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On constructing new interpolation formulas using linear operators and an operator type of quadrature rules

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Dedicated to mathematicians who work on interpolation theory and quadrature rules

Abstract

Let T, U be two linear operators mapped onto the function f such that $U(T(f))=f$ but $T(U(f))\neq f$. In this paper, interpolating the functions of type $T(U(f))$ is presented in a general case. As a special case, the linear operators $T(f)=\int_{\lambda}^{x} f(t)\,\mathrm{d}t$ and $U(f)=\mathrm{d}f(x)/\mathrm{d}x$ are considered to interpolate the family of incomplete special functions. Three new examples of interpolation formulas together with their analytic error are also given as the special samples of the mentioned operator method. Finally, by using the foresaid method, a basic class of operator type quadrature rules is defined and its properties are investigated.

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1. Introduction

Let $\{x_j\}_{j=0}^{n}\in[a,b]$ together with corresponding values $\{f_j\}_{j=0}^{n}$, which may or may not be samples of a function, say $f(x)$, be given. The main aim of interpolation is to find an appropriate model to approximate $f(x)$ at any arbitrary point of $[a,b]$ other than x_j. For this purpose, various classical methods such as Lagrange, Newton and Hermite interpolations are used [3,5]. However, we should note in each mentioned methods that the values $\{f(x_j)\}_{j=0}^{n}$ must be pre-assigned. In other words, specifying $\{f(x_j)\}_{j=0}^{n}$ as the initial data is necessary in any form.

On the other hand, there are unfortunately many incomplete special functions having extensive applications in physics and engineering mathematics that cannot be evaluated at $\{x_j\}_{j=0}^{n}$ directly, i.e., $\{f(x_j)\}_{j=0}^{n}$ are not explicitly known (or their evaluation is not easily possible). For example, one can here point to some important samples such as the incomplete beta function [9, p. 99]

$$B(x;p,q)=\int_0^x t^{p-1}(1-t)^{q-1}\,\mathrm{d}t,\quad p,q>0;\quad 0\leqslant x\leqslant 1,$$

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incomplete gamma function [9, p. 99]

$$\gamma(x;\, p) = \int_0^x t^{p-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \quad p \neq 0, -1, -2, \ldots; \quad x > 0,$$

first and second kind of incomplete elliptic functions [10, p. 142]

$$E_1(x;\, p) = \int_0^x \frac{\mathrm{d}\theta}{\sqrt{1 - p^2 \sin^2\theta}}; \quad E_2(x;\, p) = \int_0^x \sqrt{1 - p^2 \sin^2\theta}\,\mathrm{d}\theta; \quad |p| < 1,$$

error function [9, p. 100] defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t,$$

exponential, cosine and sine functions [10, p. 250] as

$$Ei(-x) = -\int_x^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t; \quad Ci(x) = -\int_x^\infty \frac{\cos t}{t}\,\mathrm{d}t; \quad Si(x) = -\int_x^\infty \frac{\sin t}{t}\,\mathrm{d}t; \quad x > 0,$$

and Fresnel functions [10, p. 248] as follows:

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)\mathrm{d}t; \quad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)\mathrm{d}t.$$

All these examples are important samples that cannot be evaluated at the initial points $\{x_j\}_{j=0}^n$ easily and consequently cannot be interpolated by the usual interpolations. A more general example of this type is the set of cumulative distribution functions [10] having various applications in mathematical statistics. They are defined by

$$F(x) = \int_\alpha^x f(t)\,\mathrm{d}t, \quad \alpha \leqslant x \leqslant \beta,$$

where

$$\int_\alpha^\beta f(t)\,\mathrm{d}t = 1 \quad \text{for } f(t) > 0.$$

As was expressed, evaluating $F(x)$ (or other given examples) at initial points is difficult although its derivative, i.e., probability density function $f(x)$ can straightforwardly be computed. Thus, to interpolate this series of special functions, one should look for a suitable polynomial to *only* interpolate $f'(x)$ at the given points $\{x_j\}_{j=0}^n$. For this purpose, it seems that we should consider an interpolation formula as

$$f(x) = \sum_{i=0}^n f'(x_i) Q_i(x) + R_n^{(1)}(x), \tag{1}$$

in which $Q_i(x)$ is an appropriate sequence of polynomials, and $R_n^{(1)}(x)$ denotes the related error function. A glance at (1) implies to have

$$Q_i'(x_j) = \delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{1.1}$$

Hence, one can induce that

$$Q_i(x) = \int L_i(x)\,\mathrm{d}x + c, \tag{2}$$

where

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j} \tag{2.1}$$

denotes the Lagrange interpolating polynomial at $x = x_i$ [6,11] and c is a constant number. But, what will happen for the error function $R_n^{(1)}(x)$ in (1)? Essentially, can we generalize the interpolation (1) in the form

$$G(x) = \sum_{i=0}^{n} G^{(m)}(x_i)P_i(x) + R_n^{(m)}(x) \quad \text{for any fixed } m \in \mathbf{N}? \tag{3}$$

There are many ways to respond to these questions. However, all of them will lead to a unified operator approach. In the next section, we respond to the interpolation problem (3) in detail and then interpolate a specific class of functions in the form $f(x) - \sum_{k=0}^{m-1} f^{(k)}(\lambda)(x-\lambda)^k/k!$; $\forall m \in \mathbf{N}$. Of course, we shall begin with the particular case $m = 1$, which is suitable for interpolating all introduced incomplete special functions and then investigate the problem for the general case $m \in \mathbf{N}$. After this stage, we can introduce a general operator method containing all described interpolation formulas in previous sections and present some special samples of it.

