On strongly $\alpha$-preinvex functions

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Received 24 June 2006
Available online 15 September 2006
Submitted by M.A. Noor

Abstract

In this paper, by means of a series of counterexamples, we study in a systematic way the relationships among (pseudo, quasi) $\alpha$-preinvexity, (strict, strong, pseudo, quasi) $\alpha$-invexity and (strict, strong, pseudo, quasi) $\alpha\eta$-monotonicity. Results obtained in this paper can be viewed as a refinement and improvement of the results of Noor and Noor [M.A. Noor, K.I. Noor, Some characterizations of strongly preinvex functions, J. Math. Anal. Appl. 316 (2006) 697–706].

Keywords: Relations; (Pseudo, quasi) $\alpha$-preinvexity; (Pseudo, quasi) $\alpha$-invexity; (Pseudo, quasi) $\alpha\eta$-monotonicity

1. Introduction

It is well known that convexity and monotonicity play an important role for studying optimization problems, equilibrium problems and variational inequality problems. In recent years, the concepts of convexity and monotonicity have been generalized and extended in several directions by using novel and innovative techniques. An important and significant generalization of convexity and monotonicity is the introduction of preinvexity, invexity and $\eta$-monotonicity, see [1–6,9] and references therein. Recently, M.A. Noor and K.I. Noor in [7] introduced more general convexity and monotonicity, which are called $\alpha$-preinvexity, $\alpha$-invexity and $\alpha\eta$-monotonicity.

$^*$ This research is supported by Colleges and Universities Science and Technology Development Foundation (20040401) of Tianjin, PR China.

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doi:10.1016/j.jmaa.2006.08.067
Motivated and inspired by ideas of Noor and Noor, in this paper, we study in a systematic way the relationships among (pseudo, quasi) \( \alpha \)-preinvexity, (strict, strong, pseudo, quasi) \( \alpha \)-invexity and (strict, strong, pseudo, quasi) \( \alpha \eta \)-monotonicity by means of a series of counterexamples. We also prove some new results. Results proved in this paper represent refinement and improvement of the results of Noor and Noor [7] in several directions.

This paper is organized as follows. In Section 2, we study the relations among \( \alpha \)-preinvexity, pseudo \( \alpha \)-preinvexity and quasi \( \alpha \)-preinvexity. In Section 3, we discuss the relations among \( \alpha \)-invexity, pseudo \( \alpha \)-invexity and quasi \( \alpha \)-invexity. In Section 4, we investigate the relations between (pseudo, quasi) \( \alpha \)-preinvexity and (pseudo, quasi) \( \alpha \)-invexity. In Section 5, we research the relations among \( \alpha \eta \)-monotonicity, pseudo \( \alpha \eta \)-monotonicity and quasi \( \alpha \eta \)-monotonicity. In last section, we consider the relations between (strict, strong, pseudo, quasi) \( \alpha \)-invexity and (strict, strong, pseudo, quasi) \( \alpha \eta \)-monotonicity.

Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \) and \( K \) be a nonempty subset of \( H \). Let \( F : K \rightarrow R \) and \( \alpha : K \times K \rightarrow R \setminus \{0\} \) be two real-valued functions and \( \eta : K \times K \rightarrow H \) be a vector-valued mapping, which is not necessarily continuous. \( \alpha(\cdot, \cdot) \) is said to be a symmetric function if \( \alpha(v, u) = \alpha(u, v), \forall u, v \in K \). \( \eta(\cdot, \cdot) \) is called a skew mapping if \( \eta(v, u) + \eta(u, v) = 0, \forall u, v \in K \). Note that all of concepts in this paper are taken from [7].

**Definition 1.1.** Let \( u \in K \). The set \( K \) is said to be \( \alpha \)-invex at \( u \) with respect to \( \eta(\cdot, \cdot) \) and \( \alpha(\cdot, \cdot) \) if
\[
F(u + t\alpha(v, u)\eta(v, u)) \leq (1 - t)F(u) + tF(v);
\]
\( K \) is said to be an \( \alpha \)-invex set with respect to \( \eta \) and \( \alpha \) if it is invex at each \( u \in K \). \( \alpha \)-invex set is also called \( \alpha \eta \)-connected set. Clearly, \( K \) is a convex set with \( \alpha(v, u) = 1, \eta(v, u) = v - u \) and is an invex set in [8] with \( \alpha(v, u) = 1 \) for all \( u, v \in K \), but the converses are not true.

From now on, unless otherwise specified, we assume that \( K \) is a nonempty \( \alpha \)-invex set with respect to \( \eta \) and \( \alpha \).

2. Relations among \( \alpha \)-preinvexity, pseudo \( \alpha \)-preinvexity and quasi \( \alpha \)-preinvexity

In this section, we investigate the relations among \( \alpha \)-preinvexity, pseudo \( \alpha \)-preinvexity and quasi \( \alpha \)-preinvexity. Firstly, we recall these notions.

