Extremes of Gaussian processes with a smooth random variance

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Received 1 November 2010; received in revised form 19 May 2011; accepted 17 June 2011
Available online 8 July 2011

Abstract

Let $\xi(t)$ be a standard stationary Gaussian process with covariance function $r(t)$, and $\eta(t)$, another smooth random process. We consider the probabilities of exceedances of $\xi(t)\eta(t)$ above a high level $u$ occurring in an interval $[0, T]$ with $T > 0$. We present asymptotically exact results for the probability of such events under certain smoothness conditions of this process $\xi(t)\eta(t)$, which is called the random variance process. We derive also a large deviation result for a general class of conditional Gaussian processes $X(t)$ given a random element $Y$.

Keywords: Gaussian process; Conditional Gaussian process; Locally stationary; Ruin probability; Random variance; Extremes; Large deviations; Fractional Brownian motion

1. Introduction

Let $(X(t), Y), \ t \in \mathbb{R}$, be a random element, where $X(t)$ is a random process taking values in $\mathbb{R}$, and $Y$ is an arbitrary random element. We say, that $X(t)$ is a conditionally Gaussian process if the conditional distribution of $X(\cdot)$ given $Y$ is Gaussian. We investigate the probabilities of large extremes,

$$P_u = P_u(T, X, Y) := P \left( \sup_{t \in [0, T]} X(t) > u \right)$$

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as \( u \to \infty \), where \( T > 0 \). We assume that \( \sup_{t \in [0, T]} X(t) < \infty \) a.s. for any \( T \). Conditioning on \( Y \), we denote the random mean of \( X \) by

\[
m(t, Y) := E(X(t) \mid Y)
\]

and the random covariance by

\[
C(s, t, Y) := E((X(s) - m(s, Y))(X(t) - m(t, Y)) \mid Y),
\]

so that

\[
V^2(t, Y) := C(t, t, Y)
\]

is the random variance of \( X \).

For application of conditionally Gaussian processes in finance, optimization and control problems see [10,3,8,9]. For examples, Doucet et al. [3] considered to model the behavior of latent variables in neural networks by Gaussian processes with random parameters; Lototsky [8] studied stochastic parabolic equations with solutions of Gaussian processes, where the coefficients are modeled by a dynamic system. To our best knowledge, the paper [2] was the first mathematical work where probabilities of large extremes of conditionally Gaussian processes were considered. As an example of stable processes, sub-Gaussian processes were considered, that is, processes of the type \( X(t) = \sqrt{\xi(t)} \), where \( \xi(t) \) is a stationary Gaussian process and \( \xi \) is a stable random variable, independent of \( \xi(\cdot) \). In our notation we have \( Y = \sqrt{\xi} \) and \( X(t) = Y\xi(t) \), hence a Gaussian process with a random variance. This paper dealt with the mean of the number of upcrossings of a level \( u \), as in the Rice formula, which can be applied for smooth Gaussian processes. The aim of the present paper and subsequent ones which are in preparation, is developing asymptotic methods for large extremes of conditionally Gaussian processes. Preliminary results are given in [12,13]. Our intention is to expand the Gaussian tools to a wider class of random processes. The asymptotic theory for large extremes of Gaussian processes and fields is already well developed, see e.g. [11,4], and references therein. We show that the asymptotic theory for large extremes of conditional Gaussian processes is mainly based on the corresponding theory for Gaussian processes which started with the well-known Pickands theorem on extremes of stationary Gaussian processes (see e.g. in [11,4]). Extensions to non-stationary Gaussian processes are mentioned in [4].

In the following paper [7], we consider stationary conditional Gaussian processes with random variance, that is, \( X(t) = \xi(t)\eta(t) \), where \( \xi(t) \) is a Gaussian stationary process and \( \eta(t) \) is a particular smooth enough process, being independent of the process \( \xi(t) \). In future works we shall consider conditionally Gaussian processes \( \xi(t) \) with random mean, and also more general classes of conditionally Gaussian random processes and fields.

We need to restrict the random mean \( m(t) \) and random variance \( V(t) \) of \( X(t) \). Suppose that \( m \) and \( V \) are uniformly finitely supported, that is, for \( t \in T \)

\[
\sigma_T^2 := \sup_{t \in [0, T]} \sigma(V^2(t, Y)) < \infty, \quad \text{and} \quad \mu_T := \sup_{t \in [0, T]} \sigma(|m(t, Y)|) < \infty
\]

where \( \sigma(Y) \) denotes for any random element \( Y \)

\[
\sigma(Y) := \text{ess sup}(Y).
\]

The function \( \sigma(V^2(t, Y)) \) plays an important role for extremes of conditionally Gaussian processes as the usual variance in the case of Gaussian processes, see Proposition 1. Thus, we may call this function the function of critical variance.
In the second main investigation we consider the exact asymptotic behavior of the probability $P_u(T)$ in Section 2, which can be derived only under more restrictive conditions on the random element $(X, Y)$. In this paper we consider the case of a random independent bounded variance, that means we consider the process $\xi(t)\eta(t)$, where $\xi(t), \ t \in \mathbb{R}$, is a Gaussian zero mean stationary process with covariance function $r(t)$ and $Y = \eta(t)$ is a particular random non-negative bounded process. We assume that the Gaussian process $\xi(t)$ satisfies

**Condition 1.** $r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \text{ as } t \to 0, \ 0 < \alpha \leq 2, \text{ and } r(t) < 1 \text{ for all } t > 0$.

