
&133 \to 78_0 + 1_0 \to 45_{0,0} + 2\times 1_{0,0} \to 24_{0,0,0} + 3\times 1_{0,0,0} + 10_{0,0,4}\\
&\qquad + 16_{0,-3} + \overline{16}_{0,3} \qquad + \overline{10}_{0,0,-4} + (10_{0,-3,-1} + \bar{5}_{0,-3,3} + 1_{0,-3,-5} + c.c.)\\
&\quad + 27_2 + \overline{27}_{-2} \quad + (16_{2,1} + 10_{2,-2} + 1_{2,4} + c.c.) \quad + (10_{2,1,-1} + \bar{5}_{2,1,3} + 1_{2,1,-5}\\
&\qquad + 5_{2,-2,2} + \bar{5}_{2,-2,-2} + 1_{2,4,0} + c.c.).
\end{aligned} \tag{B.5}$$

$$\begin{aligned}
&E_7 \to E_6\times U(1) \to SU(6)\times U(1)^2 \to SU(5)\times U(1)^2\times U(1)_S,\\
&133 \to 78_0 + 1_0 \to 35_{0,0} + 2\times 1_{00} \to 24_{0,0,0} + 3\times 1_{0,0,0} + 1_{0,\pm 2,0}\\
&\qquad + 1_{0,\pm 2} + 20_{0,1} + 20_{0,-1} \qquad + 5_{0,0,6} + \bar{5}_{0,0,-6} + 10_{0,1,-3}\\
&\qquad\qquad + \overline{10}_{0,1,3} + 10_{0,-1,-3} + \overline{10}_{0,-1,3}\\
&\quad + 27_2 + \overline{27}_{-2} \quad + (15_{2,0} + \bar{6}_{2,1} + \bar{6}_{2,-1} + c.c.) \quad + (10_{2,0,2} + 5_{2,0,-4} + \bar{5}_{2,1,-1}\\
&\qquad + 1_{2,1,5} + \bar{5}_{2,-1,-1} + 1_{2,-1,5} + c.c.).
\end{aligned} \tag{B.6}$$

$$\begin{aligned}
&E_7 \to SO(12)\times U(1) \to SO(10)\times U(1)^2 \to SU(5)\times U(1)^2\times U(1)_S,\\
&133 \to 66_0 + 1_0 + 1_{\pm 2} \to 45_{0,0} + 2\times 1_{0,0} \to 24_{0,0,0} + 3\times 1_{0,0,0} + 1_{\pm 2,0,0}\\
&\qquad + 1_{\pm 2,0} + 10_{0,2} + \overline{10}_{0,-2} \qquad + (10_{0,0,4} + c.c) + (5_{0,2,2} + \bar{5}_{0,2,-2} + c.c.)\\
&\quad + 32'_1 \quad + 16_{1,-1} + \overline{16}_{1,1} \quad + 10_{1,-1,-1} + \bar{5}_{1,-1,3} + 1_{1,-1,-5} + \overline{10}_{1,1,1} + 5_{1,1,-3} + 1_{1,1,5}\\
&\quad + 32'_{-1} \quad + 16_{-1,-1} + \overline{16}_{-1,1} \quad + 10_{-1,-1,-1} + \bar{5}_{-1,-1,3} + 1_{-1,-1,-5} + \overline{10}_{-1,1,1}\\
&\qquad + 5_{-1,1,-3} + 1_{-1,1,5}.
\end{aligned} \tag{B.7}$$

$$E_7 \to SO(12)\times U(1) \to SU(6)\times U(1)^2 \to SU(5)\times U(1)^2\times U(1)_S,$$

