Europ. J. Combinatorics (2001) **22**, 243–261
doi:10.1006/eujc.2000.0441
Available online at http://www.idealibrary.com on IDEAL®

Regular Maps on Surfaces with Large Planar Width

ROMAN NEDELA AND MARTIN ŠKOVIERA

A map is a cell decomposition of a closed surface; it is regular if its automorphism group acts transitively on the flags, mutually incident vertex-edge-face triples. The main purpose of this paper is to establish, by elementary methods, the following result: for each positive integer w and for each pair of integers $p \geq 3$ and $q \geq 3$ satisfying $1/p + 1/q \leq 1/2$, there is an orientable regular map with face-size p and valency q such that every non-contractible simple closed curve on the surface meets the 1-skeleton of the map in at least w points. This result has several interesting consequences concerning maps on surfaces, graphs and related concepts. For example, MacBeath's theorem about the existence of infinitely many Hurwitz groups, or Vince's theorem about regular maps of given type (p, q), or residual finiteness of triangle groups, all follow from our result.

1. INTRODUCTION

Tessellations of surfaces by p-gons with q of them meeting at each vertex have been studied for a long period of time as a natural generalization of the Platonic solids, extending the idea of regular polyhedra from the 2-sphere to other closed surfaces. General (p, q)-tessellations, however, may lack several important properties typical of the regular polyhedra. In particular, they may fail to be symmetrical and, because of the multiple connectivity of the supporting surface, they may have p-gonal faces with repeated occurrence of some vertices on their boundary. Thus, by restricting to symmetrical (p, q)-tessellations which do not contain self-touching faces, a more accurate generalization of the regular polyhedra can be obtained. It is the purpose of this paper to study such objects.

Let us make the above ideas more precise. A *tessellation* consists of a connected surface S without a boundary and a connected locally finite graph K embedded in S, such that each component of $S - K$ is simply connected and has compact closure; these are the *faces* of the tessellation. When S is compact (and the graph is finite), a tessellation is called a *map* on S. An *automorphism* or a symmetry of a map is a permutation of its vertices, edges and faces which can be accomplished by a homeomorphism of the surface onto itself. Equivalently, a map automorphism is an automorphism of its underlying graph which takes faces to faces. If the surface is orientable, an automorphism is often required to preserve the selected orientation of S.

At this point, let us restrict ourselves to orientable surfaces only. It is well known that the number of orientation-preserving automorphisms of each map is bounded from above by the doubled number of its edges [16]. This bound is attained precisely when the map automorphism group acts regularly on the set of *arcs* of the map, the directed edges where each edge is represented twice, once in each direction. Accordingly, such a map is called *regular*, or more precisely *orientably-regular*. Thus, in a certain sense, regular maps, such as the Platonic solids, are the most symmetrical of all maps.

Modern history of regular maps goes back at least to Klein (1878) who described in [20] a regular map of type $(7, 3)$ on the orientable surface of genus 3. In its early times, the study of regular maps was closely connected with group theory as one can see in Burnside's famous monograph [4], and more recently in Coxeter's and Moser's book [6, Chapter 8]. The present-day interest in regular maps extends to their connection to Dyck's triangle groups, Riemann surfaces, linear fractional transformations of the complex plane, algebraic curves,

0195–6698/01/020243 + 19 $35.00/0

Galois groups and other areas. Many of these links are nicely surveyed in the recent papers of Jones [15] and Jones and Singerman [17].

While regularity (in either arithmetical or group-theoretical sense) is an obvious property of any generalization of the Platonic solids, it is somewhat less obvious whether or how their planarity should be generalized. From our point of view, it is natural to replace the global planarity of the Platonic solids by a certain local variant of this notion. This should guarantee that a sufficiently 'large' neighbourhood of each face is simply connected, thereby ruling out self-touching faces. Recent works in topological graph theory (cf. [25, 28, 29, 31]) suggest the following concept as a convenient measure of local planarity. A map M on a closed surface S other than the 2-sphere is said to have *planar width* at least k, $w(M) \geq k$, if every non-contractible simple closed curve on S intersects the underlying graph of M in at least k points. Planar width (most often called 'face-width' or 'representativity') has recently received considerable attention as an important tool for the study of graph embeddings on surfaces. The present paper concentrates on the study of planar width of regular maps on orientable surfaces. Its main purpose is to show that for every hyperbolic or parabolic type (p, q), that is, $1/p + 1/q \leq 1/2$, there exists a regular map on some orientable surface having type (p, q) and arbitrarily large planar width.

MAIN THEOREM. *For every pair of integers $p \geq 3$ and $q \geq 3$ such that $1/p + 1/q \leq 1/2$ and for every integer $k \geq 2$, there exists an orientable regular map M of type (p, q) with $w(M) \geq k$.*

This theorem has several predecessors in the literature.

In 1976, Grünbaum [12] asked if for every pair of positive integers p and q with $1/p + 1/q < 1/2$ (i.e., in the hyperbolic case) there are infinitely many finite regular maps of type (p, q). He also remarked, however, that it was not even known whether for such p and q there was at least one map of that type. The question was answered in the affirmative by Vince [32] within a more general framework of higher-dimensional analogues of regular maps. His proof, based on a theorem of Mal'cev stating that every finitely generated matrix group is residually finite (see, e.g., Kaplansky [19]), was non-elementary and non-constructive. Constructive proofs of Vince's theorem were subsequently given by Gray and Wilson [10] and Wilson [35, 37] along with some refinements. Further constructions of regular maps of each type (p, q) have recently been given by Jendrol' *et al.* [14] and Archdeacon *et al.* [1].

Parallel to this development there is another line of research which is closely related to our Main Theorem. The bridge between the two streams is the observation that an orientable regular map of type (p, q) exists if and only if there is a finite group with presentation $\langle x, y;\ x^q = y^2 = (xy)^p = 1, \dots \rangle$. Indeed, given such a group H, let us view its elements as arcs of a map to be constructed. Consider the action of H on itself by left translation and represent orbits of $\langle x \rangle$, $\langle y \rangle$, and $\langle xy \rangle$ as vertices, edges and faces, respectively, with incidence being given by non-empty intersection. Taking a p-gon for each face and making the obvious identifications between the faces results in a map of type (p, q) on an orientable surface. It is easy to show that the map is regular. Conversely, the automorphism group of a regular map is a finite group admitting the required presentation.

With this relationship in mind, the solution of the above Grünbaum's (p, q)-problem can be derived from an old result (1902) of Miller [24] (rediscovered by Fox [8] in 1952) which states: *For any three integers p, q, and r, all greater than 1, there exist infinitely many pairs of permutations α, β such that α has order p, β has order q, and $\alpha\beta$ has order r, except that the three numbers are 2, 3, 3 or 2, 3, 4 or 2, 3, 5 (ordered arbitrarily) or two of the numbers are* 2. In the latter cases, the triples determine the groups uniquely: they are the tetrahedral, the octahedral and the dodecahedral group, and the dihedral groups.

The special case where $p = 7$, $q = 3$ and $r = 2$ (or its dual, with the roles of p and q interchanged) was the object of an extensive research in the area of Fuchsian groups, hyperbolic geometry and Riemann surfaces originating from the famous theorem of Hurwitz (1893): *For any Riemann surface S of genus $g \geq 2$, the number of its automorphisms* (that is, conformal homeomorphisms) *does not exceed* $84(g - 1)$. This bound is attained precisely when Aut (S) is a *Hurwitz group*, a finite group H generated by an element x of order 3 and an element y of order 2 whose product has order 7. As mentioned above, such a group gives rise to a regular map of type $(7, 3)$ on S, a (trivalent) *Hurwitz map*, whose automorphism group is isomorphic to H. In this context, the fact that there are infinitely many Hurwitz groups was first established by MacBeath in 1969 [23].

Observe, however, that MacBeath's theorem immediately follows from Vince's theorem which in turn is a consequence of our Main Theorem. Indeed, it is sufficient to take an infinite sequence of regular maps of any type (p, q) (in particular, $(7, 3)$) with increasing planar width. Thus our Main Theorem has the following two corollaries.