2. Interpolating the functions of type $f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k$ where $m \in \mathbf{N}$ and $\lambda \in \mathbf{R}$

Let $\mathbf{\Pi}_r(x)$ denote the vector space of all polynomials of degree at most r. The main problem addressed here is to find a polynomial like $Q \in \mathbf{\Pi}_r(x)$ that *only* interpolates $f^{(m)}(x)$ at the given points $\{x_j\}_{j=0}^{n}$, i.e.,

$$Q^{(m)}(x_j) = \left.\frac{\mathrm{d}^m Q(x)}{\mathrm{d}x^m}\right|_{x=x_j} = f^{(m)}(x_j); \quad j = 0, 1, \ldots, n. \tag{3.1}$$

This problem is clearly equivalent to the interpolation problem (3). However, as we pointed out, $m = 1$ is the most applicable case by which one can interpolate incomplete special functions. Hence, let us first study this particular case.

Suppose the complete form of Lagrange interpolation [6,11] is given for the arbitrary function $g(x)$ on $[a, b]$ as

$$g(x) = \sum_{i=0}^{n} g(x_i)\, L_i(x) + \frac{g^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} x - x_i; \quad a < \xi < b, \tag{4}$$

and then assume $g(t) = f'(t)$. Substituting this assumption into (4) yields

$$f'(t) = \sum_{i=0}^{n} f'(x_i)L_i(t) + \frac{f^{(n+2)}(\xi(t))}{(n+1)!} \prod_{i=0}^{n} t - x_i. \tag{5}$$

Now by integrating from both sides of (5) on $[\lambda, x]$ we have

$$\int_\lambda^x f'(t)\,\mathrm{d}t = f(x) - f(\lambda) = \sum_{i=0}^{n} f'(x_i)\left(\int_\lambda^x L_i(t)\,\mathrm{d}t\right) + \frac{1}{(n+1)!}\int_\lambda^x f^{(n+2)}(\xi(t))\left(\prod_{i=0}^{n} t - x_i\right)\mathrm{d}t, \tag{5.1}$$

which summarizes the problem (3) for $m = 1$ as

$$G(x) = f(x) - f(\lambda) = \sum_{i=0}^{n} f'(x_i)\, M_{i,1}(x;\lambda) + R_n^{(1)}(x;\lambda), \tag{6}$$

where

$$\begin{cases} M_{i,1}(x;\lambda) = \int_\lambda^x L_i(t)\,\mathrm{d}t, \\ R_n^{(1)}(x;\lambda) = \dfrac{1}{(n+1)!}\int_\lambda^x f^{(n+2)}(\xi(t))\left(\prod_{i=0}^{n} t - x_i\right)\mathrm{d}t. \end{cases} \tag{6.1}$$

Note that the degree of polynomials $M_{i,1}(x;\lambda)$ in (6.1) is $n+1$ and $f(\lambda)$ in (6) is the same constant c as in (2). According to the mean value theorem for integrals [3, p. 7], a direct result for $R_n^{(1)}(x;\lambda)$ is that

$$|R_n^{(1)}(x;\lambda)| \leqslant \frac{1}{(n+1)!}\int_\lambda^x |f^{(n+2)}(\xi(t))| \prod_{i=0}^{n} |t-x_i|\,\mathrm{d}t$$
$$\leqslant \frac{|f^{(n+2)}(\xi)|}{(n+1)!}\int_\lambda^x \prod_{i=0}^{n} |t-x_i|\,\mathrm{d}t; \quad \lambda<\xi<x. \tag{7}$$

Therefore, the interpolation (6) must be exact for all polynomials of degree at most $(n+1)$, i.e.,

$$R_n^{(1)}(x;\lambda)=0 \quad\Leftrightarrow\quad f(x)=\sum_{i=0}^{n+1} c_i^{(1)} x^i; \quad c_i^{(1)} \text{ arbitrary numbers.} \tag{7.1}$$

We can now claim that the formula (6) is a suitable tool to interpolate the incomplete special functions introduced in Section 1. For instance, if one demands to interpolate the incomplete beta function, it is sufficient in (6) to choose $\lambda=0$ and $f(x)=B(x;p,q)$ to get

$$\int_0^x t^{p-1}(1-t)^{q-1}\,\mathrm{d}t = B(x;p,q) = \sum_{i=0}^{n} x_i^{p-1}(1-x_i)^{q-1}\left(\int_0^x L_i(t)\,\mathrm{d}t\right)$$
$$+\frac{1}{(n+1)!}\int_0^x \left(\prod_{i=0}^{n} t-x_i\right)\left.\frac{\mathrm{d}^{n+1}(z^{p-1}(1-z)^{q-1})}{\mathrm{d}z^{n+1}}\right|_{z=\xi(t)}\mathrm{d}t. \tag{8}$$

Similarly, for a cumulative distribution function defined on $[\alpha,\beta]$ we have

$$\int_\alpha^x f(t)\,\mathrm{d}t = F(x) = \sum_{i=0}^{n} P_r(x=x_i)\left(\int_\alpha^x L_i(t)\,\mathrm{d}t\right) + \frac{1}{(n+1)!}\int_\alpha^x \left(\prod_{i=0}^{n} t-x_i\right) f^{(n+1)}(\xi(t))\,\mathrm{d}t. \tag{9}$$

By a similar technique the general problem (3) can also be solved. For this purpose, suppose in (4) $g(t)=f^{(m)}(t)$ to arrive at

$$f^{(m)}(t)=\sum_{i=0}^{n} f^{(m)}(x_i)L_i(t)+\frac{f^{(m+n+1)}(\xi(t))}{(n+1)!}\prod_{i=0}^{n} t-x_i. \tag{10}$$