The two processes $\xi$ and $\eta$ are independent. We consider two cases of $\eta(t)$. The first one is a random parabola, $\eta(t) = \lambda - \xi t^2$, where $\lambda, \xi$ are positive a.s. with $\lambda$ bounded from zero ($\lambda > \varepsilon$ a.s. for some $\varepsilon > 0$), such that the parabola is a.s. positive on the interval $[0, T]$. This is possible by selecting $T$ small. We have $\lambda = \eta(0)$ as maximal value. This result is applied to the more general second case, where $\eta(t)$ is a general non-negative smooth random process with constant critical variance and non-degenerated high local maxima. Other cases of $\eta(t)$ are considered in [7].

Under similar assumptions on $\eta(t)$, the Gaussian process $\xi(t) + \eta(t)$ with a random mean was considered in [12].

The main results are presented in the following section and its proofs are given in Section 3.

## 2. Main results for exact asymptotics

For the general case with $(X(t), Y)$ we derive the large deviation result which shows that $\sigma_T^2$ plays the crucial role, as mentioned in the introduction.

**Proposition 1.** Under the above assumptions,

$$
\log P_u = \log P_u(T, X, Y) = -(1 + o(1)) \frac{u^2}{2\sigma_T^2}
$$

as $u \to \infty$.

The proof of this proposition is given in Section 3.

For the second main result we consider the process $\xi(t)/\eta(t)$, where the Gaussian process $\xi(\cdot)$ and the process $\eta(\cdot)$ are independent. We consider first the parabola process $\eta(t) = \lambda - \xi t^2/2$, where $\lambda, \xi$ are non-negative random variables, being independent of $\xi(\cdot)$. We suppose that the conditions of the introduction are fulfilled, more precisely, we assume that $\sigma(\xi), \sigma(\lambda), \sigma(\xi/\lambda) < \infty$. We consider in **Theorem 1** the asymptotic behavior of the probability

$$
P_{u,1} = P_{u,1}(T, \xi, \eta) = P\left( \max_{t \in [-T,T]} \xi(t) \left( \lambda - \frac{1}{2} \xi t^2 \right) > u \right)
$$

with $T < \sqrt{2/\sigma(\xi/\lambda)}$ where $T > 0$ is chosen such that the random standard deviation $\lambda - \frac{1}{2} \xi t^2$ is a.s. positive. We denote $\Psi(x) := \varphi(x)/x$, where $\varphi$ is the standard Gaussian density, and $H_\alpha$ as the Pickands’ constant, $0 < H_\alpha < \infty$. Let us mention that

$$
H_\alpha = \lim_{T \to \infty} H_\alpha(T)/T = \lim_{T \to \infty} E\left( \exp\left\{ \sup_{0 \leq t \leq T} \chi(t) \right\} \right) / T
$$

with $\chi(t)$ the Brownian fractional motion with mean $-|t|^{\alpha}$ and covariance function $|t|^{\alpha} + |s|^{\alpha} - |t - s|^{\alpha}$ for any $0 < \alpha \leq 2$ (see [11] or [4]).
We assume the following condition on the density of the random variable $\lambda$. Suppose that the density $f_\lambda(x)$ of $\lambda$ is $k$ times continuously differentiable in a neighborhood of $\sigma := \sigma(\lambda)$, with $f_\lambda^{(l)}(\sigma) = 0$ for $l = 0, \ldots, k-1$, and $f^{(k)}(\sigma) \neq 0$, for some integer $k \geq 0$. Obviously, $f^{(0)} = f$ is assumed to be continuous.

Theorem 1. Suppose that the function $e_1(x) = E(\zeta^{-1/2}|\lambda = x)$ exists and is continuous at $x = \sigma$, with $e_1(\sigma) > 0$.

1. Let $\alpha < 2$. Then
   \[ P_{u,1} = (1 + o(1))(-1)^k \sqrt{2\pi} \sigma e_1(\sigma) \sigma^{-2(\alpha+3k+9/2)} f_\lambda^{(k)}(\sigma) u^{2/\alpha-3-2k} \Psi(u/\sigma) \]
   as $u \to \infty$.

2. Let $\alpha = 2$ and assume that $\tilde{e}_1(x) = E(\sqrt{\frac{2\lambda+\zeta}{\zeta}}|\lambda = x)$ exists and is continuous at $x = \sigma$. Then
   \[ P_{u,1} = (1 + o(1))(-1)^k \sigma^{3k+3} f_\lambda^{(k)}(\sigma) \tilde{e}_1(\sigma) u^{-2-2k} \Psi(u/\sigma) \]
   as $u \to \infty$.

The two asymptotic results hold also for $T = T_u$ tending to zero not too fast, such that $T_u^{2/\alpha} \to \infty$ as $u \to \infty$.

Remark 1. It can be seen from the proof, that if we consider as in $P_{u,1}$ only the interval $[0, T]$ instead of $[-T, T]$, one has to divide the asymptotic expressions on the right hand side by 2.

Remark 2. In the forthcoming publication [7], we consider another random variance case, namely $\lambda - \zeta t^\beta$, with any $\beta > 0$, and under different and more general conditions.

The particular result of Theorem 1 is needed to derive Theorem 2 for smooth processes $\eta(t)$, in particular for the approximation in neighborhoods of sufficiently high local maxima. We need the following assumptions.