$$
\begin{array}{llll}
133 \to 66_0 + 1_0 + \mathit{1}_{\pm 2} & \to 35_{0,0} + 2 \times 1_{00} + \mathit{1}_{\pm 2,0} & & \to 24_{0,0,0} + 3 \times 1_{0,0,0} \\
 & & & + \mathit{1}_{\pm 2,0,0} + (5_{0,0,6} + c.c.) \\
 & & + 15_{0,2} + \overline{15}_{0,-2} & + (10_{0,2,2} + 5_{0,2,-4} + c.c.) \\
 & + 32'_1 & + 15_{1,-1} + \overline{15}_{1,1} + 1_{1,\pm 3} & + 10_{1,-1,2} + 5_{1,-1,-4} \\
 & & & + \overline{10}_{1,1,-2} + \bar{5}_{1,1,4} \\
 & & & + 1_{1,\pm 3,0} \\
 & + \mathit{32}'_{-\mathit{1}} & + \mathit{15}_{-\mathit{1},-\mathit{1}} + \overline{\mathit{15}}_{-\mathit{1},\mathit{1}} + \mathit{1}_{-\mathit{1},\pm 3} & + \mathit{10}_{-\mathit{1},-\mathit{1},\mathit{2}} + \mathit{5}_{-\mathit{1},-\mathit{1},-\mathit{4}} \\
 & & & + \overline{\mathit{10}}_{-\mathit{1},\mathit{1},-2} + \bar{\mathit{5}}_{-\mathit{1},\mathit{1},\mathit{4}} \\
 & & & + \mathit{1}_{-\mathit{1},\pm \mathit{3},\mathit{0}}.
\end{array}
\tag{B.8}
$$

B.1.1. $G_S = SU(6)$

For $G_S = SU(6)$, we have the following enhancement patterns

$$SU(6) \to SU(7) \to SU(8)$$

with $G_p = SU(8)$,

$$SU(6) \to SO(12) \to SO(14)$$

with $G_p = SO(14)$,

$$SU(6) \to E_6 \to E_7$$

with $G_p = E_7$, and

$$SU(6) \to SO(12) \to E_7$$

with $G_p = E_7$.

In this case, we only get the coupling $\mathbf{5\,\bar{5}\,1}$ at $G_p = SU(8)$, and from $G_p = SO(14)$ we are able to obtain couplings $\mathbf{10\,\bar{5}\,\bar{5}}$ and $\mathbf{5\,\bar{5}\,1}$. In addition, we also get the most important one, $\mathbf{10\,10\,5}$, from $G_p = E_7$.

B.1.2. $G_S = SO(10)$

For $G_S = SO(10)$, we have following enhancement patterns

$$SO(10) \to SO(12) \to SO(14)$$

with $G_p = SO(14)$,

$$SO(10) \to E_6 \to E_7$$

with $G_p = E_7$, and

$$SO(10) \to SO(12) \to E_7$$

with $G_p = E_7$.

In this case, we have the couplings $\mathbf{10\,\bar{5}\,\bar{5}}$ and $\mathbf{5\,\bar{5}\,1}$ from $G_p = SO(14)$, and we can also obtain the most important one, $\mathbf{10\,10\,5}$, from $G_p = E_7$. Note that we are not able to get $G_p = SU(8)$ which gives rise to the coupling $\mathbf{5\,\bar{5}\,1}$. Fortunately, this coupling can found in $G_p = SO(14)$ or $G_p = E_7$ instead.

B.2. SO(10) GUT model

For $SO(10)$ GUT model, we consider G_S and G_p to be of rank six and eight, respectively. Here we only consider the case of $G_p = SO(16)$ and E_8. It is straightforward to get the following resolutions:

$G_p = SO(16)$:

$$\begin{array}{llll}
SO(16) & \rightarrow SO(14)\times U(1) & \rightarrow SO(12)\times U(1)^2 & \rightarrow SO(10)\times U(1)^2\times U(1)_S, \\
120 & \rightarrow 91_0 + 1_0 & \rightarrow 66_{0,0} + 1_{0,0} + 1_{0,0} & \rightarrow 45_{0,0,0} + 3\times 1_{0,0,0} + 10_{0,0,2} \\
 & & \quad + 12_{0,2} + \overline{12}_{0,-2} & \quad + \overline{10}_{0,0,-2} + (10_{0,2,0} + 1_{0,2,2} \\
 & & & \quad + 1_{0,2,-2} + c.c.) \\
 & \quad + 14_2 + \overline{14}_{-2} & \quad + (12_{2,0} + 1_{2,2} & \quad + (10_{2,0,0} + 1_{2,0,2} + 1_{2,0,-2} \\
 & & \quad + 1_{2,-2} + c.c.) & \quad + 1_{2,2,0} + 1_{2,-2,0} + c.c.).
\end{array} \tag{B.9}$$