By integrating from both sides of (14) frequently, we obtain

$$\int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} f^{(m)}(t)\,\mathrm{d}t\,\mathrm{d}s_1\ldots\mathrm{d}s_{m-1}$$
$$= f(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k$$
$$= \sum_{i=0}^{n} f^{(m)}(x_i)\left(\int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} L_i(t)\,\mathrm{d}t\,\mathrm{d}s_1\ldots\mathrm{d}s_{m-1}\right)$$
$$+\left(\frac{1}{(n+1)!}\int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} f^{(m+n+1)}(\xi(t))\left(\prod_{i=0}^{n} t-x_i\right)\mathrm{d}t\,\mathrm{d}s_1\ldots\mathrm{d}s_{m-1}\right), \tag{10.1}$$

which can be equated with (3) as

$$G(x)=f(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k=\sum_{i=0}^{n} f^{(m)}(x_i)M_{i,m}(x;\lambda)+R_n^{(m)}(x;\lambda), \tag{11}$$

in which

$$\begin{cases} M_{i,m}(x;\lambda) = \int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} L_i(t)\,\mathrm{d}t\,\mathrm{d}s_1 \ \ldots\ \mathrm{d}s_{m-1}, \\ R_n^{(m)}(x;\lambda) = \frac{1}{(n+1)!} \int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} f^{(m+n+1)}(\xi(t)) \left(\prod_{i=0}^n t - x_i\right) \mathrm{d}t\,\mathrm{d}s_1 \ldots \mathrm{d}s_{m-1}. \end{cases} \tag{11.1}$$

It is important in (11) to know that $G^{(m)}(x_i) = f^{(m)}(x_i)$ and $\deg(M_{i,m}(x;\lambda)) = n+m$. Moreover, according to definition $R_n^{(m)}(x;\lambda)$ in (11.1), the interpolation (11) must be exact for all polynomials of degree at most $n+m$. In this sense, if $f(x)$ is an analytic function at $x=\lambda$, Taylor expansion implies that

$$f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k = \sum_{k=m}^{\infty} \frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k. \tag{12}$$

So, by comparing (12) with (11) it is concluded that an infinite series is being interpolated by the right-side term of (11).

Remark 1. Because of rather huge calculations, Lagrange interpolation formula is attractive more for theoretical purposes than practical numerical works. Hence, since the base of the constitution of (11) is (4), to reduce the volume of computations one can first use the barycentric Lagrange interpolation (e.g. see [1]) and then implement the explained approach.

Remark 2. For large values of n, many orderings of $\{x_i\}_{i=0}^n$ lead to numerical instability. To have stability, we must select the initial points based on a certain equidistribution property [13]. The nodes of (11) should also obey this matter. In this way, the barycentric formula (mentioned in Remark 1) can be again applied to evaluate the interpolant polynomials of large degrees. For more details, see [4].

But, the interpolation (11) is not the end of the story. We can still increase the precision degree of problem somehow. To do this task, the simple Hermite (osculatory) interpolation [3, p.37] as a prevalent case or full Hermite interpolation [3, p.37] as a general case can be used. However, since the osculatory interpolation is often used in numerical analysis text books let us consider its general form here as

$$g(x) = \sum_{i=1}^n g(x_i) H_{i,1}(x) + \sum_{i=1}^n g'(x_i)\, H_{i,2}(x) + \frac{g^{(2n)}(\xi)}{(2n)!} \prod_{i=1}^n (x-x_i)^2, \tag{13}$$

in which

$$\begin{cases} H_{i,1}(x) = \left(1 - \dfrac{l''(x_i)}{l'(x_i)}(x-x_i)\right)(L_i(x))^2 \quad \text{if } l(x) = \prod_{i=1}^n x - x_i, \\ H_{i,2}(x) = (x-x_i)(L_i(x))^2, \end{cases} \tag{13.1}$$

and $L_i(x)$ denotes the Lagrange polynomial corresponding to x_i. Similar to the previous case, if $g(t) = f^{(m)}(t)$ in (13) then

$$f^{(m)}(t) = \sum_{i=1}^n f^{(m)}(x_i)\, H_{i,1}(t) + \sum_{i=1}^n f^{(m+1)}(x_i) H_{i,2}(t) + \frac{f^{(2n+m)}(\xi)}{(2n)!} \prod_{i=1}^n (t-x_i)^2. \tag{14}$$

Consequently, integrating from both sides of (14) on $[\lambda, x]$ yields

$$f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(\lambda)}{k!}(x-\lambda)^k = \sum_{i=1}^n f^{(m)}(x_i) N_{i,m}^{(1)}(x;\lambda) + \sum_{i=1}^n f^{(m+1)}(x_i) N_{i,m}^{(2)}(x;\lambda) + E_n^{(m)}(x;\lambda), \tag{15}$$

where

$$\begin{cases} N_{i,m}^{(1)}(x;\lambda) = \int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} H_{i,1}(t)\,\mathrm{d}t\,\mathrm{d}s_1 \ldots \mathrm{d}s_{m-1}, \\ N_{i,m}^{(2)}(x;\lambda) = \int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} H_{i,2}(t)\,\mathrm{d}t\,\mathrm{d}s_1 \ldots \mathrm{d}s_{m-1}, \\ E_n^{(m)}(x;\lambda) = \frac{1}{(2n)!} \int_\lambda^x \int_\lambda^{s_{m-1}} \cdots \int_\lambda^{s_1} f^{(m+2n)}(\xi(t)) \prod_{i=1}^n (t-x_i)^2\,\mathrm{d}t\,\mathrm{d}s_1 \ldots \mathrm{d}s_{m-1}. \end{cases} \tag{15.1}$$

As we said, in comparison with interpolation (11), the precision degree of this interpolation has increased up to $m+2n-1$. Also

$$\deg\, N_{i,m}^{(1)}(x;\lambda)=\deg\, N_{i,m}^{(2)}(x;\lambda)=m+2n-1. \tag{15.2}$$