**Definition 2.1.** The function \( F \) on the \( \alpha \)-invex set \( K \) is said to be
\begin{enumerate}[(i)]
  
  \item \( \alpha \)-preinvex with respect to \( \eta \) and \( \alpha \) if, for any \( u, v \in K \) and \( t \in [0, 1] \),
  \[
  F(u + t\alpha(v, u)\eta(v, u)) \leq (1 - t)F(u) + tF(v);
  \]
  \item pseudo \( \alpha \)-preinvex with respect to \( \eta \) and \( \alpha \) if there exists a strictly positive function \( b : K \times K \rightarrow R^{++} \) such that, for any \( u, v \in K \) and \( t \in [0, 1] \),
  \[
  F(v) \leq F(u) \implies F(u + t\alpha(v, u)\eta(v, u)) \leq F(u) + t(t - 1)b(u, v);
  \]
  \item quasi \( \alpha \)-preinvex with respect to \( \eta \) and \( \alpha \) if, for any \( u, v \in K \) and \( t \in [0, 1] \),
  \[
  F(u + t\alpha(v, u)\eta(v, u)) \leq \max\{F(u), F(v)\}.
  \]
\end{enumerate}

**Lemma 2.1.** \( \alpha \)-Preinvexity does not imply pseudo \( \alpha \)-preinvexity and pseudo \( \alpha \)-preinvexity does not also imply \( \alpha \)-preinvexity, that is, they are two totally different concepts.
See the following two examples:

**Example 2.1.** Let $K = R$. For any $u, v ∈ K$, let $μ(v, u) = 1$, $η(v, u) = e^v - e^u$ and $F(u) = c$, where $c ∈ R$ is a constant. Then $K$ is an $α$-invex set with respect to $α$ and $η$ and

\[ F(u + tα(v, u)η(v, u)) = (1 - t)F(u) + tF(v), \quad ∀u, v ∈ K, \quad ∀t ∈ [0, 1], \]

which indicates that $F$ is $α$-preinvex with respect to $α$ and $η$ on $K$.

On the other hand, for any $t ∈ (0, 1)$ and any strictly positive function $b: K × K → R^+$, one has $F(v) = F(u)$ and

\[ F(u) + t(t - 1)b(u, v) < F(u) = F(u + tα(v, u)η(v, u)) \]

for any $u, v ∈ K$, which shows that $F$ is not pseudo $α$-preinvex with respect to $η$ and $α$ on $K$.

**Example 2.2.** Let $K = [−1, 0)$. For any $u, v ∈ K$, let $μ(v, u) = -\frac{u}{2}$, $η(v, u) = 1$, $b(u, v) = -\frac{u}{3}$ and $F(u) = 1 - u$. Then $b(u, v)$ is a strictly positive function and, for any $t ∈ [0, 1]$,

\[ u + tα(v, u)η(v, u) = \left(1 - \frac{t}{2}\right)u ∈ K, \]

which indicates that $K$ is an $α$-invex with respect to $α$ and $η$.

Assume that $F(v) ≤ F(u)$, then $u ≤ v$ and

\[ F(u + tα(v, u)η(v, u)) - F(u) = \frac{t}{2}u < \frac{t}{3}u = -tb(u, v) ≤ t(t - 1)b(u, v). \]

Thus, $F$ is pseudo $α$-preinvex with respect to $α$ and $η$ on $K$.

On the other hand, for any $u, v ∈ K$ and $t ∈ [0, 1]$, we have

\[ F(u + tα(v, u)η(v, u)) - (1 - t)F(u) - tF(v) = t\left(v - \frac{u}{2}\right). \]

Taking $v = -\frac{1}{3}$, $u = -\frac{1}{2}$ and $t = \frac{1}{2}$, we obtain

\[ F(u + tα(v, u)η(v, u)) - (1 - t)F(u) - tF(v) = \frac{1}{40} > 0. \]

This shows that $F$ is not $α$-preinvex with respect to $η$ and $α$ on $K$.

**Lemma 2.2.** $α$-Preinvexity implies quasi $α$-preinvexity, but the converse is not true.

**Proof.** Let $F$ be $α$-preinvex with respect to $η$ and $α$ on $K$. Then for any $u, v ∈ K$ and $t ∈ [0, 1]$, it follows that

\[ F(u + tα(v, u)η(v, u)) ≤ (1 - t)F(u) + tF(v) \]

\[ ≤ (1 - t)\text{max}\{F(u), F(v)\} + t\text{max}\{F(u), F(v)\} \]

\[ = \text{max}\{F(u), F(v)\}, \]

which shows that $F$ is quasi $α$-preinvex with respect to $η$ and $α$ on $K$. □

**Example 2.3.** Let $K = [−1, 0)$. For any $u, v ∈ K$, let $μ(v, u) = -\frac{u}{2}$, $η(v, u) = 1$ and $F(u) = 1 - u$. By Example 2.2, we know that $K$ is $α$-invex and $F$ is not $α$-preinvex with respect to $η$ and $α$ on $K$. However, for any $u, v ∈ K$ and $t ∈ [0, 1]$, we have

\[ F(u + tα(v, u)η(v, u)) = 1 - u + \frac{t}{2}u ≤ F(u) ≤ \text{max}\{F(u), F(v)\}. \]

Hence, $F$ is quasi $α$-preinvex with respect to $η$ and $α$ on $K$.  □
Lemma 2.3. Pseudo $\alpha$-preinvexity does not imply quasi $\alpha$-preinvexity and quasi $\alpha$-preinvexity does not also imply pseudo $\alpha$-preinvexity, i.e., they are two totally different concepts.

See the following two examples:

Example 2.4. Let $K = (-\infty, 0)$. For any $u, v \in K$, let $\eta(v, u) = 1$, $b(u, v) = -\frac{u}{2}$, $F(u) = 1 - u$ and

$$\alpha(v, u) = \begin{cases} 2(v - u), & v < u, \\ \frac{u}{2}, & v \geq u. \end{cases}$$

Then $b(u, v)$ is a strictly positive function and, for any $t \in [0, 1]$, $u + t\alpha(v, u)\eta(v, u) = \begin{cases} u + 2t(v - u), & v < u, \\ (1 - \frac{u}{2})u, & v \geq u, \end{cases} \in K$.

This indicates that $K$ is an $\alpha$-invex set. If $F(v) \leq F(u)$, then $u \leq v$ and

$$F(u + t\alpha(v, u)\eta(v, u)) = F(u) + \frac{t}{2}u < F(u) - tb(u, v) \leq F(u) + t(t - 1)b(u, v),$$

which implies that $F$ is pseudo $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$.