COROLLARY 1 (VINCE'S THEOREM [32]). *For any pair of integers with $p \geq 2$, $q \geq 2$ and $1/p + 1/q \leq 1/2$ there exist infinitely many orientable regular maps of type (p, q).*

COROLLARY 2 (MACBEATH'S THEOREM [23]). *There exist infinitely many Hurwitz groups.*

Another notable consequence of the Main Theorem is also of group-theoretical nature. Let $T^+(2, p, q)$ be the oriented triangle group with presentation $\langle x, y; x^q = y^2 = (xy)^p = 1\rangle$. Then for any integer $w \geq 1$ there exist infinitely many finite quotients H of $T^+(2, p, q)$ such that in any presentation of H in terms of x and y all reduced identities that are not identities of $T^+(2, p, q)$ have length greater than w. The latter sentence is nothing but a reformulation of the well-known fact that triangle groups $T^+(2, p, q)$ are residually finite [19]. A group G is called *residually finite* if for each element $g \in G$ there exists a finite quotient H of G such that the epimorphism $G \to H$ does not send g onto identity. Furthermore, the group $T^+(2, p, q)$ is easily seen to be isomorphic to the even-word subgroup of the full triangle group $T(2, p, q) = \langle x, y, z;\ a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ac)^2\rangle$. Hence, by using our Main Theorem (as well as its unoriented version, Theorem 4.7) we deduce the following corollary.

COROLLARY 3. *For each pair of integers p and q such that $1/p + 1/q \leq 1/2$, both the oriented and the full triangle group $T^+(2, p, q)$ and $T(2, p, q)$ are residually finite.*

As was noted, Vince [32] proved Grünbaum's conjecture by employing the residual finiteness of triangle groups. In contrast, we have just shown that our Main Theorem implies both Vince's theorem and the residual finiteness of triangle groups. In fact, the residual finiteness of triangle groups is equivalent to the Main Theorem.

It is well known (see [13, 21, 30]) that the fundamental groups of compact surfaces are residually finite. Alternatively, this can be proved by using Corollary 3 and observing that every fundamental group of a closed surface embeds into some triangle group $T(2, p, q)$. A natural way to ensure that the fundamental group of a given surface embeds into a triangle group $T(2, p, q)$ is to try to construct a map of type (p, q) supported by S. For instance, if the surface is orientable of genus g one can take $p = 4g = q$.

High symmetry combined with high local planarity of tesellations are very restrictive conditions: in general, a map of type (p, q) will neither be highly symmetric nor highly locally planar. If these conditions are relaxed, much stronger results can be proved. Under these weaker conditions the problem of the existence of (p, q)-tessellations of closed surfaces was completely solved in 1982 by Edmonds, Ewing and Kulkarni [7] by establishing the following remarkable theorem: *Given a closed surface S with Euler characteristic* χ *there exists a* (p, q)*-tessellation of S with n vertices, m edges and r faces whenever the obvious necessary conditions are satisfied:* $n - m + r = \chi$ *and* $pr = 2m = qn$. It is important to note that in order to obtain this statement one *must* allow faces with repeated occurrence of some vertices on the boundary (the planar width of such maps is 1).

Our paper is organized as follows.

The next section contains preliminaries on maps on surfaces and their combinatorial representation. Dealing with combinatorial representations of maps rather than with topological maps themselves is essential in this paper. We introduce two different combinatorial descriptions of maps. In the case of oriented surfaces we utilize rotation-involution pairs acting on arcs, whereas in the latter case triples consisting of a longitudinal, rotary and transversal involution acting on flags, incident vertex-edge-face triples, are appropriate.

Section 3 introduces the key ingredient of the method behind the proof of our Main Theorem. It is the concept of the generic regular map over a given map, the universal smallest regular map covering it. This concept enables us to derive the required regular maps as generic regular coverings over suitable irregular maps.

The final section is devoted to the proof of the main result. In short, the proof proceeds as follows. We first take the infinite regular hyperbolic tessellation $T_{p,q}$ of type (p, q) with a distinguished fundamental polygon. We form a disc D_w consisting of all polygons whose distance from the fundamental polygon does not exceed a given value w. Then we glue two copies of D_w along their boundaries to form a map M_0 on the 2-sphere, the identified boundaries giving rise to the *Equator* of M_0. In M_0, valencies and covalencies (face-sizes) are correct everywhere except the Equator. Therefore we perform certain corrections in the vicinity of the Equator to get rid of irregularities and to obtain a map of type (p, q). These modifications are possible since, at this stage of the proof, we may allow semiedges, that is, edges having only one incident vertex. The corrections split into a number of cases most of which treat situations where $p \leq 8$ or $q \leq 6$. The modified spherical map is then lifted to its generic regular covering (which necessarily has no semiedges), and this map is shown to be the required regular map with planar width at least w.

Finally, let us note that this paper generalizes and improves the results of Jendrol' and the present authors [14] where the special case $p = 3$ (or $q = 3$) is treated. Although the main idea consisting in construction of a generic regular map over an irregular map remains the same, the proof presented here essentially differs from that in [14] even when restricted to maps of type $(3, q)$ or $(p, 3)$.

2. Topological and Combinatorial Maps

A *map* on a surface is a cellular decomposition of a closed surface into 0-cells called *vertices*, 1-cells called *edges* and 2-cells called *faces*. The vertices and edges of a map form its *underlying graph*. A map is said to be *orientable* if the supporting surface is orientable, and is *oriented* if one of two possible orientations of the surface has been specified; otherwise, a map is *unoriented*. Unless explicitly stated otherwise, all maps in this paper are oriented.

Typically, a map on a surface is constructed by embedding a connected graph in the surface. Graphs considered in this paper are finite, non-trivial and connected unless the opposite follows from the immediate context. Edges of our graphs may belong to one of three kinds: links, loops and semiedges. Multiple adjacencies are allowed. A link is incident with two vertices while a loop or a semiedge is incident with a single vertex. A link or a loop gives rise to two oppositely directed arcs that are reverse to each other. A semiedge incident with a vertex u gives rise to a single arc initiating at u that is reverse to itself. From the topological point of view, a semiedge is identical with a pendant edge except that its pendant end-point is not listed as a vertex. This property enables us to use semiedges as a convenient technical device in many constructions. Summing up, a graph seen as a topological space is just a finite one-dimensional cell complex.

With each map we can associate its *dual map* on the same surface. Given a map M, the dual map M^* is formed by placing a vertex in the interior of each face of M, and by subsequently constructing, for each edge e of M, an edge e^* of M^* which transversally crosses e and connects the new vertices in the faces on either side of e. If e is a semiedge, then e^* is also a semiedge.

Let M be a map formed by embedding a connected graph G in a closed surface S other than the sphere. Then every simple closed curve C on S (in particular, a cycle in G) falls into one of the following two classes. If C can be continuously contracted to a point in S, that is to say, if C is homotopically null, then C is said to be *contractible*; otherwise it is *non-contractible*. The *planar width* of M, denoted by $w(M)$, is the minimum of $|C \cap G|$ taken over all non-contractible simple closed curves C in S—provided that M contains no semiedges; otherwise, for convenience we set $w(M) = 1$. It can be shown that, for maps without semiedges, the planar width of M is equal to the smallest number of faces whose union together with their boundaries contains a non-contractible simple closed curve; the curve can be chosen to cross G at vertices only. Moreover, $w(M) = w(M^*)$ (see Mohar [25, Proposition 3.2]).

For the sake of technical convenience we shall usually replace topological graphs and maps by their combinatorial counterparts. Formally, a (combinatorial) *graph* is a quadruple $G = (D, V;\ I, L)$ where $D = D(G)$ and $V = V(G)$ are disjoint non-empty finite sets, $I : D \to V$ is a surjective mapping, and $L = L_G$ is an involutory permutation on D. The elements of D and V are *arcs* and *vertices*, respectively, I is the incidence function assigning to every arc its *initial vertex*, and L is the *arc-reversing involution*; the orbits of the group $\langle L \rangle$ on D are *edges* of G. If an arc x is a fixed point of L, that is, $L(x) = x$, then the corresponding edge is a *semiedge*. If $IL(x) = I(x)$ but $L(x) \neq x$, then the edge is a *loop*. The remaining edges are *links*. Note that the same type of graphs are considered in Jones and Singerman [16] and in our paper [26].