Condition 2. Assume that $\eta(t)$ is non-negative and
\[ \sigma(\eta(t)) \equiv \sigma > 0. \]

We consider in Theorem 2 the asymptotic behavior of
\[ P_{u,2} := P\left( \max_{t \in [0,T]} \xi(t) \eta(t) > u \right). \]

By Proposition 1 we have for any $T$,
\[ \log P_{u,2} = -(1 + o(1)) \frac{u^2}{2\sigma^2} \]
as $u \to \infty$. From the proof of Proposition 1 one can notice that, in order to get the exact asymptotic behavior of $P_{u,2}$, we have to consider the trajectories which lie partially in a narrow band near $\sigma$,
\[ B(\delta, \varepsilon) := [-\delta, T + \delta] \times [\sigma - \varepsilon, \sigma], \]
for small $\varepsilon, \delta > 0$. We need to restrict the smoothness of $\eta(t)$ in $B(\delta, \varepsilon)$.
Condition 3. Assume for some \( \varepsilon, \delta > 0 \), that \( \eta''(t) \) exists for all \( t \) with \( (t, \eta(t)) \in B(\delta, \varepsilon) \), and that
\[
\sup_{(t, \eta(t)) \in B(\delta, \varepsilon)} |\eta''(t)| \leq C
\]
a.s., for some \( C < \infty \). Moreover, assume that the second derivative \( \eta''(t) \) is equicontinuous in the following sense:
\[
\omega(h) := \sup_{(t, \eta(t)) \in B(\delta, \varepsilon)} \sup_{s \in [0, h]} \sigma(|\eta''(t + s) - \eta''(t)|) \to 0
\]
as \( h \to 0 \).

We deal with the point process of local maxima of the process \( \eta(t) \) with values close to \( \sigma \).

Condition 4. Assume that for some \( \varepsilon, \delta > 0 \), the vector \( (\eta(t), \eta'(t), \eta''(t)) \) has a conditional density \( f_{\eta(t), \eta'(t), \eta''(t)}(\eta(t) \in [\sigma - \varepsilon, \sigma]) \) which is bounded for any \( t \in [-\delta, T + \delta] \).

In Theorem 2 and its proof we deal only with densities conditioned on \( \eta(t) \in [\sigma - \varepsilon, \sigma] \), therefore we shall omit it in the conditions. Notice that with this convention, \( f_{\eta(t)}(x) = 0 \) for \( x < \sigma - \varepsilon \).

Suppose further that the process \( \eta(t) \) is "sufficiently random", that is, given \( \eta(t) \) close to \( \sigma \), local maxima are possible, with some non-degeneracy of the points of high local maxima, which is stated precisely in Condition 5.

Condition 5. Assume for some \( \varepsilon, \delta, \kappa > 0 \), that \( \eta''(t) \leq -\kappa \), a.s., for any \( (t, x) \in B(\delta, \varepsilon) \) such that \( \eta'(t) = 0 \) and \( \eta''(t) < 0 \). Moreover, the function
\[
e_2(t, x) := \int_{-C}^{-x} |z|^{1/2} f_{\eta(t), \eta''(t)}(\eta(t) = x, z)dz
\]
is continuous at \( x = \sigma \) uniformly in \( t \), with \( \int_0^T e_2(t, \sigma) dt > 0 \).

Remark 3. Instead of Condition 5 we formulate the following two conditions:

1. There exist \( \varepsilon, \delta > 0 \) such that \( f_{\eta''(t)|\eta(t)}(0|x) > 0 \), for any \( (t, x) \in B(\delta, \varepsilon) \); moreover, \( \lim_{x \uparrow \sigma} f_{\eta''(t)|\eta(t)}(0|x) = f_{\eta''(t)}(0) > 0 \), uniformly in \( t \).

   In particular, it implies that \( \text{supp} f_{\eta''(t)}(\cdot) = \text{supp} f_{\eta''(t)|\eta(t)}(0|\cdot) \), hence \( \sigma = \text{ess sup}(\text{supp} f_{\eta''(t)}(\cdot)) \).

2. There exist \( \varepsilon, \delta, \kappa > 0 \) such that for any \( (t, x) \in B(\delta, \varepsilon) \), if \( \eta'(t) = 0 \) and \( \eta''(t) < 0 \), then \( \eta''(t) \leq -\kappa \), a.s.. Moreover, the function
\[
\tilde{e}_2(t, x) := E((|\eta''(t)|)^{1/2} | \eta(t) = x, \eta'(t) = 0)
\]
is continuous at \( \sigma \) uniformly in \( t \), with \( \int_0^T \tilde{e}_2(t, \sigma) dt > 0 \).

The two assumptions imply Condition 5.

Condition 5 implies that the random set \( \mathcal{N}(\varepsilon) \) of local maximum points of the process \( \eta(t) \), which are above \( \sigma - \varepsilon \), is a regular point process with intensity
\[
v(t) = \int_{\sigma - \varepsilon}^\sigma \int_{-\infty}^0 |z| f_{\eta(t), \eta'(t), \eta''(t)}(x, 0, z)dx dz
\]
(see for example [1]). We may also consider the point process of local maxima as a point process in \([-\delta, T + \delta] \times [\sigma - \varepsilon, \sigma] \times (-\infty, 0]\), that is,

\[ \{(t, \eta(t), \eta''(t)), \ t \in \mathcal{N}(\varepsilon)\}. \]

Its intensity is

\[ \nu(t, x, z) = |z|1_{|z|<0}f_{\eta(t), \eta'(t), \eta''(t)}(x, 0, z). \]

Recall that \( f \) is in fact the conditional density given \( \eta(t) \in [\sigma - \varepsilon, \sigma] \). Let us mention that for this process we have for any bounded function \( F(t, x, z) \)

\[
E \sum_{\mathcal{N}(\varepsilon) \cap [0,T]} F(t, \eta(t), \eta''(t)) = \int_0^T \int_{\sigma-\varepsilon}^\sigma \int_{-\infty}^0 F(t, x, z) \nu(t, x, z) d\nu dx dz. \tag{3}
\]

The second main theorem considers processes satisfying the Conditions 1–5 which means that the random variance process is rather smooth, quite different to the case dealt with in Theorem 1. We derive the asymptotically exact probabilities of exceedances of such a process.