As an applied important case, $m=1$ in (15) gives

$$f(x)-f(\lambda)=\sum_{i=1}^{n} f'(x_i)\left(\int_{\lambda}^{x} H_{i,1}(t)\,\mathrm{d}t\right)+\sum_{i=1}^{n} f''(x_i)\left(\int_{\lambda}^{x} H_{i,2}(t)\,\mathrm{d}t\right)+E_n^{(1)}(x;\lambda), \tag{16}$$

where

$$E_n^{(1)}(x;\lambda)=\frac{1}{(2n)!}\int_{\lambda}^{x} f^{(2n+1)}(\xi(t))\prod_{i=1}^{n}(t-x_i)^2\,\mathrm{d}t=\frac{f^{(2n+1)}(\xi_x)}{(2n)!}\int_{\lambda}^{x}\prod_{i=1}^{n}(t-x_i)^2\,\mathrm{d}t. \tag{16.1}$$

A straightforward result from (16.1) is that

$$|E_n^{(1)}(x;\lambda)|\leqslant\frac{|f^{(2n+1)}(\xi_1)|}{(2n)!}\int_{\lambda}^{x}\prod_{i=1}^{n}(t-x_i)^2\,\mathrm{d}t;\quad \lambda<\xi_1<x. \tag{16.2}$$

By using this result we have recently introduced a new type of weighted quadrature rules in [7] and obtained its relationship with usual Gaussian quadratures and orthogonal polynomials, see also Section 4.

It is now a good position to introduce the main method based on linear operators.

3. A generic operator method to interpolate the functions

As it is known, interpolation theory is concerned with reconstructing functions on the basis of certain functional information. In many cases, functionals are considered linear. Thus, by assuming that X is a linear vector space of dimension n, its elements should be functions of x.

Now suppose $S_1, S_2, \ldots, S_n$ are n given linear functionals defined on X so that the following interpolation holds for the arbitrary function g,

$$g(x)=\sum_{i=1}^{n} S_i(g(x))\,\varphi_i(x)+E(x;g), \tag{17}$$

where $E(x;g)$ denotes the error function and $\{\varphi_i(x)\}_{i=1}^{n}$ are to be obtained.

In order to be satisfied the general interpolation (17) the sequence $\{\varphi_i(x)\}_{i=1}^{n}$ must be selected in such a way that

$$S_j(\varphi_i(x))=\delta_{j,i}=\begin{cases}0 & \text{if } j\neq i,\\ 1 & \text{if } j=i.\end{cases} \tag{18}$$

In this way we must also have

$$S_j(E(x;g))=0 \quad \text{for } j=0,1,\ldots,n. \tag{19}$$

Now let T, U be two linear operators mapped onto the function f such that $U(T(f))=f$ but $T(U(f))\neq f$. Also, let $g(t)=U(f(t))$ and replace it in (17) to get

$$U(f(t))=\sum_{i=1}^{n} S_i(U(f))\,\varphi_i(t)+E(t;U(f)). \tag{20}$$

Since $S_i(U(f(t)))$ are just numbers independent of t and T is a *linear* operator, taking this operator on both sides of (20) yields

$$T(U(f(t)))=\sum_{i=1}^{n} S_i(U(f))T(\varphi_i(t))+T(E(t;U(f))). \tag{21}$$

This is the first basic operator formula for interpolating the functions of type $T(U(f)) \neq f$. Similarly, to derive the second type formula we should apply the transform $g(t) = T(f(t))$ on (17) to eventually reach

$$U(T(f(t))) = f(t) = \sum_{i=1}^{n} S_i(T(f))U(\varphi_i(t)) + U(E(t; T(f))). \tag{22}$$

Fortunately, many new formulas in the theory of functions interpolation can be constructed by using these two basic rules (21) and (22). Here we introduce only three applied samples.

Example 1. *An interpolation formula for statistical purposes*: First, we should note that the Lagrange interpolation could be represented in terms of the components (17) as

$$S_i(g) = g(x_i); \quad \varphi_i(x) = L_i(x) \quad \text{and} \quad E(x; f) = \frac{f^{(n)}(\xi)}{n!} \prod_{i=1}^{n} x - x_i. \tag{23}$$

If for this special example the linear operators T and U are defined as

$$T(f) = \int_{\lambda}^{x} f(t)\,\mathrm{d}t \quad \text{and} \quad U(f) = \frac{\mathrm{d}}{\mathrm{d}x} f(x), \tag{24}$$

then by substituting the assumptions (23) in the basic formula (22) and noting the definitions (24) we obtain the interpolation formula

$$U(T(f)) = f(x) = \sum_{i=1}^{n} \left(\int_{\lambda}^{x_i} f(t)\,\mathrm{d}t \right) \frac{\mathrm{d}}{\mathrm{d}x}(L_i(x)) + \frac{1}{n!} \frac{\mathrm{d}}{\mathrm{d}x} \left(f^{(n-1)}(\xi(x)) \prod_{i=1}^{n} x - x_i \right). \tag{25}$$