The usual graph-theoretical concepts such as cycles, connectedness, etc., easily translate to our present formalism. In particular, the *valency* of a vertex v is the number of arcs having v as their initial vertex, and a *cycle* in a graph is a connected 2-valent subgraph.

As far as maps on surfaces are concerned, there are two essentially different approaches to their combinatorial description. The first approach, based on a rotation-involution pair acting on arcs, involves the orientation of the supporting surface and so is suitable only for maps on orientable surfaces [11, 16]. The corresponding combinatorial structure is called a combinatorial (or, sometimes, algebraic) oriented map. The other approach, using three involutions acting on mutually incident (vertex, edge, face)-triples called flags, is orientation insensitive and thus allows us to represent maps on non-orientable surfaces as well [18, Section 2]. The resulting combinatorial structure will be called a combinatorial unoriented map. Although we will exclusively deal with maps on orientable surfaces, we will also occasionally employ the three-involutions approach. It should be emphasized, however, that the primary object for us is a topological map, and a combinatorial map is just a convenient technical device for proving results about topological maps. Accordingly, we shall normally employ the same notation for a topological map and for the corresponding combinatorial structure on it.

We start with necessary definitions concerning oriented maps. By a (combinatorial) *oriented map* we henceforth mean a triple $M = (D;\ R, L)$ where $D = D(M)$ is a non-empty finite set of *arcs*, and R and L are two permutations of D such that L is an involution and the group $\mathrm{Mon}\,(M) = \langle R, L\rangle$ acts transitively on D. The group $\mathrm{Mon}\,(M)$ is called the oriented *monodromy group* of M. The permutation R is called the *rotation* of M. The orbits of the group $\langle R\rangle$ are the *vertices* of M, and elements of an orbit v of $\langle R\rangle$ are the arcs *radiating* (or *emanating*) from v, that is, v is their initial vertex. The cycle of R permuting the arcs emanating from v is the *local rotation* R_v at v. The permutation L is the *arc-reversing involution* of M, and the orbits of $\langle L\rangle$ are the *edges* of M. The orbits of $\langle RL\rangle$ define the face-boundaries of M. The incidence between vertices, edges and faces is given by non-trivial set intersection. The vertices, arcs and the incidence function define the *underlying graph* M, which is always connected due to the transitive action of the monodromy group.

An oriented map can be equivalently described as a pair $M = (G; R)$ where $G = (D; V, I, L)$ is a connected graph and R is a permutation of the arc-set of G cyclically permuting arcs with the same initial vertex, that is, $IR(x) = I(x)$ for every arc x of G.

Combinatorial unoriented maps are built from three involutions acting on a non-empty finite set F of *flags* [18]. A (combinatorial) *unoriented map* is a quadruple $M = (F;\ \lambda, \rho, \tau)$ where λ, ρ and τ are fixed-point free involutory permutations of $F = F(M)$ called the *longitudinal*, the *rotary* and the *transversal involution*, respectively, which satisfy the following conditions:

(i) $\lambda\tau = \tau\lambda$; and

(ii) the group $\langle\lambda, \rho, \tau\rangle$ acts transitively on F.

This group is the unoriented *monodromy group* $\mathrm{Mon}^\natural\,(M)$ of M.

We define the *vertices* of M to be the orbits of the subgroup $\langle\rho, \tau\rangle$, the *edges* of M to be the orbits of $\langle\lambda, \tau\rangle$, and the *face-boundaries* to be the orbits of $\langle\rho, \lambda\rangle$ under the action on F, the incidence being given by non-trivial set intersection. Note that each orbit z of $\langle\lambda, \tau\rangle$ has cardinality 2 or 4 according to whether z is a semiedge or not.

Clearly, the even-word subgroup $\langle\rho\tau, \tau\lambda\rangle$ of $\mathrm{Mon}^\natural\,(M)$ always has index at most two. If the index is two, then M is said to be *orientable*.

With every oriented map $M = (D;\ R, L)$ we associate the *corresponding unoriented map* $M^\natural = (F^\natural;\ \lambda^\natural, \rho^\natural, \tau^\natural)$ by setting $F^\natural = D \times \{1, -1\}$ and defining for a flag $(x, j) \in D \times \{1, -1\}$:

$$\lambda^\natural(x, j) = (L(x), -j), \qquad \rho^\natural(x, j) = (R^j(x), -j), \qquad \text{and} \qquad \tau^\natural(x, j) = (x, -j).$$

Conversely, from an unoriented map $M = (F;\ \lambda, \rho, \tau)$ we can construct a pair of oriented maps $M' = (D;\ R, L)$ and $M'' = (D;\ R^{-1}, L)$ that are the *mirror image* of each other. We take D to be the set F/τ of orbits of τ on F, and for an arc $\{z, \tau(z)\} = [z]$, where $z \in F$, we set $R([z]) = [\rho\tau(z)]$ and $L([z]) = [\lambda\tau(z)]$. (The definitions are readily verified to be correct.) Instead of R we could have taken the rotation $R'([z]) = [\tau\rho(z)]$, but since $R' = R^{-1}$ we get nothing but the mirror image—as expected. Of course, $(M')^\natural \cong (M'')^\natural \cong M$.

The concept of the dual map M^* for a map M can be combinatorially introduced as follows: if $M = (D;\ R, L)$ is an oriented map, we set $M^* = (D;\ RL, L)$. In the unoriented case, for $M = (F;\ \lambda, \rho, \tau)$ we set $M^* = (F;\ \tau, \rho, \lambda)$, that is, the roles of λ and τ are interchanged. The dual concept to valency of a vertex is the *covalency* of a face, the valency of the corresponding vertex of the dual map.

Let $M_1 = (D_1;\ R_1, L_1)$ and $M_2 = (D_2;\ R_2, L_2)$ be two oriented maps. A *homomorphism* $\varphi : M_1 \to M_2$ of oriented maps is a mapping $\varphi : D_1 \to D_2$ such that

$$\varphi R_1 = R_2\varphi \qquad \text{and} \qquad \varphi L_1 = L_2\varphi.$$

Analogously, a *homomorphism* $\varphi : M_1 \to M_2$ of unoriented maps $M_1 = (F_1;\ \lambda_1, \rho_1, \tau_1)$ and $M_2 = (F_2;\ \lambda_2, \rho_2, \tau_2)$ is a mapping $\varphi : F_1 \to F_2$ such that

$$\varphi\lambda_1 = \lambda_2\varphi, \qquad \varphi\rho_1 = \rho_2\varphi \qquad \text{and} \qquad \varphi\tau_1 = \tau_2\varphi.$$

The properties of homomorphisms of both varieties of maps are similar except that homomorphisms of unoriented maps ignore orientation. Every map homomorphism induces an epimorphism of the corresponding monodromy groups. Furthermore, transitive actions of the monodromy groups ensure that every map homomorphism is surjective and that it also induces an epimorphism of the underlying graphs. Topologically speaking, a map homomorphism is a graph preserving branched covering projection of the supporting surfaces with branch points possibly at vertices, face centres or free ends of semiedges. Therefore we can say that a map $\tilde{M}$ *covers* M if there is a homomorphism $\tilde{M} \to M$.

With map homomorphisms we use also isomorphisms and automorphisms. The automorphism group $\mathrm{Aut}\,(M)$ of an oriented map $M = (D;\ R, L)$ consists of all permutations in the full symmetry group $S(D)$ of D which commute with both R and L. Similarly, the automorphism group $\mathrm{Aut}^{\natural}(M)$ of an unoriented map $M = (F;\ \lambda, \rho, \tau)$ is formed by all permutations in the symmetry group $S(F)$ which commute with each of λ, ρ and τ. Hence, in both cases the automorphism group is nothing but the centralizer of the monodromy group in the full symmetry group of the supporting set of the map (cf. [16, Proposition 3.3(i)]). In particular, $\mathrm{Aut}\,(M^*) = \mathrm{Aut}\,(M)$ and $\mathrm{Aut}^{\natural}\,(M^*) = \mathrm{Aut}^{\natural}\,(M)$.