On the other hand, for $u, v \in K$: $v < u$ and $t = \frac{3}{4}$, we have $F(u) < F(v)$ and

$$F\left(u + \frac{3}{4}\alpha(v, u)\eta(v, u)\right) - \max\{F(u), F(v)\} = F(u + t\alpha(v, u)\eta(v, u)) - F(v) = -\frac{1}{2}(v - u) > 0.$$

This shows that $F$ is not quasi $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$.

Example 2.5. Let $K, \alpha(v, u), \eta(v, u)$ and $F(u)$ be same as in Example 2.1. By Example 2.1, we know that $K$ is an $\alpha$-invex set with respect to $\alpha$ and $\eta$ and $F$ is not pseudo $\alpha$-preinvex with respect to $\alpha$ and $\eta$ on $K$. However, for any $u, v \in K$ and $t \in [0, 1]$,

$$F(u + t\alpha(v, u)\eta(v, u)) = c = \max\{F(u), F(v)\},$$

where $c \in R$ is a constant. Hence, $F$ is quasi $\alpha$-preinvex with respect to $\alpha$ and $\eta$ on $K$.

3. Relations among $\alpha$-invexity, pseudo $\alpha$-invexity and quasi $\alpha$-invexity

In this section, we study the relations among $\alpha$-invexity, pseudo $\alpha$-invexity and quasi $\alpha$-invexity. Let the function $F : K \to R$ be differentiable on $K$.

Definition 3.1. $F$ on the $\alpha$-invex set $K$ is said to be

(i) $\alpha$-invex with respect to $\alpha$ and $\eta$ if

$$F(v) - F(u) \geq \langle \alpha(v, u)F'(u), \eta(v, u) \rangle, \quad \forall u, v \in K,$$

where $F'(u)$ denotes the differential of $F$ at $u \in K$;

(ii) strictly $\alpha$-invex with respect to $\alpha$ and $\eta$ if

$$F(v) - F(u) > \langle \alpha(v, u)F'(u), \eta(v, u) \rangle, \quad \forall u, v \in K;$$
(iii) strongly $\alpha$-invex with respect to $\alpha$ and $\eta$ if there exists a constant $\mu > 0$ such that, for any $u, v \in K$,
\[ F(v) - F(u) \geq [\alpha(v, u)F'(u), \eta(v, u)] + \mu \|\eta(v, u)\|^2; \]
(iv) pseudo $\alpha$-invex with respect to $\alpha$ and $\eta$ if, for any $u, v \in K$,
\[ \langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0 \implies F(v) \geq F(u); \]
(v) strictly pseudo $\alpha$-invex with respect to $\alpha$ and $\eta$ if, for any $u, v \in K$,
\[ \langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0 \implies F(v) > F(u); \]
(vi) strongly pseudo $\alpha$-invex with respect to $\alpha$ and $\eta$ if there exists a constant $\mu > 0$ such that, for any $u, v \in K$,
\[ \langle \alpha(v, u)F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^2 \geq 0 \implies F(v) \geq F(u); \]
(vii) quasi $\alpha$-preinvex with respect to $\alpha$ and $\eta$ if, for any $u, v \in K$,
\[ F(v) \leq F(u) \implies \langle \alpha(v, u)F'(u), \eta(v, u) \rangle \leq 0. \]

**Lemma 3.1.** It is clear that $\alpha$-invexity implies pseudo $\alpha$-invexity, but the converse is not true.

**Example 3.1.** Let $K = \mathbb{R}$. For any $u, v \in K$, let $\alpha(v, u) = \frac{3}{2}$, $\eta(v, u) = v - u$ and $F(u) = \frac{1}{3}u$. Then, $K$ is an $\alpha$-invex set with respect to $\alpha$ and $\eta$.

Assume that $\langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0$, then $v - u \geq 0$ and $F(v) - F(u) = \frac{1}{3}(v - u) \geq 0$.

This shows that $F$ is pseudo $\alpha$-invex with respect to $\alpha$ and $\eta$ on $K$.

On the other hand, for $\bar{u} = 0$ and $\bar{v} = 1$, we have
\[ F(\bar{v}) - F(\bar{u}) - \langle \alpha(\bar{v}, \bar{u})F'(\bar{u}), \eta(\bar{v}, \bar{u}) \rangle = -\frac{1}{6} < 0. \]

So, $F$ is not $\alpha$-invex with respect to $\alpha$ and $\eta$ on $K$.

**Lemma 3.2.** It is clear that $\alpha$-invexity implies quasi $\alpha$-invexity, but the converse is not true.

**Example 3.2.** Let $K, \alpha(v, u), \eta(v, u)$ and $F(u)$ are same as in Example 3.1. By Example 3.1, we know that $F$ is not $\alpha$-invex with respect to $\alpha$ and $\eta$ on the $\alpha$-invex set $K$.

However, for any $u, v \in K$, if $F(v) \leq F(u)$, then $v \leq u$ and
\[ \langle \alpha(v, u)F'(u), \eta(v, u) \rangle = \frac{1}{2}(v - u) \leq 0, \]

which shows that $F$ is quasi $\alpha$-invex with respect to $\alpha$ and $\eta$ on $K$.

**Lemma 3.3.** Pseudo $\alpha$-invexity does not imply quasi $\alpha$-invexity and quasi $\alpha$-invexity does not also imply pseudo $\alpha$-invexity, that is, they are two different notions.