**Theorem 2.** Let the Conditions 1–5 be fulfilled. Then, for \( \alpha < 2 \),

\[
\lim_{u \to \infty} \frac{P_{u,2}}{\Psi(u/\sigma) u^{2/\alpha - 3 - 2k}} = \sqrt{2\pi} H_{\alpha} \sigma^{3k+9/2-2/\alpha} \int_0^T (-1)^k f_{\eta(t)}^{(k)}(\sigma) e_2(t, \sigma) dt;
\]

and for \( \alpha = 2 \),

\[
\lim_{u \to \infty} \frac{P_{u,2}}{\Psi(u/\sigma) u^{-2-2k}} = \sigma^{3k+3} \int_0^T e_2^*(t, \sigma)(-1)^k f_{\eta(t)}^{(k)}(\sigma) dt,
\]

where \( e_2^*(t, x) := \int_{-\infty}^{-x} \sqrt{|z|} \frac{1}{\sigma} f_{\eta(t), \eta'(t)|\eta(t)=x}(0, z) \)

3. **Proofs**

We use a simple fact concerning saddle-point approximation (given in [4]), which we state here for easier reading and understanding of the steps of the proofs.

**Proposition 2.** Let \( g(x), x \in [0, \sigma] \), be a bounded function, which is \( k \) times continuously differentiable in a neighborhood of \( \sigma \), such that \( g^{(r)}(\sigma) = 0 \) for \( r = 0, 1, \ldots, k-1 \) and \( g^{(k)}(\sigma) \neq 0 \). Then for any \( \varepsilon \in (0, \sigma) \),

\[
\int_\varepsilon^\sigma g(x) \Psi(u/x) dx = (-1)^k \sigma^{3k+3} g^{(k)}(\sigma) u^{-2-2k} \Psi(u/\sigma)(1 + o(1)) \tag{4}
\]

as \( u \to \infty \). If \( g(x) = g_1(x) g_2(x) \), where \( g_1(x) \) is continuous at \( \sigma \) with \( g_1(\sigma) > 0 \), and \( g_2(x) \) satisfies the above conditions on \( g \), one can change \( g^{(k)}(\sigma) \) in (4) to \( g_1(\sigma) g_2^{(k)}(\sigma) \).

To prove this it is sufficient to change the variable \( x \) to \( y = u^2(x - \sigma) \) in the integral, and apply simple calculus, or see [5].

**Proof of Proposition 1.** We introduce the random modulus of continuity in square mean, given \( Y \)

\[
\phi(h, Y) = \sup_{(s, t) \in T \times T, |s-t| \leq h} \sqrt{C(s, s, Y) - 2C(s, t, Y) + C(t, t, Y)}
\]

and suppose that \( \sigma (\int_1^\infty \phi(e^{-x^2}, Y) dx) < \infty \).
Using Theorem 4.1.1 in [6], taking the conditional expectations given $Y$, we get for any $p \geq 2$ and all $x \geq \sqrt{1 + 4 \ln p}$,

$$
P \left( \sup_{t \in [0, T]} |\xi(t)| \geq \mu_T + x \left[ \sigma_T + (2 + \sqrt{2}) \sigma \left( \int_1^\infty \phi(p^{-u^2}, Y) dv \right) \right] \right) \leq \frac{5}{2} p^2 \int^\infty_x e^{-u^2/2} dv.
$$

(5)

Now choose $x$ sufficiently large (in relation to $u$), to find that for fixed $T$

$$
\limsup_{u \to \infty} u^{-2} \log P_u \leq - \frac{1}{2\sigma^2_T}.
$$

For the lower bound, choose a small $\epsilon > 0$ and select $t(\epsilon) \in [0, T]$ such that $\sigma(t(\epsilon)) \geq \sigma_T - \epsilon > 0$. We have, for any small positive $\epsilon'$, some $c > 0$ and all sufficiently large $u$,

$$
P_u \geq P(\xi(t(\epsilon)) > u) = E(\xi(t(\epsilon)) > u \mid Y) \\
\geq E(E(1_{[\xi(t(\epsilon)) > u]} 1_{V(t(\epsilon), Y) \geq \sigma_T - \epsilon} \mid Y)) \\
= E(1_{V(t(\epsilon), Y) \geq \sigma_T - \epsilon} E(1_{[\xi(t(\epsilon)) > u]} \mid Y)) \\
\geq c \exp \left( - \frac{u^2}{2(\sigma^2(t(\epsilon)) - \epsilon')} \right) P(V(t(\epsilon), Y) \geq \sigma_T - \epsilon).
$$

It follows that for fixed $T$

$$
\liminf_{u \to \infty} u^{-2} \log P_u \geq - \frac{1}{2\sigma^2(T)}.
$$

Together, we get the large deviation statement of Proposition 1. □

3.1. Proof of Theorem 1

(1) Let us consider the case 1 with $\alpha < 2$. We will condition first on $\lambda$. Thus, we start considering the case $\lambda = 1$ a.s.. We show for $P_{u,1}$ for any positive $T$, with $T < \sqrt{2}/\sigma(\xi)$, that

$$
P \left( \max_{t \in [-T, T]} \xi(t) \left( 1 - \frac{1}{2} \xi^2 \right) > u \right) = \left( 1 + o(1) \right) \sqrt{2\pi} H_a u^{2/\alpha - 1} \Psi(u) E(\xi^{-1/2})
$$