It is well known and easy to see that $|\,\mathrm{Aut}\,(M)| \leq |D(M)|$ for every oriented map M and $|\,\mathrm{Aut}^{\natural}\,(M)| \leq |F(M)|$ for every unoriented map M (see, e.g., [9, 16, 18]). If the equality is attained, then the monodromy group acts regularly on the supporting set, and therefore the map is called *orientably-regular* or *regular*, respectively. Our use of the term *regular map* thus agrees with that of Gardiner *et al*. [9] and Wilson [36], but is not yet standard. For instance, [18] uses the term 'reflexible', and [34] calls such maps 'reflexible symmetrical'. On the other hand, our orientably-regular maps are called 'regular' in Coxeter and Moser [6], 'symmetrical' in [3] and [34], and 'rotary' in Wilson [36].

In a general situation, a map that has the set of face-covalencies P and the set of vertex-valencies Q will be said to have *pattern* $(P;\ Q)$ and *type* (p, q), where p and q are the least common multiple of the elements of P and Q, respectively.

For each homomorphism $\varphi : M_1 \to M_2$ of oriented maps there is the corresponding homomorphism $\varphi^{\natural} : M_1^{\natural} \to M_2^{\natural}$ defined by $\varphi^{\natural}(x, i) = (\varphi(x), i)$. If $M_1 = M_2 = M$, that is, φ is an automorphism, then this definition and the assignment $\varphi \mapsto \varphi^{\natural}$ yield the isomorphic embedding of $\mathrm{Aut}\,(M) \to \mathrm{Aut}^{\natural}\,(M^{\natural})$. This allows us to treat $\mathrm{Aut}\,(M)$ as a subgroup of $\mathrm{Aut}^{\natural}\,(M^{\natural})$ and, consequently, say that every orientable regular map is orientably-regular (but not necessarily vice versa). It is easy to see that the index $|\,\mathrm{Aut}^{\natural}\,(M^{\natural}) : \mathrm{Aut}\,(M)|$ is at most two. If it is two, then the map M is said to be *reflexible*, otherwise it is *chiral*. In the former case, there is an isomorphism ψ of the map $M = (D;\ R, L)$ with its mirror image $(D;\ R^{-1}, L)$ called a *reflection* of M. Clearly, $\psi^{\natural}$ is an automorphism that extends $\mathrm{Aut}\,(M)$ to $\mathrm{Aut}^{\natural}\,(M^{\natural})$. Topologically speaking, oriented map automorphisms preserve the chosen orientation of the supporting surface whereas reflections reverse it.

3. Generic Regular Maps

It is well known that every map can be covered by a finite regular map—oriented or unoriented—see [16, Theorem 6.7, Corollary 6.8] and [18, Theorem 3]. Among the regular maps that cover a given map M there is a map N smallest in the sense of the following universal property. There exists a covering $\pi : N \to M$ such that for every regular map $\tilde{M}$ and a covering $\varphi : \tilde{M} \to M$ there exists a unique covering $\varphi' : \tilde{M} \to N$ such that the diagram

$$\begin{tikzcd} \tilde{M} \arrow[rr, "\varphi'"] \arrow[dr, "\varphi"'] & & N \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes, that is to say, $\varphi = \pi\varphi'$. If M is an oriented map (in which case the covering projections are represented by oriented map homomorphisms), then the map N is denoted by $M^{\#}$ and is called the *generic orientably-regular map* over M; in the unoriented case, N is denoted by M^{+} and called the *generic regular map* over M.

The above definitions do not provide any hint of how the generic map can be constructed from a given oriented or unoriented map. It is a consequence of the theory of Schreier representations of maps developed in [26, Section 3] that to construct the generic map $M^{\#} = (D^{\#}; R^{\#}, L^{\#})$ for an oriented map $M = (D; R, L)$ it is sufficient to set $D^{\#} = \mathrm{Mon}\,(M)$, $R^{\#}(x) = Rx$, and $L^{\#}(x) = Lx$ for any $x \in D^{\#}$. Observe that the automorphisms of $M^{\#}$ are just the right translations of $D^{\#} = \mathrm{Mon}\,(M)$ by the elements of $\mathrm{Mon}\,(M)$, and so $M^{\#}$ is indeed an orientably-regular map. Similarly, if $M = (F; \lambda, \rho, \tau)$ is an unoriented map, then the generic regular map $M^{+} = (F^{+}; \lambda^{+}, \rho^{+}, \tau^{+})$ over M can be constructed by setting $F^{+} = \mathrm{Mon}^{\natural}\,(M)$, $\lambda^{+}(x) = \lambda x$, $\rho^{+}(x) = \rho x$, and $\tau^{+}(x) = \tau x$, for any $x \in F^{+}$. Again, the map automorphisms are given by the right translations of F^{+} by the elements of $\mathrm{Mon}^{\natural}\,(M) = F^{+}$.

It is obvious that if M is orientable, then so is M^{+}. Moreover, the above diagram with $N = M^{\#}$ and $\tilde{M} = M^{+}$ implies that M^{+}, as a topological map, covers $M^{\#}$.

EXAMPLE. Recall from the Introduction that a trivalent *Hurwitz map* is an orientably-regular map of type $(7, 3)$, and a *Hurwitz group* is a group isomorphic to its automorphism group. Below we give two examples of Hurwitz maps as generic maps over spherical quotients.

(1) Let M_1 be the spherical map represented in Figure 1. Clearly, M_1 is not reflexible. Nevertheless, the generic map $M_1^{\#}$ is a reflexible Hurwitz map of genus 3 whence $M_1^{+} \cong M_1^{\#}$ (as topological maps). The dual of this map is known as Klein's triangulation of genus 3. The automorphism group of this map is the smallest Hurwitz group and is isomorphic to $\mathrm{PSL}(2, 7)$.

(2) Let M_2 be the spherical map shown in Figure 2. Again, M_2 is not reflexible. It can be shown that the generic orientably-regular map $M^{\#}$ is a Hurwitz map of genus 17 whereas the generic regular map M^{+} has genus 129. Thus $M_2^{\#}$ is not isomorphic to M_2^{+} in this case and, of course, is chiral.

We conclude this section by an easy but important lemma which will be crucial for the proof of our main result.

PROPOSITION 3.1. *For any map M, the type of the generic maps $M^{\#}$ and M^{+} is the same as the type of M.*

PROOF. We prove the lemma only in the case of oriented maps; the unoriented case is analogous. Let the map $M = (D; R, L)$ and its generic map $M^{\#}$ have types (p, q) and (p', q'), respectively. The vertices in $M^{\#}$ have valency q' which, by the definition of $M^{\#}$ equals the length of an arbitrary cycle of $R^{\#}$. Since the cycles of $R^{\#}$ have the form $(x, Rx, R^2x, \dots)$ for some $x \in D^{\#}$, q' equals the order of R, that is, the least common multiple of the lengths

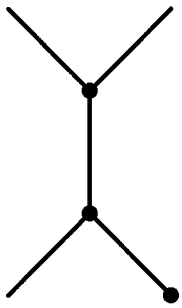


FIGURE 1. An example of an irreflexible map M with $M^+ \simeq M^\#$.

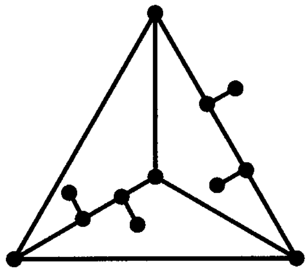


FIGURE 2. An example of a map M with $M^+ \not\simeq M^\#$.

of cycles in R. The length of any cycle of R coincides with the valency of the corresponding vertex and so q' equals the least common multiple of vertex-valencies in M, that is, q. The proof that $p' = p$ is similar. $\square$

4. PROOF OF THE MAIN RESULT

We start with a series of preparatory lemmas. The *n-semistar Ss_n* is the graph consisting of a single vertex and n incident semiedges. Up to isomorphism, it has only one embedding and this embedding forms a regular and reflexible map.