See the following two examples:

**Example 3.3.** Let $K = \mathbb{R}$. For any $u, v \in K$, let $\eta(v, u) = 1$, $F(u) = u$ and
\[ \alpha(v, u) = \begin{cases} -1, & v \neq u, \\ 1, & v = u. \end{cases} \]
Then $K$ is an $\alpha$-invex set with respect to $\eta$ and $\alpha$ and

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle = \begin{cases} -1, & v \neq u, \\ 1, & v = u. \end{cases}$$

Consequently,

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0 \quad \Rightarrow \quad \langle \alpha(v, u)F'(u), \eta(v, u) \rangle = 1$$

$$\iff \quad v = u$$

$$\iff \quad F(u) = F(v),$$

which implies that $F$ is pseudo $\alpha$-invex, but not quasi $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$.

**Example 3.4.** Let $K =\mathbb{R}$. For any $u, v \in K$, let $F(u) = u$ and

$$\alpha(v, u) = \begin{cases} 1, & v \geq u, \\ -1, & v < u, \end{cases} \quad \eta(v, u) = \begin{cases} 1, & v > u, \\ -1, & v = u, \\ 0, & v < u. \end{cases}$$

Then $K$ is an $\alpha$-invex set with respect to $\eta$ and $\alpha$ and

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle = \begin{cases} 1, & v > u, \\ -1, & v = u, \\ 0, & v < u. \end{cases}$$

Consequently,

$$F(u) \leq F(v) \quad \Rightarrow \quad v \leq u \quad \Rightarrow \quad \langle \alpha(v, u)F'(u), \eta(v, u) \rangle \leq 0,$$

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle = 0 \quad \Rightarrow \quad F(v) < F(u),$$

which shows that $F$ is quasi $\alpha$-invex, but not pseudo $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$.

4. Relations between (pseudo, quasi) $\alpha$-preinvexity and (pseudo, quasi) $\alpha$-invexity

In this section, we discuss the relations between pseudo and quasi $\alpha$-preinvexity and pseudo and quasi $\alpha$-invexity. Concerned with the relation between $\alpha$-preinvexity and $\alpha$-invexity, see [7, Theorem 3.1].

**Lemma 4.1.** Pseudo $\alpha$-preinvexity implies pseudo $\alpha$-invexity, but the converse is not true.

**Proof.** Let $F$ be pseudo $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$. Then there exists a strictly positive function $b: K \times K \to \mathbb{R}^{++}$ such that (1) holds for any $u, v \in K$ and $t \in [0, 1]$.

If $\langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0$, then there exists $t_0 \in (0, 1)$ such that

$$F(u + t\alpha(v, u)\eta(v, u)) - F(u) \geq 0, \quad \forall t \in (0, t_0),$$

since

$$\lim_{t \downarrow 0} \frac{F(u + t\alpha(v, u)\eta(v, u)) - F(u)}{t} = \langle \alpha(v, u)F'(u), \eta(v, u) \rangle.$$

(3)

Consequently,

$$F(u + t\alpha(v, u)\eta(v, u)) > F(u) + t(t - 1)b(u, v), \quad \forall t \in (0, t_0).$$

By (1), we get $F(v) > F(u)$ and then $F$ is pseudo $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$. \qed
Example 4.1. Let $K = \mathbb{R}$. For any $u, v \in K$, let $F(u) = u$ and

$$\alpha(v, u) = \begin{cases} 1, & v > u, \\ -1, & v \leq u, \end{cases} \quad \eta(v, u) = \begin{cases} 1, & v \neq u, \\ 0, & v = u. \end{cases}$$

Then $K$ is an $\alpha$-invex set with respect to $\eta$ and $\alpha$ and

$$\alpha(v, u)\eta(v, u) = \begin{cases} 1, & v > u, \\ 0, & v = u, \\ -1, & v < u. \end{cases}$$

Consequently,

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0 \implies v \geq u \implies F(v) \geq F(u),$$

which shows that $F$ is pseudo $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$.

On the other hand, if $F(v) = F(u)$, then $v = u$ and, for any strictly positive function $b : K \times K \to \mathbb{R}^+$ and any $t \in (0, 1)$,

$$F\left(u + t\alpha(v, u)\eta(v, u)\right) - F(u) = 0 > t(t - 1)b(u, v),$$

which shows that $F$ is not pseudo $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$.

Lemma 4.2. Pseudo $\alpha$-preinvexity implies quasi $\alpha$-invexity, but the converse is not true.

Proof. Let $F$ be pseudo $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$. Then there exists a strictly positive function $b : K \times K \to \mathbb{R}^+$ such that (1) holds for any $u, v \in K$ and $t \in [0, 1]$.

Take arbitrarily $u, v \in K$ and let $F(v) \leq F(u)$. By (1), it follows that

$$\frac{F(u + t\alpha(v, u)\eta(v, u)) - F(u)}{t} \leq (t - 1)b(u, v), \quad \forall t \in (0, 1).$$

Letting $t \downarrow 0$, by (3), we have

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle \leq -b(u, v) < 0.$$

Thus, $F$ is quasi $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$. \qed

Example 4.2. Let $K = [0, +\infty)$. For any $u, v \in K$, let $F(u) = u$, $\alpha(v, u) = 1$ and

$$\eta(v, u) = \begin{cases} 1, & v > u, \\ 0, & v \leq u. \end{cases}$$

Then, for any $t \in [0, 1]$,

$$u + t\alpha(v, u)\eta(v, u) = \begin{cases} u + t, & v > u, \\ u, & v \leq u, \end{cases} \quad \in K,$$

which indicates that $K$ is an $\alpha$-invex set with respect to $\eta$ and $\alpha$.