Although the precision degree of (25) is $n - 2$, it may be a suitable formula to interpolate some important probability density functions of continuous random variables such as normal, gamma, beta, T-Student and F distribution functions [10], because the probability values

$$\begin{aligned} \int_{\lambda}^{x_i} f(x)\,\mathrm{d}x &= P_r(\lambda \leqslant x \leqslant x_i) = P_r(a \leqslant x \leqslant x_i) - P_r(a \leqslant x \leqslant \lambda) \\ &= F(x_i) - F(\lambda); \quad a \leqslant x \leqslant b, \end{aligned} \tag{25.1}$$

are essentially available (for an appropriate λ) in any statistical table, while it is rather hard to directly use the Lagrange interpolation in comparison with (25), because usually the probability values $f(x_i)$ are not easily accessible in the statistical tables. Of course, if in this direction one can somehow provide a sample table of the mentioned values $f(x_i) = P_r(x = x_i)$, then the interpolation formula (25) can be improved and its precision degree can be increased. For this purpose, it is enough to consider the elements of simple Hermite interpolation corresponding to representation (17) and substitute them into the basic formula (22) to finally get

$$\begin{aligned} f(x) = &\sum_{i=1}^{n} \left(\int_{\lambda}^{x_i} f(t)\,\mathrm{d}t \right) \frac{\mathrm{d}}{\mathrm{d}x}(H_{i,1}(x)) + \sum_{i=1}^{n} f(x_i) \frac{\mathrm{d}}{\mathrm{d}x}(H_{i,2}(x)) \\ &+ \frac{1}{(2n)!} \frac{\mathrm{d}}{\mathrm{d}x} \left(f^{(2n-1)}(\xi(x)) \prod_{i=1}^{n} (x - x_i)^2 \right). \end{aligned} \tag{26}$$

This formula is exact for all polynomials of degree at most $2n - 2 \geqslant n - 2$.

Example 2. *Mixed interpolation formulas of Taylor type*: By applying different operators in the basic formulas (21) and (22), many new mixed interpolation formulas of Taylor type can be derived. For instance, if the Taylor interpolation (as a special case of (17) for $S_i(g) = g^{(i)}(\theta)$, $\varphi_i(x) = (x - \theta)^i / i!$ and $E(x; g) = g^{(n+1)}(\xi)\,(x - \theta)^{n+1}/(n + 1)!$) is considered at $x = \theta$, then by defining (for example) the following linear operators:

$$T(f) = \int_{\lambda}^{x} f(t)\,\mathrm{d}t \quad \text{and} \quad U(f) = \frac{\mathrm{d}}{\mathrm{d}x}(xf(x)) = xf'(x) + f(x), \tag{27}$$

and replacing them in the main formula (21) we get

$$T(U(f)) = \int_{\lambda}^{x} D(tf(t))\,\mathrm{d}t = xf(x) - \lambda f(\lambda)$$

$$= \sum_{k=0}^{n} \frac{[(t-\theta)^{k+1}]_{\lambda}^{x}}{(k+1)!} \left.\frac{\mathrm{d}^{k+1}(zf(z))}{\mathrm{d}z^{k+1}}\right|_{z=\theta} + \frac{1}{(n+1)!}\int_{\lambda}^{x} (t-\theta)^{n+1} \left.\frac{\mathrm{d}^{n+2}(zf(z))}{\mathrm{d}z^{n+2}}\right|_{z=\xi(t)} \mathrm{d}t. \tag{28}$$

On the other hand since

$$(xf(x))^{(m)} = xf^{(m)}(x) + mf^{(m-1)}(x), \tag{29}$$

(28) is simplified in the final form

$$xf(x) = \lambda f(\lambda) + \sum_{k=0}^{n} \frac{(\theta f^{(k+1)}(\theta) + (k+1)f^{(k)}(\theta))}{(k+1)!}((x-\theta)^{k+1} - (\lambda-\theta)^{k+1})$$

$$+\frac{1}{(n+1)!}\int_{\lambda}^{x} (t-\theta)^{n+1} \left.\left(zf^{(n+2)}(z) + (n+2)f^{(n+1)}(z)\right)\right|_{z=\xi(t)} \mathrm{d}t. \tag{30}$$

Note that (30) is only a small sample of mixed Taylor interpolation formulas and by defining other linear operators one can derive infinite other interpolation formulas of this type. Furthermore, let us add that constructing *new expansions* of functions using linear operators is also possible [in preprint].

The remainder term in (30) shows that the formula is exact for all polynomials of degree at most n.

Example 3. *Mixed Taylor–Lagrange and mixed Taylor–Hermite interpolation formulas*: The obtained relation (11) can be called "mixed Taylor–Lagrange interpolation formula" because if we choose the following options in (17):

$$\begin{cases} S_i(g) = g^i(\lambda) & \text{for } i = 0, 1, \ldots, m-1 \quad \text{and} \quad S_{m+j}(g) = f^{(m)}(x_j) \quad \text{for } j = 0, 1, \ldots, n, \\ \varphi_i(x) = \dfrac{(x-\lambda)^i}{i!} & \text{for } i = 0, 1, \ldots, m-1 \quad \text{and} \quad \varphi_{m+j}(x) = M_{j,m}(x; \lambda) \quad \text{for } j = 0, 1, \ldots, n, \end{cases} \tag{31}$$

and define the corresponding linear operators as

$$T(f) = \int_{\lambda}^{x}\int_{\lambda}^{s_{m-1}} \cdots \int_{\lambda}^{s_1} f(t)\,\mathrm{d}t\,\mathrm{d}s_1 \ldots \mathrm{d}s_{m-1} \quad \text{and} \quad U(f) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} f(x), \tag{32}$$

then replacing these assumptions in the basic formula (21) gives the modified form of (9) as

$$f(x) = \sum_{i=0}^{m-1} f^{(i)}(\lambda)\frac{(x-\lambda)^i}{i!} + \sum_{i=0}^{n} f^{(m)}(x_i)M_{i,m}(x; \lambda) + R_n^{(m)}(x; \lambda). \tag{33}$$

An important note in this formula is that the distributed nodes obey the specific form $\{\lambda, \lambda, \ldots, \lambda, x_0, x_1, \ldots, x_n\}$ where λ is repeated m times, while the distribution of derivatives index in (33) is in the form $\{0, 1, \ldots, m-1, m, m, \ldots, m\}$ in which m is repeated $n+1$ times.