LEMMA 4.1. *The only regular or orientably-regular maps containing semiedges are the n-semistars Ss_n embedded into the 2-sphere.*

LEMMA 4.2. *For each pair of positive integers c and d with $c > d$ there exists a rooted tree $S_{c,d}$ with c arcs (semiedges allowed) in which each vertex is of valency d or 1 and the root is of valency 1.*

PROOF. Since $c > d$, there exist integers $k \geq 1$ and $r \geq 0$, with $r < d$, such that $c = kd+r$. Let P be a path on k vertices with end-vertices u and v.

If $k = 1$, then $u = v$, and $r \geq 1$ because $c \neq d$. Form $S_{c,d}$ by attaching r pendant links and $d - r$ semiedges to v (see Figure 3). Clearly, the resulting tree has r vertices of valency 1 and a single vertex of valency d; the number of arcs is therefore $2r + (d - r) = c$.

If $k \geq 2$, then $u \neq v$. Now we form $S_{c,d}$ by attaching $d-1$ semiedges to u, $d-2$ semiedges to each inner vertex of P, and r pendant links and $d - r - 1$ semiedges to v (see Figure 4). The resulting tree has every vertex of valency d or 1, and the total number of its arcs is the valency sum, i.e., $d + (k - 2)d + (1 + r + d - r - 1) + r = kd + r = c$. $\square$

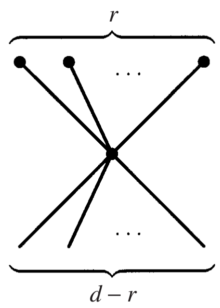


FIGURE 3.

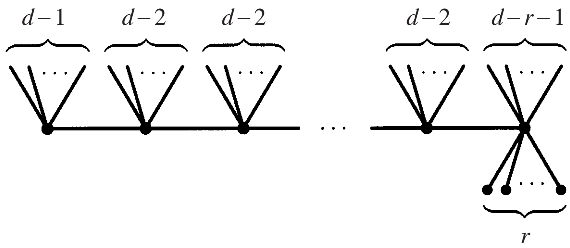


FIGURE 4.

When embedded into the 2-sphere, the tree $S_{c,d}$ gives rise to a map of type (c, d). The resulting tree-like maps are essential for a simple proof of the following result answering Grünbaum's problem of the existence of regular maps of arbitrary type [12].

THEOREM 4.3. *For every pair of integers $p \geq 2$ and $q \geq 2$ there exists an orientable map of type (p, q) not containing semiedges. Moreover, the map can be chosen to be regular and reflexible.*

PROOF. As shown in Lemma 4.2, for every pair of integers c and d with $c > d$ there exists a spherical map M of type (c, d). By Proposition 3.1, the generic map M^+ is regular and has the same type, proving the theorem in the case when $c > d$. Taking into account that the dual map to M^+ has type (d, c) and is also regular, it remains to construct a regular map of type (c, c) for each $c \geq 2$. Consider a map obtained from the embedding of a cycle of length 2 in the sphere by adding $c - 2$ semiedges to each vertex in such a way that both faces contain exactly $c - 2$ of them on the boundary (see Figure 5). Clearly, the resulting map has type (c, c). By Proposition 3.1, the corresponding generic map is regular and has the same type (c, c). As it has at least two vertices, Lemma 4.1 implies that it contains no semiedges. □

Let $T_{p,q}$ be the infinite regular tessellation of the hyperbolic plane by p-gons, q meeting at a vertex (for the relevant notions see [5, 6]). Thus $1/p + 1/q < 1/2$. Let $B_{p,q}$ be the underlying infinite graph, the 1-skeleton of $T_{p,q}$. In $T_{p,q}$ we may define the distance between two polygons P and Q to be $\mathrm{dist}(P, Q) = \min |J \cap B_{p,q}|$ where the minimum is taken over all simple arcs J in the hyperbolic plane starting in the interior of P and ending in the interior of Q.

For every integer $k \geq 0$ we construct a closed disk D_k in the hyperbolic plane as follows. Fix an arbitrary closed p-gon D_0 of the tessellation $T_{p,q}$, the *fundamental polygon*. If $k \geq 1$, let

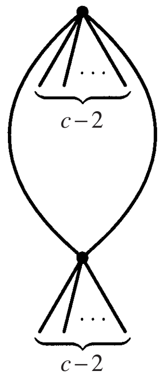


FIGURE 5.

D_k be the union of D_{k-1} and all the closed p-gons of $T_{p,q}$ which have a point in common with the boundary of D_{k-1}. In other words, D_k consists of all closed polygons of the tessellation whose distance from the fundamental polygon D_0 does not exceed k. The *boundary polygons* of D_k are those contained in D_k but not in D_{k-1}.

Let us now look at the boundary of the disk $D_k \subseteq T_{p,q}$. Our analysis splits into two cases according to whether $q = 3$ or $q \geq 4$.

First, if $q \geq 4$, there are two kinds of boundary polygons in D_k:

(a) a boundary *polygon over a vertex* which intersects the boundary of D_{k-1} in a vertex;
(b) a boundary *polygon over an edge* which intersects the boundary of D_{k-1} in an edge.

If $q = 3$, the boundary polygons in D_k are two other kinds:

(a′) a boundary *polygon over a path of length* 2 which intersects the boundary of D_{k-1} in a path of length 2;
(b′) a boundary *polygon over an edge* which intersects the boundary of D_{k-1} in an edge.

The structure of the boundary of D_k in $T_{p,q}$ is now described by the following two straightforward lemmas.

LEMMA 4.4. *Let D_k be the closed disk in the regular hyperbolic tessellation $T_{p,q}$, and assume that $q \geq 4$ and $k \geq 1$. Then all the vertices on the boundary of D_k are either* 2*-valent or* 3*-valent. Moreover, any two vertices of valency* 3 *on the boundary of D_k are separated by a sequence of $p-3$ or $p-4$ vertices of valency* 2*, according to whether they are contained in the boundary polygon over a vertex or over an edge.*

LEMMA 4.5. *Let D_k be the closed disk in the regular hyperbolic tessellation $T_{p,3}$ (that is, $q = 3$), and assume that $k \geq 1$. Then all the vertices on the boundary of D_k are either* 2*-valent or* 3*-valent. Any two vertices of valency* 3 *on the boundary of D_k are separated by a sequence of $p-4$ or $p-5$ vertices of valency* 2*, according to whether they are contained in the boundary polygon over an edge or over a path of length* 2*. Moreover, two polygons over a path of length* 2 *are separated by a sequence of $p-5$ polygons over an edge.*

Finally, we need the following topological lemma.

LEMMA 4.6. *Let $\psi : \tilde{S} \to S$ be a branched covering projection of surfaces. Let $U \subseteq S$ be an open disk which contains no branching point of ψ, and let $\tilde{U}$ be any connected component of $\psi^{-1}(U)$. Then the restriction $\psi|\tilde{U} : \tilde{U} \to U$ is a homeomorphism.*

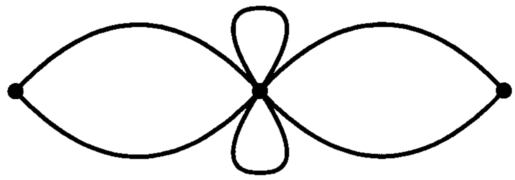


FIGURE 6.

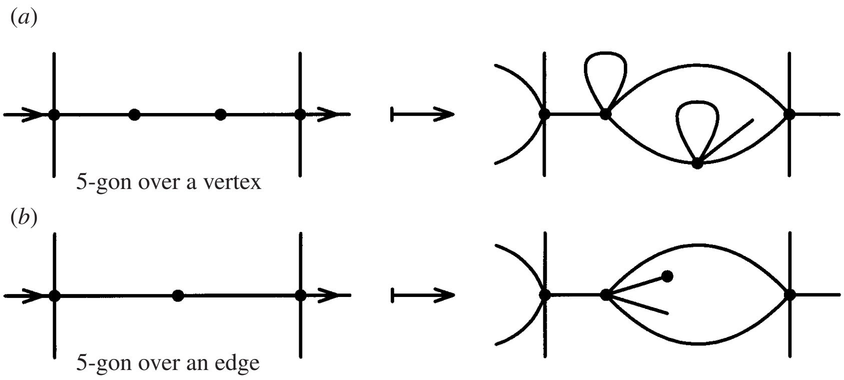


FIGURE 7. (*a*) Type (5,5), the modification of a path of length 3 on the Equator. (*b*) Type (5,5), the modification of a path of length 2 on the Equator.