If $F(v) \leq F(u)$, then $v \leq u$ and, for any strictly positive function $b : K \times K \to \mathbb{R}^+$ and any $t \in (0, 1)$,

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle = 0,$$

$$F\left(u + t\alpha(v, u)\eta(v, u)\right) - F(u) = 0 > t(t - 1)b(u, v),$$

which shows that $F$ is quasi $\alpha$-invex, but not pseudo $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$.\[\square\]
Lemma 4.3. Quasi $\alpha$-preinvexity implies quasi $\alpha$-invexity, but the converse is not true.

Proof. Let $F$ be quasi $\alpha$-preinvex with respect to $\alpha$ and $\eta$ on $K$. Then (2) holds for any $u, v \in K$ and $t \in (0, 1)$.

Assume that $F(v) \leq F(u)$. By (2) and (3), we can deduce that

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle \leq 0,$$

which indicates that $F$ is quasi $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$. □

Example 4.3. Let $K, F(u), \alpha(v, u)$ and $\eta(v, u)$ be same as in Example 4.1. By Example 4.1, we know that $K$ is an $\alpha$-invex set with respect to $\eta$ and $\alpha$ and

$$F(v) \leq F(u) \Rightarrow \alpha(v, u)\eta(v, u) \leq 0 \Rightarrow \langle \alpha(v, u)F'(u), \eta(v, u) \rangle \leq 0.$$

Thus, $F$ is quasi $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$.

On the other hand, taking arbitrarily $u \in K$ and $t \in (0, 1)$ and letting $v = u + \frac{t}{2}$, we have

$$F(u + t\alpha(v, u)\eta(v, u)) = u + t > u + \frac{t}{2} = \max\{F(u), F(v)\},$$

which implies that $F$ is not quasi $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$.

Lemma 4.4. Quasi $\alpha$-preinvexity does not imply pseudo $\alpha$-invexity and pseudo $\alpha$-invexity does not also imply quasi $\alpha$-preinvexity.

See the following two examples:

Example 4.4. Let $K, F(u), \alpha(v, u)$ and $\eta(v, u)$ be same as in Example 4.1. By Examples 4.1 and 4.3, we know that $F$ is pseudo $\alpha$-invex, but not quasi $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$.

Example 4.5. Let $K = \mathbb{R}$. For any $u, v \in K$, let $F(u) = u$ and

$$\alpha(v, u) = \begin{cases} -1, & v \geq u, \\ 1, & v < u, \end{cases} \quad \eta(v, u) = \begin{cases} 1, & v \geq u, \\ 0, & v < u. \end{cases}$$

Then $K$ is an $\alpha$-invex set with respect to $\eta$ and $\alpha$ and

$$\alpha(v, u)\eta(v, u) = \begin{cases} -1, & v \geq u, \\ 0, & v < u. \end{cases}$$

Consequently,

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0 \Rightarrow \alpha(v, u)\eta(v, u) = 0 \Rightarrow v < u \Rightarrow F(v) < F(u),$$

which shows that $F$ is not pseudo $\alpha$-invex with respect to $\eta$ and $\alpha$ on $K$.

On the other hand, for any $u, v \in K$ and $t \in [0, 1]$,

$$F(u + t\alpha(v, u)\eta(v, u)) = \begin{cases} u - t \leq F(v), & v \geq u, \\ u = F(u), & v < u, \end{cases} \leq \max\{F(u), F(v)\}.$$

Hence, $F$ is quasi $\alpha$-preinvex with respect to $\eta$ and $\alpha$ on $K$. 

5. Relations among $\alpha\eta$-, pseudo $\alpha\eta$- and quasi $\alpha\eta$-monotonicity

In this section, we consider the relations among $\alpha\eta$-monotonicity, pseudo $\alpha\eta$-monotonicity and quasi $\alpha\eta$-monotonicity.

**Definition 5.1.** An operator $T : K \to H$ on the $\alpha$-invex set $K$ is said to be

(i) $\alpha\eta$-monotone if, for any $u, v \in K$,
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle + \langle \alpha(u, v)T(v), \eta(u, v) \rangle \leq 0;
\]

(ii) strictly $\alpha\eta$-monotone if, for any $u, v \in K$,
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle + \langle \alpha(u, v)T(v), \eta(u, v) \rangle < 0;
\]

(iii) strongly $\alpha\eta$-monotone if there exists a constant $\beta > 0$ such that, for any $u, v \in K$,
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle + \langle \alpha(u, v)T(v), \eta(u, v) \rangle \leq -\beta \left\{ \| \eta(v, u) \|^2 + \| \eta(u, v) \|^2 \right\};
\]

(iv) pseudo $\alpha\eta$-monotone if, for any $u, v \in K$,
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle \geq 0 \implies \langle \alpha(u, v)T(v), \eta(u, v) \rangle \leq 0;
\]

(v) strictly pseudo $\alpha\eta$-monotone if, for any $u, v \in K$,
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle \geq 0 \implies \langle \alpha(u, v)T(v), \eta(u, v) \rangle < 0;
\]

(vi) strongly pseudo $\alpha\eta$-monotone if there exists a constant $\mu > 0$ such that, for any $u, v \in K$,
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle + \mu \| \eta(v, u) \|^2 \geq 0 \implies \langle \alpha(u, v)T(v), \eta(u, v) \rangle \leq 0;
\]

(vii) quasi $\alpha\eta$-monotone if, for any $u, v \in K$,
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle > 0 \implies \langle \alpha(u, v)T(v), \eta(u, v) \rangle \leq 0.
\]

**Lemma 5.1.** It is clear that strict (respectively strong) $\alpha\eta$-monotonicity implies $\alpha\eta$-monotonicity, but the converse is not true.