(6)

using $E(\xi^{-1/2}) = \epsilon(1) < \infty$. Since $\xi > 0$ a.s., we apply Theorem D.3 in [11], to get

$$
\lim_{u \to \infty} \frac{P \left( \max_{t \in [-T, T]} \xi(t) \left( 1 - \frac{1}{2} \xi^2 \right) > u \mid \xi \right)}{\sqrt{2\pi} H_a u^{2/\alpha - 1} \Psi(u)} = \xi^{-1/2}.
$$

(7)

Since we have to take the expectation, we have to derive an upper bound for the application of the dominating convergence theorem. We use the idea for the upper estimation in the proof of Theorem D.3, case 1 in [11]. We split the interval $[-T, T]$ into subintervals $\Delta_k = [ku^{-a}, (k + 1)u^{-a}]$ with length $u^{-a}$, where $a \in (1, 2/\alpha)$. Given $\xi$ we get by Theorem D.1 in [11] the upper bound

$$
P \left( \max_{\Delta_k} \xi(t) > u \left( 1 - \frac{1}{2} \kappa^2 \xi^{-2a} \right) \mid \xi \right)
$$
Proposition 2 follows.

By Theorem D.3(ii) in (2) Now we turn to the case 

By dominating convergence,

where \( \gamma(u) \downarrow 0 \) as \( u \uparrow 0 \), not depending of \( k \) and \( \zeta \). Summing on \( k \), we get

By the conditions of the theorem, the right hand side has a finite expectation and \( C_2 \zeta^{-1/2} \) dominates the fraction on the left hand side of (7), with some \( C_2 > 0 \). By dominating convergence, (6) follows.

Using \( \lambda \in [\varepsilon, \sigma] \) a.s. for a sufficiently small \( \varepsilon \), we have furthermore by conditioning on \( \lambda \)

Applying now Proposition 2 for the integral of (8), with \( g_1(x) = x^{-2/\alpha+3/2}e_1(x) \) and \( g_2(x) = f_\lambda(x) \), we get the first assertion of Theorem 1.

(2) Now we turn to the case \( \alpha = 2 \) and consider again the probability on the left hand side of (6). By Theorem D.3(ii) in [11], we have

\[
\lim_{u \to \infty} \frac{P \left( \max_{t \in [-T,T]} \xi(t) \left( 1 - \frac{1}{2} \xi t^2 \right) > u \right)}{\Psi(u)} = \lim_{S \to \infty} E \left( \exp \left( \max_{t \in [-S,S]} \left( \sqrt{2}Ut - t^2 \left( 1 + \frac{1}{2} \right) \right) \right) \right)
\]
where $U$ is a standard normal random variable, not depending on $\zeta$. The representation in the right hand side of (9) holds for the fractional Brownian motion in the case $\alpha = 2$. The exponent under the expectation on the right hand side increases to $\exp(U^2/(2 + \zeta))$, as $S \to \infty$, so that by Fatou’s Theorem, this limit equals now $E(\exp(U^2/(2 + \zeta)))|\zeta) = \sqrt{(2 + \zeta)/\zeta}$.

Again, we consider an upper bound for the approximation in (9), for applying the dominating convergence theorem. The probability in the left side of Eq. (9) is at most

$$P \left( \max_{t \in [-Su^{-1},Su^{-1}]} \xi(t) \left( 1 - \frac{1}{2} \xi t^2 \right) > u \mid \zeta \right)$$

$$+ \sum_{k \geq 1 \text{ or } k < -1} P \left( \max_{t \in [kSu^{-1},(k+1)Su^{-1}]} \xi(t) \left( 1 - \frac{1}{2} \xi t^2 \right) > u \mid \zeta \right)$$

(10)

for $S > 0$. The first term of (10) is bounded again by Theorem D.3(ii) in [11] and the argument after (9), for all sufficiently large $u$.

$$P \left( \max_{t \in [-Su^{-1},Su^{-1}]} \xi(t) \left( 1 - \frac{1}{2} \xi t^2 \right) > u \mid \zeta \right)$$

$$\leq 2 \Psi(u) E \left( \exp \left( \max_{t \in [-S,S]} \left( \sqrt{2ut} - t^2 \left( 1 + \frac{1}{2} \zeta \right) \right) \right) \right)$$

$$\leq 2 \Psi(u) \sqrt{(2 + \zeta)/\zeta}.$$  

For the other terms ($k \geq 1$ and $k < -1$) of (10) we have

$$P \left( \max_{t \in [kSu^{-1},(k+1)Su^{-1}]} \xi(t) \left( 1 - \frac{1}{2} \xi t^2 \right) > u \mid \zeta \right)$$

$$\leq P \left( \max_{t \in [kSu^{-1},(k+1)Su^{-1}]} \xi(t) \left( 1 - \frac{1}{2} \zeta k^2 S^2 u^{-2} \right) > u \mid \zeta \right)$$

$$\leq P \left( \max_{t \in [0,Su^{-1}]} \xi(t) > u \left( 1 + \frac{1}{2} \zeta k^2 S^2 u^{-2} \right) \mid \zeta \right).$$

By Lemma 6.1 in [11], the last probability is at most

$$(1 + \gamma(u)) H_2(S) \Psi \left( u + \frac{1}{2} \zeta k^2 S^2 u^{-1} \right) \leq (1 + \gamma(u)) H_2(S) \Psi(u)e^{-\zeta k^2 S^2/2},$$

where $\gamma(u) \downarrow 0$ as $u \to \infty$ and does not depend of $k$, $\zeta$ and $S$.