$G_p = E_8$:

$$\begin{array}{llll}
E_8 & \rightarrow E_7\times U(1) & \rightarrow E_6\times U(1)^2 & \rightarrow SO(10)\times U(1)^2\times U(1)_S, \\
248 & \rightarrow 133_0 + 1_0 & \rightarrow 78_{0,0} + 2\times 1_{0,0} & \rightarrow 45_{0,0,0} + 3\times 1_{0,0,0} + 16_{0,0,-3} \\
 & \quad + 1_{\pm 2} & \quad + 27_{0,2} + \overline{27}_{0,-2} + 1_{\pm 2,0} & \quad + \overline{16}_{0,0,3} + (16_{0,2,1} + 10_{0,2,-2} \\
 & & & \quad + 1_{0,2,4} + c.c.) + 1_{\pm 2,0,0} \\
 & \quad + 56_1 & \quad + 27_{1,-1} + \overline{27}_{1,1} + 1_{1,\pm 3} & \quad + 16_{1,-1,1} + 10_{1,-1,-2} \\
 & & & \quad + 1_{1,-1,4} + \overline{16}_{1,1,-1} + \overline{10}_{1,1,2} \\
 & & & \quad + 1_{1,1,-4} + 1_{1,\pm 3,0} \\
 & \quad + 56_{-1} & \quad + 27_{-1,-1} + \overline{27}_{-1,1} & \quad + 16_{-1,-1,1} + 10_{-1,-1,-2} \\
 & & \quad + 1_{-1,\pm 3} & \quad + 1_{-1,-1,4} + \overline{16}_{-1,1,-1} + \overline{10}_{-1,1,2} \\
 & & & \quad + 1_{-1,1,-4} + 1_{-1,\pm 3,0}.
\end{array} \tag{B.10}$$

$$\begin{array}{llll}
E_8 & \rightarrow E_7\times U(1) & \rightarrow SO(12)\times U(1)^2 & \rightarrow SO(10)\times U(1)^2\times U(1)_S, \\
248 & \rightarrow 133_0 + 1_0 & \rightarrow 66_{00} + 2\times 1_{00} + 1_{0,\pm 2} & \rightarrow 45_{0,0,0} + 3\times 1_{0,0,0} + (10_{0,0,2} \\
 & & & \quad + c.c.) + 1_{0,\pm 2,0} + 16_{0,1,-1} \\
 & \quad + 1_{\pm 2} & \quad + 32'_{0,1} + 32'_{0,-1} + 1_{\pm 2,0} & \quad + \overline{16}_{0,1,1} + 16_{0,-1,-1} + \overline{16}_{0,-1,1} \\
 & \quad + 56_1 & \quad + 32_{1,0} + 12_{1,1} + 12_{1,-1} & \quad + 1_{\pm 2,0,0} + 16_{1,0,1} + \overline{16}_{1,0,-1} \\
 & & & \quad + 10_{1,1,0} + 1_{1,1,\pm 2} \\
 & & & \quad + 10_{1,-1,0} + 1_{1,-1,\pm 2} \\
 & \quad + 56_{-1} & \quad + 32_{-1,0} + 12_{-1,1} & \quad + 16_{-1,0,1} + \overline{16}_{-1,0,-1} + 10_{-1,1,0} \\
 & & \quad + 12_{-1,-1} & \quad + 1_{-1,1,\pm 2} + 10_{-1,-1,0} \\
 & & & \quad + 1_{-1,-1,\pm 2}.
\end{array} \tag{B.11}$$

B.2.1. $G_S = SO(12)$

For $G_S = SO(12)$, we have following enhancement patterns

$SO(12) \rightarrow SO(14) \rightarrow SO(16)$

with $G_p = SO(16)$, and

$$SO(12) \to E_7 \to E_8$$

with $G_p = E_8$.

In this case, at $G_p = SO(16)$, we have couplings $\mathbf{10}\,\overline{\mathbf{10}}\,\mathbf{1}$ and $\mathbf{10}\,\mathbf{10}\,\mathbf{1}$, and at $G_p = E_8$, we can obtain $\mathbf{16}\,\mathbf{16}\,\mathbf{10}$.

B.2.2. $G_S = E_6$

For $G_S = E_6$, we have the following enhancement pattern

$$E_6 \to E_7 \to E_8$$

with $G_p = E_8$.

In this case, the only G_p we get is E_8, which gives rise to the couplings $\mathbf{16}\,\mathbf{16}\,\mathbf{10}$.

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