Similarly, the formula obtained in (15) can be interpreted as a *mixed Taylor–Hermite* interpolation.

Clearly many other interpolation formulas can be constructed by the basic formulas (21) and (22). However, let us claim that for any known interpolation formula of type (17), there exist two unique linear operators T and U that minimize either the error function of (21), i.e., $T(E(t; U(f)))$ or the error function of (22) on $[a, b]$.

4. Application of defined operator method in quadrature rules

Recently in [7] we introduced a new type of weighted quadrature rules as

$$\int_{\alpha}^{\beta} \rho(x)\,(f(x) - P_{m-1}(x;\, f))\,\mathrm{d}x = \sum_{i=1}^{n} a_{i,m} f^{(m)}(b_{i,m}) + R_n^{(m)}[f], \tag{34}$$

in which $P_{m-1}(x;\, f) = \sum_{j=0}^{m-1} f^{(j)}(\lambda)\,(x-\lambda)^j/j!$; $\lambda \in \mathbf{R}$; $m \in \mathbf{N}$; $\rho(x)$ is a positive function; $f^{(m)}$ denotes the mth derivative of $f(x)$ and $R_n^{(m)}[f]$ is the error value. We computed the unknowns $\{a_{i,m}, b_{i,m}\}_{i=1}^{n}$ explicitly so that (34) is exact for all polynomials of degree at most $2n+m-1$. However, according to [7], to derive the unknowns $\{a_{i,m}, b_{i,m}\}_{i=1}^{n}$ we should solve a key integral equation in order to be able to connect (34) to the usual weighted quadrature rules. Although we determined the error function $R_n^{(m)}[f]$ by the Peano theorem [5], a glance at (34) shows that it is exactly a consequence of constructive interpolation (15) where $\int_{\alpha}^{\beta} \rho(x)\, N_{i,m}^{(2)}(x;\,\lambda)\,\mathrm{d}x = 0$. Let us study a specific example here. Consider the interpolation (16) and (after multiplying $\rho(x)$ on both sides of equality) integrate from both sides of the result on $[\alpha, \beta]$ to get

$$\begin{aligned}\int_{\alpha}^{\beta} \rho(x)(f(x) - f(\lambda))\,\mathrm{d}x &= \sum_{i=1}^{n} f'(x_i) \int_{\alpha}^{\beta} \rho(x) \left(\int_{\lambda}^{x} H_{i,1}(t)\,\mathrm{d}t \right) \mathrm{d}x \\ &\quad + \sum_{i=1}^{n} f''(x_i) \int_{\alpha}^{\beta} \rho(x) \left(\int_{\lambda}^{x} H_{i,2}(t)\,\mathrm{d}t \right) \mathrm{d}x \\ &\quad + \frac{1}{(2n)!} \int_{\alpha}^{\beta} \rho(x) f^{(2n+1)}(\xi_x) \left(\int_{\lambda}^{x} \prod_{i=1}^{n} (t - x_i)^2\,\mathrm{d}t \right) \mathrm{d}x. \end{aligned} \tag{35}$$

We shall prove in (35) that

$$I = \int_{\alpha}^{\beta} \int_{\lambda}^{x} \rho(x)\, H_{i,2}(t)\,\mathrm{d}t\,\mathrm{d}x = 0. \tag{36}$$

For this purpose, first for convenience suppose in (36) that $\lambda = \alpha$ and $\rho(x) = w'(x)$. So, it changes to

$$I = \int_{\alpha}^{\beta} \int_{\alpha}^{x} w'(x) H_{i,2}(t)\,\mathrm{d}t\,\mathrm{d}x = \int_{\alpha}^{\beta} (w(\beta) - w(t)) H_{i,2}(t)\,\mathrm{d}t. \tag{37}$$

Now, if $w^*(t) = w(\beta) - w(t)$ is a weight function corresponding to a sequence of orthogonal polynomials, say $P_n(x)$, having the orthogonality property [2,12]

$$\int_{\alpha}^{\beta} w^*(x) P_n(x) P_m(x)\,\mathrm{d}x = \left(\int_{\alpha}^{\beta} w^*(x) P_n^2(x)\,\mathrm{d}x \right) \delta_{n,m} \tag{38}$$

on $[\alpha, \beta]$ then according to the theory of weighted quadratures (35) will be transformed to the following rule:

$$\int_{\alpha}^{\beta} w'(x)(f(x) - f(\alpha))\,\mathrm{d}x = \sum_{i=1}^{n} A_i\, f'(x_i) + \frac{f^{(2n+1)}(\xi)}{(2n)!} \int_{\alpha}^{\beta} w'(x) \left(\int_{\alpha}^{x} \prod_{i=1}^{n} (t - x_i)^2\,\mathrm{d}t \right) \mathrm{d}x, \tag{39}$$

where

$$A_i = \int_{\alpha}^{\beta} w'(x) \left(\int_{\alpha}^{x} H_{i,1}(t)\,\mathrm{d}t \right) \mathrm{d}x, \tag{39.1}$$

and $\{x_i\}_{i=0}^n$ are the zeros of a sequence of polynomials that are orthogonal with respect to the weight function $w^*(t) = w(\beta) - w(t)$ on $[\alpha, \beta]$. For example, suppose $w(x) = x$; $0 \leqslant x \leqslant 1$. Then (39) becomes

$$\int_0^1 (f(x) - f(0))\,\mathrm{d}x = \sum_{i=1}^n A_i^{(1)} f'(x_i) + \frac{f^{(2n+1)}(\xi)}{(2n)!} \int_0^1 (1-x)\bar{P}_n^2(x)\,\mathrm{d}x, \tag{40}$$

where

$$A_i^{(1)} = \int_0^1 \left(\int_0^x H_{i,1}(t)\,\mathrm{d}t \right) \mathrm{d}x = \int_0^1 (1-t)\,H_{i,1}(t)\,\mathrm{d}t, \tag{40.1}$$

and $\bar{P}_n(x)$ is the monic type of a sequence of polynomials orthogonal with respect to the weight function $w^*(x) = w(1) - w(x) = 1 - x$ on $[0, 1]$. Fortunately these polynomials are known as a special case of shifted Jacobi polynomials $P_n^{(\alpha,\beta)}(2x-1)$ for $\alpha = 1$ and $\beta = 0$ on $[-1, 1]$. So, the following corollary is valid.