Summing now the bounds of the second term in (10), we get the upper bound

$$C_0 \Psi(u) \sum_{k \geq 1} e^{-\zeta k^2 S^2/2} \leq C_0 \Psi(u) \left( e^{-\zeta S^2/2} + \int_{1}^{\infty} e^{-\zeta y^2 S^2/2} dy \right),$$

with some $C_0 > 0$. The expression in the brackets is finite, by taking the expectation on $\zeta$. Thus by dominated convergence we can take the expectation in (9) to get

$$P \left( \max_{t \in [-T,T]} \xi(t) \left( 1 - \frac{1}{2} \xi t^2 \right) > u \right) = (1 + o(1)) E \left( \sqrt{\frac{2 + \zeta}{\zeta}} \right) \Psi(u).$$

(11)

as $u \to \infty$ and $S \to \infty$.  

To derive the statement on $P_{u,1}$, we condition on $\lambda$ and apply again Proposition 2.

$P_{u,1} = \int_\varepsilon^\sigma \mathbb{P} \left( \max_{t \in [-T,T]} \xi(t) \left( 1 - \frac{1}{2} \zeta t^2 / x \right) > u / \lambda = x \right) f_\lambda(x)dx$

$= (1 + o(1)) \int_\varepsilon^\sigma E \left( \frac{2 + \zeta / x}{\zeta / x} \bigg| \lambda = x \right) \Psi(u/x)f_\lambda(x)dx$

$= (1 + o(1))(-1)^k E \left( \frac{2\sigma + \zeta}{\zeta} \bigg| \lambda = \sigma \right) \Psi(u/\sigma)(-1)^k \sigma^{3k+3} u^{-2-2k} f_\lambda^{(k)}(\sigma)$

as $u \to \infty$ which is our claim. \(\Box\)

We note the following asymptotic result from the equivalence (11) and its derivation.

**Corollary 1.** Under the assumptions of Theorem 1, we have

$$P \left( \max_{t \in [-T,T]} \xi(t) \left( \lambda - \frac{1}{2} \zeta t^2 \right) > u / \lambda, \zeta \right) = (1 + o(1)) \sqrt{\frac{2\lambda + \zeta}{\zeta}} \Psi(u/\lambda) \quad \text{as } u \to \infty.$$  

### 3.2. Proof of Theorem 2

(1) Upper bound. Let $t$ be a point of local maximum of $\eta(t)$ with $\eta(t) \geq \sigma - \varepsilon(u)$, with $0 < \varepsilon(u) \leq \varepsilon$, where $\varepsilon(u) \to 0$ (as $u \to \infty$) is chosen later. We say that $s$ is connected with $t$ if

$$(v, \eta(v)) \in B(\delta, \varepsilon), \quad \forall v \in [s, t] \cup [t, s],$$

where we suppose that $[s, t] = \emptyset$ if $s > t$.

By Conditions 3 and 5, we get for $s$ connected with $t$,

$$\eta''(s) \leq -\kappa + w(|t - s|). \quad (12)$$

From the left inequality we get by integration, that

$$\eta(s) \geq \eta(t) - \frac{C}{2}(t - s)^2 \geq \sigma - \varepsilon(u) - \frac{C}{2}(t - s)^2.$$

If the right hand part exceeds $\sigma - \varepsilon$, then any $s$ with $|s - t| \leq \sqrt{2(\varepsilon - \varepsilon(u))/C}$ is connected with $t$. Take $u$ large to have $\varepsilon(u) < \varepsilon/2$ and $h > 0$ such that $\omega(h) < \min(\kappa/2, \sqrt{\varepsilon/C})$. From the right inequality of (12) we get by integration separately for $s \in [t - h, t]$ and $s \in [t, t + h]$, that $|\eta''(s)| \geq \kappa|s - t|/2$. It means that there are no local maxima in $[t - h, t + h]$ but $t$, with trajectories in $B(\delta, \varepsilon)$. Thus all points of local maxima in $B(\delta, \varepsilon(u))$ are separated by at least $2h$, for all sufficiently large $u$. We can assume that $h \leq \delta$, so that all conditions stated for $B(\delta, \varepsilon)$ are valid for $B(h, \varepsilon)$. For $s$ connected with $t$ with $|s - t| < h$ we have also the inequalities

$$\eta(s) \leq \eta(t) + \frac{1}{2}(t - s)^2(\eta''(t) + w(h)) \quad (13)$$

and

$$\eta(s) \geq \eta(t) + \frac{1}{2}(t - s)^2(\eta''(t) - w(h)). \quad (14)$$
We get that the set
\[ D_u(\delta) := \{(t, x) \in B(\delta, \varepsilon(u)) : x = \eta(t)\} \]
consists of small “hats” with only one point of maximum for each “hat”, these points of maximum are separated by at least 2h. Since for the “hats”, |η'(s)| > κ|s − t|/2, we have \( \sigma - \varepsilon(u) \leq \eta(s) \leq \eta(t) - \kappa(t - s)^2/4 \leq \sigma - \kappa(t - s)^2/4 \), hence |t − s| \leq 2√ε(u)/κ and hence the width of the base of any “hat” is at most 2ε(u) := 4√ε(u)/κ.