**Example 5.1.** Let $K = [0, 1]$. For any $u, v \in K$, let $\alpha(v, u) = 1$, $\eta(v, u) = v - u$ and $T(u) = 1$. Then $K$ is an $\alpha$-invex set with respect to $\alpha$ and $\eta$ and $T$ is $\alpha\eta$-monotone on $K$. But $T$ is neither strictly nor strongly $\alpha\eta$-monotone.

**Lemma 5.2.** Strict $\alpha\eta$-monotonicity does not imply strong $\alpha\eta$-monotonicity and strong $\alpha\eta$-monotonicity does not also imply strict $\alpha\eta$-monotonicity.

See the following two examples:

**Example 5.2.** Let $K = R$. For any $u, v \in K$, let $\alpha(v, u) = 1$, $T(u) = 1$ and
\[
\eta(v, u) = \begin{cases} 
  u - v, & v > u, \\
  -1, & v = u, \\
  0, & v < u.
\end{cases}
\]
Then, for any \( t \in [0, 1] \),
\[
\begin{align*}
u + t \alpha(v, u) \eta(v, u) &= \begin{cases} 
(1 + t)u - tv, & v > u, \\
u - t, & v = u, \\
u, & v < u,
\end{cases} \quad \in K,
\end{align*}
\]
\[
\begin{align*}
\langle \alpha(v, u) T(u), \eta(v, u) \rangle + \langle \alpha(u, v) T(v), \eta(u, v) \rangle &= \begin{cases} 
u - v, & v > u, \\
-2, & v = u, \\
v - u, & v < u,
\end{cases} < 0,
\end{align*}
\]
which shows that \( K \) is an \( \alpha \)-invex set with respect to \( \eta \) and \( \alpha \) and \( T \) is strictly \( \alpha \eta \)-monotone on \( K \).

On the other hand, for any \( \beta > 0 \) and any \( u \in K \), take \( v = u + \frac{2}{\beta} \). We can deduce that
\[
\begin{align*}
\langle \alpha(v, u) T(u), \eta(v, u) \rangle + \langle \alpha(u, v) T(v), \eta(u, v) \rangle &= -\frac{2}{\beta},
\end{align*}
\]
\[
-\beta \left\{ \| \eta(v, u) \|^2 + \| \eta(u, v) \|^2 \right\} = -\frac{8}{\beta}.
\]
Thus, \( T \) is not strongly \( \alpha \eta \)-monotone on \( K \).

Example 5.3. Let \( K = (-\infty, -1] \). For any \( u, v \in K \), let \( \alpha(v, u) = 1, T(u) = 1 \) and
\[
\eta(v, u) = \begin{cases} 
0, & v \neq u, \\
-1, & v = u.
\end{cases}
\]
Then \( K \) is an \( \alpha \)-invex set with respect to \( \eta \) and \( \alpha \). For \( \beta = \frac{1}{2} \) and any \( u, v \in K \), we can deduce that
\[
\begin{align*}
\langle \alpha(v, u) T(u), \eta(v, u) \rangle + \langle \alpha(u, v) T(v), \eta(u, v) \rangle &= \begin{cases} 
0, & v \neq u, \\
-2, & v = u,
\end{cases}
\end{align*}
\]
\[
-\beta \left\{ \| \eta(v, u) \|^2 + \| \eta(u, v) \|^2 \right\} = \begin{cases} 
0, & v \neq u, \\
-1, & v = u.
\end{cases}
\]
Therefore, \( T \) is strongly but not strictly \( \alpha \eta \)-monotone on \( K \).

Lemma 5.3. It is clear that \( \alpha \eta \)-monotonicity implies pseudo (respectively quasi) \( \alpha \eta \)-monotonicity, but the converse is not true.

Example 5.4. Let \( K = R \). For any \( u, v \in K \), let \( \eta(v, u) = 1, T(u) = 1 \) and
\[
\alpha(v, u) = \begin{cases} 
1, & v > u, \\
-\frac{1}{2}, & v \leq u.
\end{cases}
\]
Then \( K \) is an \( \alpha \)-invex set with respect to \( \eta \) and \( \alpha \) and
\[
\begin{align*}
\langle \alpha(v, u) T(u), \eta(v, u) \rangle = \alpha(v, u) = \begin{cases} 
1, & v > u, \\
-\frac{1}{2}, & v \leq u.
\end{cases}
\end{align*}
\]
Consequently, we have
\[
\langle \alpha(v, u) T(u), \eta(v, u) \rangle \geq 0 \quad \Rightarrow \quad v > u \quad \Rightarrow \quad \langle \alpha(u, v) T(v), \eta(u, v) \rangle < 0,
\]
\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle + \langle \alpha(u, v)T(v), \eta(u, v) \rangle = \begin{cases} 
\frac{1}{2}, & v \neq u, \\
-1, & v = u.
\end{cases}
\]

Thus, \( T \) is both pseudo and quasi \( \alpha \eta \)-monotone, but not \( \alpha \eta \)-monotone on \( K \).

**Lemma 5.4.** It is clear that pseudo \( \alpha \eta \)-monotonicity implies quasi \( \alpha \eta \)-monotonicity, but the converse is not true.