Corollary. *The quadrature formula*

$$\int_0^1 (f(x) - f(0))\,\mathrm{d}x = \sum_{i=1}^n \left(\int_0^1 (1-t) H_{i,1}^{(b_i)}(t)\,\mathrm{d}t \right) f'(b_i) + \frac{(n\,!)^4 (n+1)^2}{(2n)!(2n+1)!(2n+2)!} f^{(2n+1)}(\xi) \tag{41}$$

is exact for all polynomials of degree at most $2n$ *if and only if* b_i *are the zeros of polynomial*

$$Q_n(x) = P_n^{(1,0)}(2x-1) = \sum_{k=0}^n \binom{n+1+k}{k} \binom{n}{n-k} (-1)^{n-k} x^k, \tag{41.1}$$

and $H_{i,1}^{(b_i)}(t)$ *are the polynomials* (13.1) *corresponding to the zeros* b_i.

Consequently, we made some conditions to be able to connect the described quadrature rules to usual Gauss rules (although they are much more extensive than usual weighted quadratures). This process can also happen for Newton–Cotes quadrature rules and also other models of numerical integration of interpolatory type, see e.g., [8] in which the nodes of integration are chosen based on a statistical distribution.

In other words, since the precision degree of interpolation (6) is $n+1$, the construction of a quadrature corresponding to it can be interpreted as an analogue of Newton–Cotes rules. Hence, we should first assume that integration nodes are equidistant and then consider the following problem:

Let x_j; $j = 0, 1, \ldots, n$ be $(n+1)$ distinct points in $[a, b]$ where $x_{j+1} - x_j = h$, $x_0 = a$ and $x_n = b$. Find the explicit form of the coefficients $\{w_i\}_{i=0}^n$ so that the approximate formula

$$\int_a^b (f(x) - f(\lambda))\,\mathrm{d}x \cong \sum_{i=0}^n w_i f'(a + ih); \quad h = \frac{b-a}{n}, \tag{42}$$

is precise for any polynomial of degree at most $(n+1)$or equivalently for the basis $\{x^j\}_{j=0}^{n+1}$.

There are generally two ways to solve this problem. The direct way to compute $\{w_i\}_{i=0}^n$ is to integrate from both sides of relation (6) on $[a, b]$ to finally obtain

$$w_i = \int_a^b M_{i,1}(x;\lambda)\,\mathrm{d}x = \int_a^b \left(\int_\lambda^x L_i(t)\,\mathrm{d}t \right) \mathrm{d}x. \tag{42.1}$$

For instance, when $n = 1, 2$ the quadrature (42) takes the forms

$$\begin{aligned} \int_{x_0}^{x_1} (f(x) - f(\lambda))\,\mathrm{d}x \cong{} & (\tfrac{1}{2}x_0^2 + (h-\lambda)x_0 + \tfrac{1}{3}h^2 + \tfrac{1}{2}\lambda^2 - h\lambda) f'(x_0) \\ & + (-\tfrac{1}{2}x_1^2 + (h+\lambda)x_1 - \tfrac{1}{3}h^2 - \tfrac{1}{2}\lambda^2 - h\lambda) f'(x_1), \end{aligned} \tag{42.2}$$

Table 1
Numerical results for estimating the integrals I_1–I_{10}

I_i	Exact value	Approx. trape.	Approx. (42.2)	|Error (42.2)|	|Error trape.|
I_1	0.0175230963	0.0163746150	0.0176998973	0.0001768010	0.0011484812
I_2	0.0003081225	0.0005724334	0.0005008792	0.0001927566	0.0002643108
I_3	0.0050887275	0.0075079392	0.0047894954	0.0002992321	0.0024192116
I_4	−0.0022104330	−0.0032129917	−0.0019732091	0.0002372239	0.0010025587
I_5	−0.0229593117	−0.0333675736	−0.0205128810	0.0024464307	0.0104082618
I_6	−0.0126003906	−0.0116884264	−0.0128778372	0.0002774465	0.0009119642
I_7	−0.0361115770	−0.1074314090	−0.0350612910	0.0010502860	0.0713198319
I_8	−0.0120398169	−0.0365202822	−0.0125000001	0.0004601831	0.0244804652
I_9	0.9700342363	0.9098204125	0.9673674284	0.0026668079	0.0602138238
I_{10}	0.0111264920	0.0346182830	0.0126192182	0.0014927260	0.0234917920

$$\begin{aligned}\int_{x_0}^{x_2} (f(x) - f(\lambda))\,\mathrm{d}x \cong{}& \left(\frac{1}{3h}x_0^3 + \left(\frac{3}{2} - \frac{\lambda}{h}\right)x_0^2 + \left(2h - 3\lambda + \frac{\lambda^2}{h}\right)x_0\right.\\ &\left. + \left(\frac{2}{3}h^2 - 2h\lambda + \frac{3}{2}\lambda^2 - \frac{\lambda^3}{3h}\right)\right) f'(x_0)\\ &+ \left(-\frac{2}{3h}x_1^3 + \frac{2\lambda}{h}x_1^2 + 2\left(h - \frac{\lambda^2}{h}\right)x_1 + 2\left(-h\lambda + \frac{\lambda^3}{3h}\right)\right) f'(x_1)\\ &+ \left(\frac{1}{3h}x_2^3 - \left(\frac{3}{2} + \frac{\lambda}{h}\right)x_2^2 + \left(2h + 3\lambda + \frac{\lambda^2}{h}\right)x_2\right.\\ &\left. - \left(\frac{2}{3}h^2 + 2h\lambda + \frac{3}{2}\lambda^2 + \frac{\lambda^3}{3h}\right)\right) f'(x_2), \end{aligned} \tag{42.3}$$

which are exact for any parabolic and cubic polynomials respectively.