Now we are in position to prove the main result of this paper.

THEOREM 4.7. *For every pair of integers* $p \geq 3$ *and* $q \geq 3$ *such that* $1/p + 1/q \leq 1/2$ *and for every integer* $w \geq 2$ *there exists a regular map* M *of type* (p, q) *with* $w(M) \geq w$. *Moreover,* M *can be required to be reflexible.*

PROOF. Let us first deal with the parabolic case, that is, $1/p + 1/q = 1/2$. Clearly, (p, q) is one of the pairs $(3, 6)$, $(4, 4)$ and $(6, 3)$, and the required maps must be toroidal.

Let us consider a rectangular grid Q with each side of length w. By identifying the opposite sides of Q we obtain a map of type $(4, 4)$ with planar width w. The map is easily seen to be regular and reflexible (see [6, Section 8.3]).

In order to obtain the maps of type $(3, 6)$ take, for each w, the map of type $(4, 4)$ already constructed in the previous step and add the 'main' diagonal into each face. Again, the map has planar width w and is both regular and reflexible. Finally, the dual of this map is of type $(6, 3)$ and has all the desired properties (again, see [6, Section 8.4]).

We proceed to the heart of the proof, the hyperbolic case $1/p + 1/q < 1/2$. We fix w and assume that $p \geq q \geq 3$. The latter assumption is possible in view of surface duality.

We first describe the general strategy and then finish the proof by applying our strategy to several particular cases we have to distinguish. The general part of the proof starts by taking two differently oriented copies D' and D'' of the disk D_w and identifying the boundaries isomorphically. The isomorphism need not be the identity—the choice of this isomorphism will depend upon the particular type (p, q) considered. In each case, the result is a spherical map M_0 with a distinguished cycle, the *Equator*. The centre of the fundamental polygon D_0 in one of D' and D'' is selected as the *North Pole* of M_0.

In M_0, the valencies and covalencies are correct everywhere except the vertices and faces incident with the Equator. The next step is therefore to perform certain modifications near to

(*a*)

(*b*)

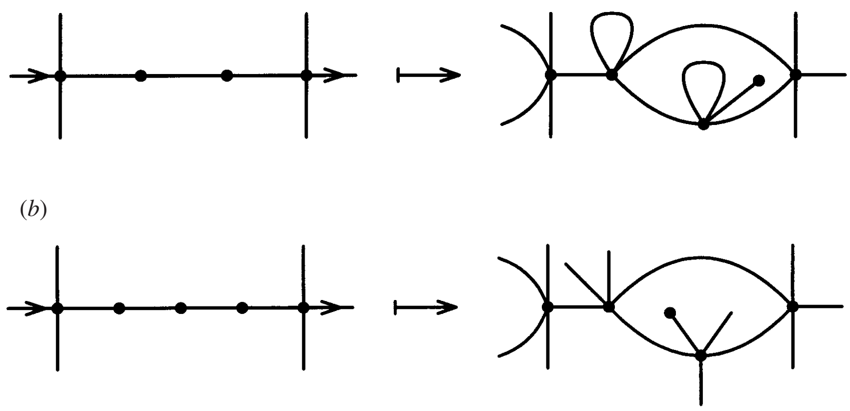


FIGURE 8. (*a*) Type (6,5), the modification of a path of length 3 on the Equator. (*b*) Type (6,5), the modification of a path of length 4 on the Equator.

the Equator in order to amend the valencies and covalencies and thereby create a map M_1 of type (p, q) with the property that $D_{w-1} \subseteq M_1$. These modifications vary according to the particular type (p, q).

We show that the generic map $(M_1)^{\#}$ or $(M_1)^{+}$ is the required regular map M with $w(M) \geq w$. Suppose this is not the case. Then there exists a non-contractible simple closed curve C with $|C \cap G| < w$ where G is the underlying graph of M. Since M is orientably-regular or regular, it is face-transitive, and so we may assume that C contains a preimage $\tilde{n}$ of the North Pole. Consider the branched covering projection $\pi : M \to M_1$. Let B be the copy of D_{w-1} in M_1 containing the North Pole, $U = \mathrm{Int}(B)$, and let $\tilde{U}$ be the component of $\pi^{-1}(U)$ containing $\tilde{n}$. Since $U \subseteq M_1$ is an open disk containing no branching points of π, Lemma 4.6 yields that $\tilde{U}$ is homeomorphic to U. Hence $\tilde{U}$ is also an open disk. But $\tilde{n} \in C$ and $|C \cap G| \leq w - 1$, so $C \subseteq \tilde{U}$. Thus C is contractible, contrary to our assumption. This proves that $w(M) \geq w$.

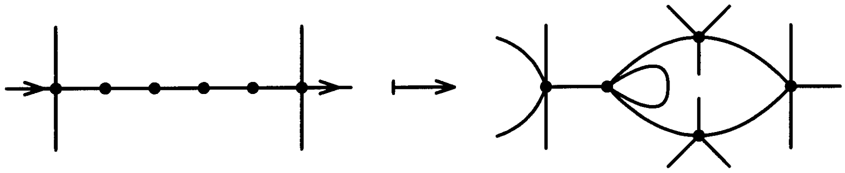
To finish the proof we describe in detail how the map M_1 of type (p, q) can be constructed. We distinguish eight cases.

Type (p, q), $p \geq q \geq 7$. In this general case, the map M_0 is constructed by glueing the boundaries of D' and D'' identically. It follows that every vertex on the Equator of M_0 has valency 2 or 4. By replacing every edge on the Equator by two parallel edges we obtain a map with pattern $(p, 2; q, 4, 6)$. All the digons of the latter map and all the vertices of valency 4 or 6 are located on the Equator. The digons form a chain $F_1, F_2, \ldots, F_k$ in which, for $1 \leq i \leq k-1$, F_i and F_{i+1} share a vertex u_i of valency $d = 4$ or 6. The idea is to insert into each F_i a planar map T having the following properties:

- the outer face of T has size $p - 2$;
- the sizes of all other faces (if any) divide p;
- exactly one vertex u on the boundary of the outer face, the *root* of T, has valency $q - d$;
- the valencies of all other vertices in T (if any) divide q.

By placing T in the interior of F_i and identifying u with u_i the digon will transform into a face of size p and the vertex u_i into a vertex of valency q. The desired map M_1 of type (p, q) is obtained by performing the above procedure for every vertex u_i.

(*a*)



(*b*)

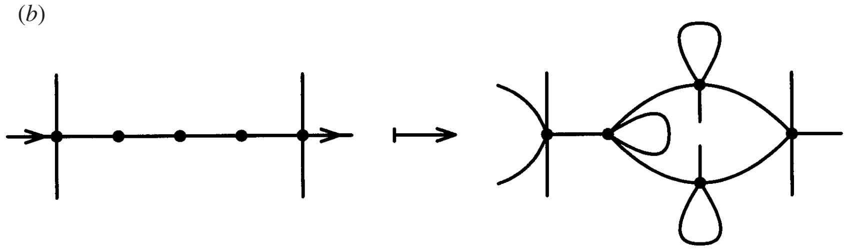


FIGURE 9. (*a*) Type (7,5), the modification of a path of length 5 on the Equator. (*b*) Type (7,5), the modification of a path of length 4 on the Equator.

We have to show that the planar map T satisfying the above properties does exist for all $p \geq q \geq 7$ and $d \in \{4, 6\}$. There are two subcases according to whether $d = 4$ or $d = 6$.