**Example 5.5.** (See [4, Example 3.3].) Let \( K = R \). For any \( u, v \in K \), let \( \alpha(v, u) = 1, \eta(v, u) = e^v - e^u \) and \( T(u) = \begin{cases} 
0, & u \geq 0, \\
-u, & u < 0.
\end{cases} \)

Then \( K \) is an \( \alpha \)-invex set with respect to \( \eta \) and \( \alpha \) and

\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle = \begin{cases} 
0, & u \geq 0, \\
-u(e^v - e^u), & u < 0,
\end{cases}
\begin{cases} 
= 0, & u \geq 0 \text{ or } u = v, \\
> 0, & u < 0 \text{ and } v > u, \\
< 0, & u < 0 \text{ and } v < u.
\end{cases}
\]

If \( \langle \alpha(v, u)T(u), \eta(v, u) \rangle > 0 \), then \( v > u \) and

\[
\langle \alpha(u, v)T(v), \eta(u, v) \rangle = \begin{cases} 
0, & v \geq 0, \\
-v(e^u - e^v), & v < 0,
\end{cases} \leq 0,
\]

which shows that \( T \) is quasi \( \alpha \eta \)-monotone on \( K \).

On the other hand, for \( u, v \in K: v < 0 < u \), we have

\[
\langle \alpha(v, u)T(u), \eta(v, u) \rangle = 0 \quad \text{and} \quad \langle \alpha(u, v)T(v), \eta(u, v) \rangle > 0.
\]

So, \( T \) is not pseudo \( \alpha \eta \)-monotone on \( K \).

### 6. Relations between \( \alpha \)-invexity and \( \alpha \eta \)-monotonicity

In this section, we research the relations between (strict, strong, pseudo, quasi) \( \alpha \)-invexity of the differentiable function \( F(u) \) and (strict, strong, pseudo, quasi) \( \alpha \eta \)-monotonicity of its differential \( F'(u) \).

Using the similar proof way of the first consequence in [7, Theorem 3.2], we have the following relation.

**Lemma 6.1.** If \( F \) is (respectively strictly, strongly) \( \alpha \)-invex, then its differential \( F'(u) \) is (respectively strictly, strongly) \( \alpha \eta \)-monotone.

We prove the following result, which is a modification of a result of Noor and Noor [7, Theorem 3.2].

**Theorem 6.1.** Let \( F'(u) \) be strongly \( \alpha \eta \)-monotone with modulus \( \beta > 0 \) on \( K \). If the following assumptions hold:

(i) \( \alpha \) is a symmetric function such that

\[
\alpha(u, u + t\alpha(v, u)\eta(v, u)) = t\alpha(v, u), \quad \forall u, v \in K, \; t \in [0, 1];
\]

(ii) \( \alpha \) is a symmetric function such that

\[
\alpha(\alpha(v, u)T(u), \eta(v, u)) = 0, \quad \forall u, v \in K.
\]

Then \( F \) is (respectively strictly, strongly) \( \alpha \)-invex on \( K \).
(ii) $\eta$ is a skew mapping such that
\[ \eta(u, u + t\alpha(v, u)\eta(v, u)) = -t\eta(v, u), \quad \forall u, v \in K, \ t \in [0, 1]; \tag{5} \]

(iii) $F(u + \alpha(v, u)\eta(v, u)) \leq F(v), \forall u, v \in K.$

Then $F$ is strongly $\alpha$-invex with modulus $2\beta$ on $K.$

**Proof.** Take arbitrarily $u, v \in K$ and $t \in [0, 1]$ and let $v_t = u + t\alpha(v, u)\eta(v, u).$ By the $\alpha$-invexity of $K$ and the strong $\alpha\eta$-monotonicity of $F'(u),$ we know that $v_t \in K$ and
\[ \{\alpha(v_t, u)F'(u), \eta(v_t, u)\} + \{\alpha(u, v_t)F'(v_t), \eta(u, v_t)\} \leq -\beta\{\|\eta(v_t, u)\|^2 + \|\eta(u, v_t)\|^2\}. \]

From (4) and (5), it follows that
\[ \{\alpha(v, u)F'(u), \eta(v, u)\} \geq \{\alpha(v, u)F'(u), \eta(v, u)\} + 2\beta\|\eta(v, u)\|^2. \tag{6} \]

Let $g(t) = F(u + t\alpha(v, u)\eta(v, u)), \forall t \in [0, 1].$ Then, it follows from (6) that
\[
g'(t) = \{\alpha(v, u)F'(v_t), \eta(v, u)\} \\
\geq \{\alpha(v, u)F'(u), \eta(v, u)\} + 2\beta\|\eta(v, u)\|^2. \\
\]

Consequently,
\[
g(1) - g(0) = \int_{0}^{1} g'(t) \, dt \geq \{\alpha(v, u)F'(u), \eta(v, u)\} + 2\beta\|\eta(v, u)\|^2. \\
\]

By the assumption (iii), we get
\[ F(v) - F(u) \geq \{\alpha(v, u)F'(u), \eta(v, u)\} + 2\beta\|\eta(v, u)\|^2, \]

which shows that $F$ is strongly $\alpha$-invex with the modulus $2\beta.$ \hfill $\Box$

With the similar way in Theorem 6.1, we have the following consequence.

**Theorem 6.2.** Let $F'(u)$ be (respectively strictly) $\alpha\eta$-monotone on $K.$ If the hypotheses (i)--(iii) in Theorem 6.1 hold, then $F$ is (respectively strictly) $\alpha$-invex on $K.$

The following relation is an direct consequence of the definitions of strictly pseudo $\alpha$-invexity and strictly pseudo $\alpha\eta$-monotonicity.

**Lemma 6.2.** If $F$ is strictly pseudo $\alpha$-invex on $K,$ then its differential $F'(u)$ is strictly pseudo $\alpha\eta$-monotone on $K.$

Concerning the converse relation, we have the following theorem.