Let s₁ be the first time of local maximum in \( D_u(0) \) (that is in \([0, T]\) with \( \eta(s_1) \geq \sigma - \varepsilon(u) \)) and \( s_M \) be the last one. Introduce the random sets
\[ L = [0, T] \cap \bigcup_{s \in \mathcal{N}(\varepsilon(u))} [s − \delta(u), s + \delta(u)], \]
\[ L_+ = L \cup [0, s_1]_{\eta(0) \geq \sigma - \varepsilon(u), \eta'(0) < 0} \cup [s_M, T]_{\eta(T) \geq \sigma - \varepsilon(u), \eta'(T) > 0} \cap \mathcal{N}(\varepsilon(u)). \]

We have, using Fernique’s inequality,
\[
P_{u,2} = E \left( P \left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \mid \eta \right) \right) = E \left( P \left( \max_{L} \xi(t) \eta(t) > u \mid \eta \right) \right) + O(\exp(-u^2/2(\sigma - \varepsilon(u))^2)).
\]

We choose now \( \varepsilon(u) \) appropriately: \( \varepsilon(u) = \sigma^2 A u^{-2} \log u \), with a large positive \( A \), such that the above second term is of order \( u^{-A} \exp(-u^2/2\sigma^2) \), as \( u \to \infty \). It implies the upper bound
\[
P_{u,2} \leq E \left( \sum_{t \in \mathcal{N}(\varepsilon(u)) \cap [0, T]} P \left( \max_{s \in [s - h, s + h]} \xi(s) \eta(s) > u \mid \eta \right) \right) + E \left( P \left( \max_{s \in [0, s_1 - h]} \xi(s) \eta(s) > u, \eta(0) \geq \sigma - \varepsilon(u), \eta'(0) < 0 \mid \eta \right) \right) + E \left( P \left( \max_{s \in [s_M + h, T]} \xi(s) \eta(s) > u, \eta(T) \geq \sigma - \varepsilon(u), \eta'(T) > 0 \mid \eta \right) \right) + O(u^{-A} \exp(-u^2/2\sigma^2)),
\]
by setting the maximum equals \(-\infty\) on an empty set. Consider first the last two probabilities in (15). If \( \eta'(0) < 0 \) and \( \eta(0) \geq \sigma - \varepsilon(u) \) then the largest negative point of local maximum, say, \( s_0 \), is bigger than \(-\delta(u)\) by the structure of the set \( D_u(\delta) \). In the same way the first local maximum after \( T \), say \( s_{M+1} \), in case \( \eta(T) \geq \sigma - \varepsilon(u) \) and \( \eta'(T) > 0 \) is smaller than \( T + \delta(u) \). Thus these two probabilities in (15) can be bounded above by the probabilities related to local maxima of \( \eta(t) \) in intervals \([-\delta(u), 0]\) and \([T, T + \delta(u)]\). Thus we get with \( \mathcal{N} = \mathcal{N}(\varepsilon(u)) \cap [-\delta(u), T + \delta(u)] \)
\[
P \left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \right) \leq E \left( \sum_{t \in \mathcal{N}} P \left( \max_{s \in [s - h, s + h]} \xi(s) \eta(s) > u \mid \eta \right) \right) + O(u^{-A} \exp(-u^2/2\sigma^2)).
\]
(i) For $\alpha < 2$, using (13) and Theorem D.3 in [11]

$$
P\left( \max_{s \in [t-h, t+h]} \xi(s) \eta(s) > u / \eta \right) \leq P\left( \max_{s \in [t-h, t+h]} \xi(s) \left( \eta(t) + \frac{1}{2} (t-s)^2 \right) \left( \eta''(t) + w(h) \right) > u / \eta \right) \leq \sqrt{2\pi} H_\alpha (-\eta''(t))^{-1/2} (1 - w(h)/\kappa)^{-1/2} \eta(t)^{3/2-2/\alpha} u^{2/\alpha-1} \Psi(u/\eta(t))(1 + \gamma(u)) ,
$$

where $\gamma(u)$ can be chosen non-random. For, since $\eta(t) \leq \sigma$ and $\eta(s) \geq \sigma - \varepsilon$ with $|\eta''(t)| \in [\kappa, C]$, take $\gamma(u)$ based on $\eta''(t) = -\kappa$ and $\eta(t) = \sigma$, to get the above estimate with non-random $\gamma(u)$. Applying (3), we derive

$$
P\left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \right) \leq \frac{(1 + \gamma(u))\sqrt{2\pi} H_\alpha u^{2/\alpha-1}}{\sqrt{1 - w(h)/\kappa}} \times \int_{\delta(u)}^{T+\delta(u)} \int_{-\varepsilon(u)}^{\sigma} \int_{-C}^{\sigma} \frac{\Psi(u/\sigma)}{|z|^{1/2} x^{3/2-2/\alpha}} f_{\eta(t), \eta''(t)}(x, 0, z) dz dx dt + O(u^{-A} \exp(-u^2/2\sigma^2)).
$$

By Proposition 2 where $g_1 = x^{3/2-2/\alpha} e_2(t, x)$ and $g_2 = f_{\eta(t)}(x)$, the conditions of the theorem and using

$$
f_{\eta(t), \eta''(t)}(x, 0, z) = f_{\eta(t)}(x) f_{\eta''(t)}(0, z) = f_{\eta(t)}(0, z),
$$

we derive the bound

$$
P\left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \right) \leq \frac{(1 + \gamma(u) + \gamma_1(u))\sqrt{2\pi} H_\alpha \sigma^{9/2-2/\alpha+3k}}{\sqrt{1 - w(h)/\kappa}} \times u^{2/\alpha-3-2k} \Psi(u/\sigma) \int_{\delta(u)}^{T+\delta(u)} (-1)^k f_{\eta(t)}^{(k)}(\sigma) e_2(t, \sigma) dt + O(u^{-A} \exp(-u^2/2\sigma^2)),
$$

where $\gamma_1(u) \to 0$ as $u \to \infty$, since the inner integral $\int |z|^{1/2} f_{\eta(t)} \eta''(t) = e_2(0, z) dz = e_2(t, x)$. Thus it implies that

$$
\limsup_{u \to \infty} P\left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \right) / u^{2/\alpha-3-2k} \Psi(u/\sigma) \to 0
$$

tends to the stated asymptotic expression as $h \to 0$.