Subcase 1: $d = 6$. Let us consider the tree $T_{p+4,q}$ constructed in Lemma 4.2. If $p \geq 2q - 4$, the tree has at least two vertices of valency q. Since $q \geq 7$, there are $q - 1 \geq 6$ semiedges attached to the vertex u. By removing six of them we produce a tree which in turn determines a planar map satisfying all the required properties. Thus we have to deal with the cases where $7 \leq q \leq p \leq 2q - 5$. First of all, the 1-vertex tree having r pendant links and $q - 6 - r$ semiedges, r ranging from 6 to $q - 6$, covers the situations where $q \leq p \leq 2q - 10$. To complete the proof of the subcase we have to consider the situations with $2q - 9 \leq p \leq 2q - 5$. If $p = 2q - 4 - s$ and $1 \leq s \leq 3$, we take a planar map with two vertices joined by a single link in which the root vertex is incident with $q - 7$ semiedges and with the link, while the second vertex is incident with $q - 2s - 1$ semiedges, s loops and with the link. If $p = 2q - 8$ or $2q - 9$, we take a planar map with two vertices joined by a single link such that there are $q - 9$ semiedges and one loop incident with the root and there are three, respectively four, loops and $q - 7$, respectively $q - 9$, semiedges incident with the other vertex. The only situation not covered by the above constructions is $p = q = 8$. This is solved by the planar map represented in Figure 6.

Subcase 2: $d = 4$. If $p \geq 2q - 2$, then the tree $T_{p+2,q}$ constructed in Lemma 4.2 contains $q - 1$ semiedges incident with the vertex u. To obtain the required planar map it suffices to remove four of them. Further, the 1-vertex tree $T_{p-2,q-4}$ covers the cases where $q \leq p \leq 2q - 6$. Thus we are left with the situation where $p = 2q - 2 - s$ and s ranges from 1 to 3. Here we take a planar map with two vertices joined by a single link such that the root is incident with $q - 5$ semiedges and with the link, whereas the other vertex is incident with s loops, $q - 2s - 1$ semiedges and with the link.

Type $(p, 4)$, $p \geq 4$. Again, the map M_0 is constructed by identifying the boundaries of the disks D' and D'' according to the identity isomorphism. This time, however, no further modifications are made; in other words, $M_1 = M_0$. It follows that every vertex on the Equator

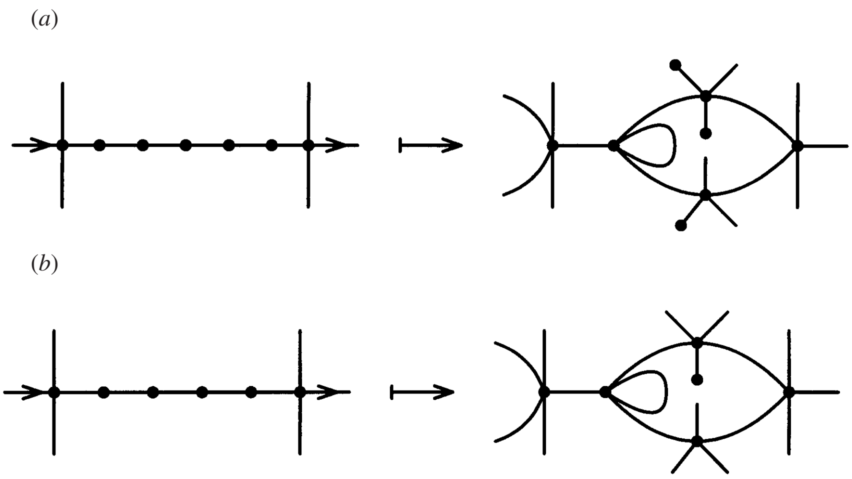


FIGURE 10. (a) Type (8,5), the modification of a path of length 6 on the Equator. (b) Type (8,5), the modification of a path of length 5 on the Equator.

of M_1 is of valency 2 or 4. Hence M_1 is of type $(p, 4)$, as required.

Type $(p, 6)$, $p \geq 6$. Before glueing the boundaries of D' and D'', the disk D' is rotated one step counterclockwise. No further modifications on the Equator are made afterwards, i.e., $M_1 = M_0$. By Lemma 4.4, every 3-valent vertex of one of the disks is identified with a 2-valent vertex of the other disk. Therefore M_1 has pattern $(p; 2, 3, 6)$ and type $(p, 6)$.

Type $(p, 5)$, $p \geq 9$. We glue the boundaries of D' and D'' identically. Thus, in this case, the map M_0 has pattern $(p; 2, 4, 5)$. The Equator of M_0 is formed by paths of length at least 3 where the internal vertices are of valency 2 and the end-vertices are of valency 4. Let $P = u_0, e_1, u_1, e_2, \dots, e_k, u_k$ be one of these paths. For $2 \leq i \leq k$ we replace each edge e_i by a digon F_i. Since $p \geq 9$, Lemma 4.2 guarantees that we can take the rooted trees $T = S_{p-2,5}$ and $T' = S_{p-3,5}$. Furthermore, in T attach a semiedge to the root to obtain a tree T''. Now we place T'' into F_2 and identify the root with u_1. Similarly, for $i \geq 3$ we place T' into F_i and identify the root with u_{i-1}. By repeating this procedure for each of the paths P on the Equator we obtain the required map M_1 of type $(p, 5)$.

Types $(5, 5)$, $(6, 5)$, $(7, 5)$, *and* $(8, 5)$. In each of the singular types $(p, 5)$, $p \in \{5, 6, 7, 8\}$, the map M_0 is constructed by glueing the disks D' and D'' identically. As above, the Equator is a union of paths with end-vertices of valency 4 and internal vertices of valency 2 in M_0. Let P be one of these paths. Then the length of P is either $p - 2$ or $p - 3$ depending on whether P is a part of a p-gon over a vertex or over an edge. The necessary modifications leading to the map M_1 are shown in Figures 7, 8, 9 and 10, respectively. It is easy to check that all the resulting maps have the required type.

Type $(p, 3)$, $p \geq 9$. Before glueing the boundaries of D' and D'', the disk D' is rotated one step counterclockwise. It follows that every vertex on the Equator of the map M_0 has valency 2 or 3, and between any two 3-valent vertices there are $p - 5$ or $p - 6$ vertices of valency 2, depending on whether they are contained in a p-gon over an edge or in a p-gon over a path of length 2, respectively. We modify the Equator in two steps. First, we replace

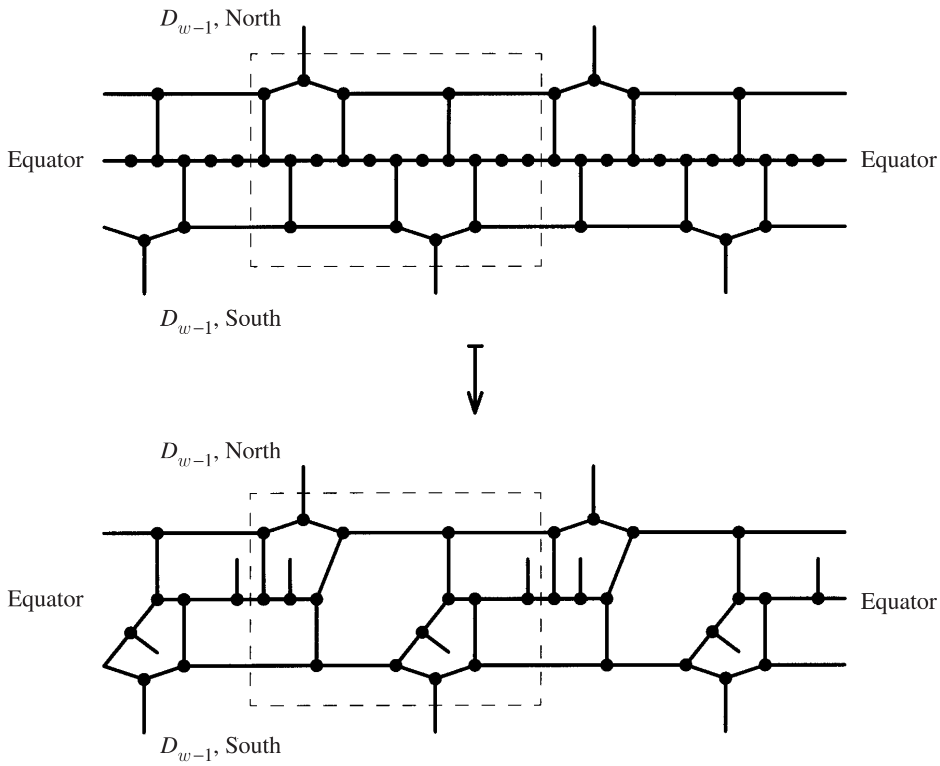


FIGURE 11. Type (7,3).