**Theorem 6.3.** Let $F'(u)$ be (respectively strictly) pseudo $\alpha\eta$-monotone on $K.$ If the hypotheses (i)--(iii) in Theorem 6.1 hold, then $F$ is (respectively strictly) pseudo $\alpha$-invex on $K.$
Proof. Take arbitrarily \( u, v \in K \) and \( t \in [0, 1] \) and let \( \langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0 \). Then
\[
\langle \alpha(v_t, u)F'(u), \eta(v_t, u) \rangle = t^2 \langle \alpha(v, u)F'(u), \eta(v, u) \rangle \geq 0,
\]
where \( v_t = u + t\alpha(v, u)\eta(v, u) \in K \). As \( F'(u) \) is pseudo \( \alpha\eta \)-monotone, we have
\[
\langle \alpha(u, v_t)F'(v_t), \eta(u, v_t) \rangle = -t^2 \langle \alpha(v, u)F'(v_t), \eta(v, u) \rangle \leq 0,
\]
which can be written as \( \langle \alpha(v, u)F'(v_t), \eta(v, u) \rangle \geq 0 \),
\[
\langle \alpha(u, v_t)F'(v_t), \eta(u, v_t) \rangle = \begin{cases} 
-1, & v \geq u, \\
1, & v < u.
\end{cases}
\]
As \( F'(u) \) is pseudo \( \alpha\eta \)-monotone, we have
\[
\langle \alpha(v, u)F'(u), \eta(v, u) \rangle = -t^2 \langle \alpha(v, u)F'(v_t), \eta(v, u) \rangle \leq 0,
\]
which can be written as \( \langle \alpha(v, u)F'(v_t), \eta(v, u) \rangle = 0 \).

Let \( g(t) = F(u + t\alpha(v, u)\eta(v, u)) \), \( \forall t \in [0, 1] \). Then
\[
F(v) - F(u) \geq g(1) - g(0) = \int_0^1 g'(t) dt = \int_0^1 \langle \alpha(v, u)F'(v_t), \eta(v, u) \rangle dt \geq 0.
\]
Thus, \( F \) is pseudo \( \alpha \)-invex.

The following example shows that the result of [7, Theorem 3.5] may not be true.

Example 6.1. Let \( K = R \). For any \( u, v \in K \), let \( F(u) = u, \eta(v, u) = v - u \) and
\[
\alpha(v, u) = \begin{cases} 
-1, & v \geq u, \\
1, & v < u.
\end{cases}
\]
Then \( K \) is an \( \alpha \)-invex set with respect to \( \eta \) and \( \alpha \) and
\[
\alpha(v, u)\eta(v, u) = \begin{cases} 
-u, & v \geq u, \\
u - v, & v < u.
\end{cases}
\]

Let \( \langle \alpha(v, u)F'(u), \eta(v, u) \rangle + \|\eta(v, u)\|^2 \geq 0 \). Then \( \alpha(v, u)(v - u) + (v - u)^2 \geq 0 \) and then
\[
\langle \alpha(u, v)F'(v), \eta(u, v) \rangle = \begin{cases} 
0, & v = u, \\
-v - u \leq -1, & v > u, \\
v - u \leq -1, & v < u,
\end{cases}
\]
which shows that \( F'(u) \) is strongly pseudo \( \alpha\eta \)-monotone with modulus \( \mu = 1 \) on \( K \).

Since
\[
F(u + \alpha(v, u)\eta(v, u)) = \begin{cases} 
2u - v, & v \geq u, \\
v, & v < u.
\end{cases}
\]
\[
\eta(u, u + t\alpha(v, u)\eta(v, u)) = \begin{cases} 
t\eta(v, u), & v \geq u, \\
-t\eta(v, u), & v < u,
\end{cases}
\]
\[
\eta(v, u + t\alpha(v, u)\eta(v, u)) = \begin{cases} 
(1 + t)\eta(v, u), & v \geq u, \\
(1 - t)\eta(v, u), & v < u,
\end{cases}
\]
we know that all the assumptions of [7, Theorem 3.5] hold for any \( u, v \in K : v < u \) and \( t \in [0, 1] \).

However, for \( u, v \in K : v < u \) and \( t \in [0, 1] \), if \( \langle \alpha(v, u)F'(u), \eta(v, u) \rangle + \|\eta(v, u)\|^2 \geq 0 \), then \( F(v) \leq F(u) - 1 < F(u) \). This shows that \( F \) is not strongly pseudo \( \alpha \)-invex on \( K \).

We modify the result of [7, Theorem 3.5] by the following theorem.

Theorem 6.4. Let \( F'(u) \) be strongly pseudo \( \alpha\eta \)-monotone with modulus \( \mu > 0 \) on \( K \). If the hypotheses (i)–(iii) in Theorem 6.1 hold, then \( F \) is strongly pseudo \( \alpha \)-invex with same modulus \( \mu \) on \( K \).
Proof. For any $u, v \in K$ and $t \in [0, 1]$, let $v_t = u + t\alpha(v, u)\eta(v, u)$ and

$$\langle \alpha(v, u)F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^2 \geq 0.$$ 

Then, by the $\alpha$-invexity of $K$, $v_t \in K$ and

$$\langle \alpha(v_t, u)F'(u), \eta(v_t, u) \rangle + \mu \|\eta(v, u)\|^2 = t^2(\langle \alpha(v, u)F'(u), \eta(v, u) \rangle + \mu \|\eta(v, u)\|^2) \geq 0.$$ 

From the strongly pseudo $\alpha\eta$-monotonicity of $F'(u)$, it follows that (7) holds. By using the technique in Theorem 6.3, we can prove that $F$ is strongly pseudo $\alpha$-invex with modulus $\mu$ on $K$. \qed

The following relation is a direct consequence of the definitions of quasi $\alpha$-invexity and quasi $\alpha\eta$-monotonicity.

Lemma 6.3. If $F$ is quasi $\alpha$-invex on $K$, then its differential $F'(u)$ is quasi $\alpha\eta$-monotone on $K$.

References