It may be interesting to know that the quadrature (42) will be more accurate if λ is a known root of the integrand function, because in this case the precision degree in (42) increases up to one degree in comparison with usual Newton–Cotes rules and it is therefore better to use the approximation (42) rather than usual Newton–Cotes formulas. Fortunately, not only this specific λ is available for almost all incomplete special functions but also for other difficult integrals. For example, Table 1 shows a high preference of approximation (42.2) with respect to the usual trapezoidal approximation for the following integrals:

$I_1 = \int_0^{0.2} x\,\mathrm{e}^{-x}\,\mathrm{d}x = \gamma(0.2; 2)$ with $\lambda = 0$, which evaluates a special case of incomplete gamma function,

$I_2 = \int_0^{0.2} x^3\sqrt{(1-x)^3}\,\mathrm{d}x = B(0.2; 4, 5/2)$ with $\lambda = 0$, which evaluates a special case of incomplete beta function,

$I_3 = \int_0^{0.5}\left(\frac{1}{\sqrt{1-(\sin x)^2/4}} - 1\right)\mathrm{d}x = E_1(0.5; \frac{1}{2}) - \frac{1}{2}$ with $\lambda = 0$, which evaluates a special case of the first kind of incomplete elliptic function,

$I_4 = \int_0^{0.5}\left(\sqrt{1-(\sin x)^2/9} - 1\right)\mathrm{d}x = E_2(0.5; \frac{1}{3}) - \frac{1}{2}$ with $\lambda = 0$, which evaluates a special case of the second kind of incomplete elliptic function,

$I_5 = \frac{2}{\sqrt{\pi}}\int_0^{0.4}(\exp(-x^2) - 1)\,\mathrm{d}x = \mathrm{erf}(0.4) - \frac{0.8}{\sqrt{\pi}}$ with $\lambda = 0$, which evaluates a special case of the error function,

$I_6 = \int_1^{1.2}\left(\frac{\exp(-x)}{x} - \frac{1}{e}\right)\mathrm{d}x = \int_1^{1.2}\frac{\exp(-x)}{x}\,\mathrm{d}x - \frac{0.2}{e}$ with $\lambda = 1$,

$I_7 = \int_0^1(\cos x - \cos(\frac{1}{2})\,\mathrm{d}x = \int_0^1 \cos x\,\mathrm{d}x - \cos(\frac{1}{2})$ with $\lambda = \frac{1}{2}$,

$I_8 = \int_1^2(\arctan x - \arctan(\frac{3}{2}))\,\mathrm{d}x = \int_1^2 \arctan x\,\mathrm{d}x - \arctan(\frac{3}{2})$ with $\lambda = \frac{3}{2}$,

$I_9 = \int_0^{0.9} \arccos x \, dx$ with $\lambda = 1$ and finally
$I_{10} = \int_2^4 (\sqrt{1+x^2} - \sqrt{10}) \, dx = \int_2^4 \sqrt{1+x^2} \, dx - 2\sqrt{10}$ with $\lambda = 3$.

Note that all introduced integrands in I_1 to I_{10} are equal to zero for the given λ and because of this (42.2) is prior to trapezoidal rule. Also, it is clear that by the above table one can estimate for instance the integrals $\int_1^{1.2} \frac{\exp(-x)}{x} \, dx = I_6 + \frac{0.2}{e}$ or $\int_1^2 \arctan x \, dx = I_8 + \arctan(\frac{3}{2})$.

But, all introduced quadratures in (34) and (42) are only special cases of a basic class of operator type quadrature rules. To construct this class we should reconsider the basic interpolations (21) and (22), multiply the positive measure $w(x)$ on both sides of them and finally integrate on an interval, say $[a, b]$, to respectively get

$$\int_a^b w(x) T(U(f(x))) \, dx = \sum_{i=1}^n S_i(U(f)) \left(\int_a^b w(x) T(\phi_i(x)) \, dx \right) + \int_a^b w(x) T(E(x; U(f))) \, dx, \tag{43}$$

$$\int_a^b w(x) U(T(f(x))) \, dx = \sum_{i=1}^n S_i(T(f)) \left(\int_a^b w(x) U(\phi_i(x)) \, dx \right) + \int_a^b w(x) U(E(x; T(f))) \, dx. \tag{44}$$

These are two basic classes of operator type quadrature rules that many new numerical integration rules can be constructed by means of them and defining appropriate linear operators T and U. For instance, suppose the constructive interpolation (25) is given. Integrating on both sides of it yields

$$\int_a^b w(x) f(x) \, dx = \sum_{i=1}^n \left([w(x) L_i(x)]_a^b - \int_a^b w'(x) L_i(x) \, dx \right) \int_\lambda^{x_i} f(x) \, dx + \frac{1}{n!} \int_a^b w(x) \frac{d}{dx} \left(f^{(n-1)}(\xi(x)) \prod_{i=1}^n x - x_i \right) dx. \tag{45}$$

Although the precision degree of above constructive quadrature is $n - 2$, it is however approximating a weighted integral by a linear combination of simple (non-weighted) elements $\int_\lambda^{x_i} f(x) \, dx$, which is an important advantage for the quadrature (45).

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