(ii) For $\alpha = 2$, we use Corollary 1 to get an upper bound for all sufficiently large $u$ with some non-random $\gamma(u)$ where $\gamma(u) > 0$ and $\lim_{u \to \infty} \gamma(u) = 0$, 

$$
P\left( \max_{s \in [t-h, t+h]} \xi(s) \left( \eta(t) + \frac{1}{2} (t-s)^2 \right) \left( \eta''(t) + w(h) \right) > u / \eta \right) \leq P\left( \max_{s \in [t-h, t+h]} \xi(s) \left( \eta(t) - \frac{1 - w(h)/\kappa}{2} |\eta''(t)| s^2 \right) > u / \eta \right) \leq (1 + \gamma(u)) \Psi(u/\eta(t)) \frac{2\eta(t) + (1 - w(h)/\kappa)|\eta''(t)|}{(1 - w(h)/\kappa)|\eta''(t)|}.
$$
by using stationarity. Averaging on \( \eta \) and \( \eta'' \) with the above representation for the density \( f \) we get by applying again Proposition 2,

\[
P \left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \right) \leq \frac{1 + \gamma(u)}{\sqrt{1 - w(h)/k}} \int_{\delta(u)}^{T+\delta(u)} \int_{T-\delta(u)}^{\sigma} |z|^{1/2} \times \sqrt{2x + (1 - w(h)/k)z} \Psi(u/x) f_{\eta(t)}(x) f_{\eta'(t)}(x) f_{\eta''(t)}(x) (0, z|x) dz dx dr \\
+ O(u^{-A} \exp(-u^2/2\sigma^2)) \\
= (1 + \gamma(u)) \int_{-\delta(u)}^{T+\delta(u)} \int_{-\epsilon(u)}^{\sigma} e^{\sigma/2} \left(t, 2x \frac{1 - w(h)/k}{1 - w(h)/k} \right) \\
\times \Psi(u/x) f_{\eta(t)}(t) dx dt + O(u^{-A} \exp(-u^2/2\sigma^2)) \\
= (1 + \gamma_1(u)) \Psi(u/\sigma) \int_{-\delta(u)}^{T+\delta(u)} e^{\sigma/2} \left(t, \sigma + O(w(h)) \right) (-1)^k \\
\times f_{\eta(t)}(\sigma)^{2k+3} u^{-2-2k} dt,
\]

for some \( \gamma_1(u) \to 0 \) as \( u \to \infty \). Thus we have that

\[
\limsup_{u \to \infty} P \left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \right) / u^{-2-2k} \Psi(u/\sigma)
\]
tends to the stated asymptotic expression as \( h \to 0 \).

For the lower bound we use the definition of the set \( D_u \), if \( t \) and \( s \) both are points of local maxima of \( \eta(t) \) above \( \sigma - \epsilon(u) \), then \( |t - s| \geq h \). Thus, there are at most \( T/2h \) points of such local maxima in the interval \([0, T]\). We have by using now \( \mathcal{N} = \mathcal{N}(\epsilon(u)) \cap [\delta(u), T - \delta(u)] \)

\[
E \left( P \left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \mid \eta \right) \right) \geq E \left( P \left( \max_{v \in [s-h/2, s+h/2]} \xi(v) \eta(v) > u \right) \right) \\
\geq E \left( \sum_{s \in \mathcal{N}} P \left( \max_{v \in [s-h/2, s+h/2]} \xi(v) \eta(v) > u \mid \eta \right) \right) \\
- E \left( \sum_{s \in \mathcal{N}} P \left( \max_{v \in [s-h/2, s+h/2]} \xi(v) \eta(v) > u, \max_{v \in [s-h/2, s+h/2]} \xi(v) \eta(v) > u \mid \eta \right) \right),
\]

where the last sum is taken over the set \( \{ s, t \in \mathcal{N}(\epsilon(u)) \cap [\delta(u), T + \delta(u)], s \neq t \} \). Using (14) and the above arguments for upper bound, we get for the first sum a lower estimate which tends to the upper bound as \( u \to \infty \) and then \( h \to 0 \). One can estimate the double sum from above by standard methods, using that the distances between the considered intervals are at least \( h \), which implies that the double sum is at most \( O(\exp(-u^2\sigma^{-2}/(1 + a))) \) for any \( a \in (\max_{t \geq h} r(t), 1) \). Thus we get that

\[
\liminf_{u \to \infty} P \left( \max_{t \in [0, T]} \xi(t) \eta(t) > u \right) / u^{-2-2k} \Psi(u/\sigma)
\]
also tends to the stated asymptotic expression as \( h \to 0 \). \( \square \)
Acknowledgments

We are very grateful to the reviewers of the first version of this paper, for their careful reading and comments on improving the presentation of our results and proofs.

The first and second authors were supported by a grant of Swiss scientific foundation.

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