the paths of length $p-4$ or $p-5$ joining the consecutive 3-valent vertices on the Equator by paths of length 3, thereby reducing the size of the boundary polygons in D' and D''. Let P be one of these new paths. Clearly, it constitutes the intersection of a polygon $Q' \subseteq D'$ with the corresponding polygon $Q'' \subseteq D''$. As the second step we take the rooted tree $T = S_{c,3}$, where $c = p-7$ or $c = p-8$ according to whether Q' (and also Q'') is a polygon over an edge or a polygon over a path of length 2; the existence of the tree is guaranteed by Lemma 4.2 and the assumption that $p \geq 9$. Let u' and u'' be the two 2-valent vertices on P. We place a copy of T into the interior of Q' and another copy into the interior of Q'' and identify the root of each copy with u' and u'', respectively. This turns both polygons Q' and Q'' back into p-gons. The desired map M_1 of type $(p, 3)$ is now obtained by repeating the procedure for each pair of the corresponding boundary polygons.

Type $(7, 3)$. To construct the map M_0 we first rotate the disk D' five steps counterclockwise and then identify the boundary of D' with the boundary of D''. Lemma 4.5 implies that in D_w any two boundary 7-gons over a path of length 2 are separated by two 7-gons over an edge. A part of the map M_0 near the Equator including boundary polygons of D' and D'' is depicted in the upper part of Figure 11. It follows that to construct M_1 it suffices to indicate how the marked part of M_0, which cyclically repeats along the Equator, is to be modified. The corresponding modification is shown in the bottom of Figure 11. It is easy to check that M_1 thus constructed has indeed type $(7, 3)$.

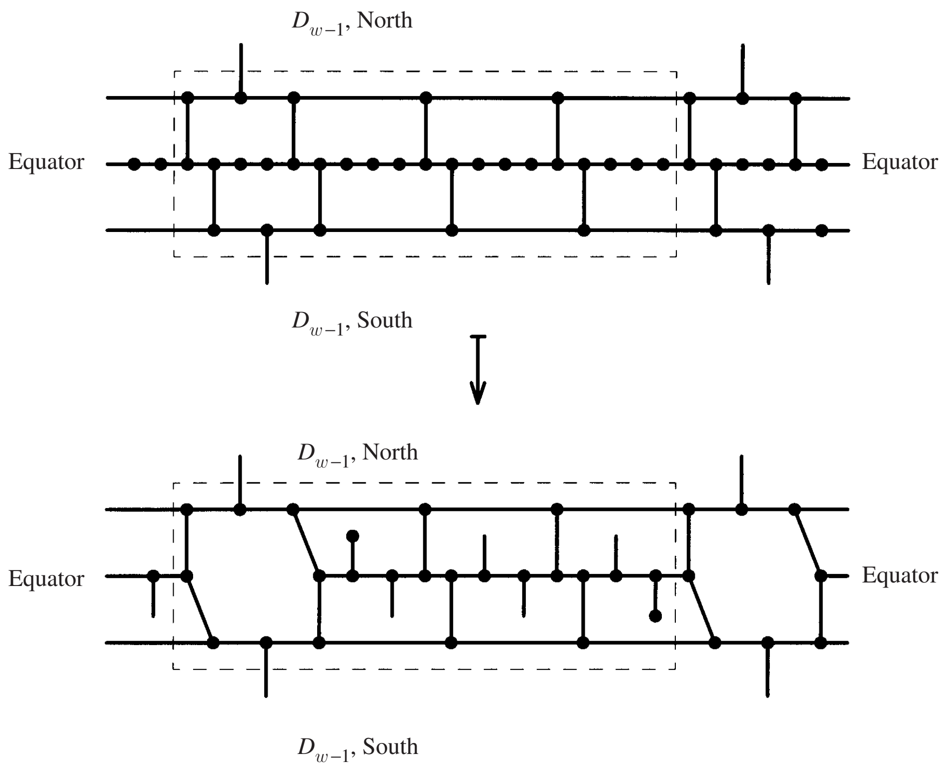


FIGURE 12. Type (8,3).

Type $(8, 3)$. Before identifying the boundary of D' with the boundary of D'', the disk D' is rotated one step counterclockwise. By Lemma 4.5 any two boundary 8-gons of D_w that are attached to a path of length 2 are separated by three 8-gons over an edge. As in the previous case, the rest can easily be completed by employing Figure 12.

In the above cases we have described the construction of a map M_1 of any hyperbolic type (p, q). The proof is complete. □

Note that given integers p and q, the right choice of planar width in our main result may lead to some additional properties of the constructed maps. By employing this idea we may, for example, strengthen classical work on the existence of regular graphs with large girth [2, 22, 33] and guarantee the precise value of girth. A detailed proof together with some other applications of the Main Theorem can be found in our paper [27].

THEOREM 4.8. *For every pair of integers $k \geq 3$ and $g \geq 3$ there exists a k-valent arc-transitive graph of valency k and girth g; for $k = 3$ the graph can be required to be 2-arc-transitive. Moreover, if $1/k + 1/g \leq 1/2$, or if $(g, k) = (3, 4)$, there are infinitely many such graphs.*

ACKNOWLEDGEMENTS

Research was partially supported by the VEGA grants 1/7152/20(^{2}) and 1/6132/99(^{1,2}).

REFERENCES

1. D. Archdeacon, P. Gvozdjak and J. Širáň, Constructing and forbidding automorphisms in lifted maps, *Math. Slovaca*, **47** (1997), 113–129.
2. N. L. Biggs, Girth and residual finiteness, *Combinatorica*, **8** (1988), 307–312.
3. N. L. Biggs and A. T. White, *Permutation Groups and Combinatorial Structures*, Cambridge University Press, Cambridge, 1979.
4. W. Burnside, *Theory of Groups of Finite Order*, 2nd edn, Dover, New York, 1955, reprinted.
5. H. S. M. Coxeter, *Non-Euclidean Geometry*, University of Toronto Press, Toronto, 1957.
6. H. S. M. Coxeter and W. O. J. Moser, *Generators and Relations for Discrete Groups*, 4th edn, Springer-Verlag, Berlin, 1984.
7. A. L. Edmonds, J. H. Ewing and R. S. Kulkarni, Regular tesselations of surfaces and $(p, q, 2)$-triangle groups, *Ann. Math.*, **116** (1982), 113–132.
8. R. H. Fox, On Fenchel's conjecture about F-groups, *Mat. Tidskrift B*, (1952), 61–65.
9. A. Gardiner, R. Nedela, J. Širáň and M. Škoviera, Characterization of graphs which underlie regular maps on closed surfaces, *J. London Math. Soc.*, **59** (1999), 100–108.
10. A. Gray and S. Wilson, A more elementary proof of Grünbaum's conjecture, *Congr. Numer.*, **72** (1990), 25–32.
11. J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley, New York, 1987.
12. B. Grünbaum, Regularity of graphs, complexes and designs, Problèmes combinatoires et théorie des graphes, *Colloq. int.* C.N.R.S. 260, Paris, (1978), 191–197.
13. J. Hempel, Residual finiteness of surface groups, *Proc. Am. Math. Soc.*, **32** (1972), 323.
14. S. Jendroľ, R. Nedela and M. Škoviera, Constructing regular maps and graphs from planar quotients, *Math. Slovaca*, **47** (1997), 155–170.
15. G. A. Jones, Maps on surfaces and Galois groups, *Math. Slovaca*, **47** (1997), 1–33.
16. G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. London Math. Soc. (3)*, **37** (1978), 273–307.
17. G. A. Jones and D. Singerman, Belyĭ functions, hypermaps and Galois groups, *Bull. London Math. Soc.*, **28** (1996), 561–590.
18. G. A. Jones and J. S. Thornton, Operations on maps, and outer automorphisms, *J. Comb. Theory, Ser. B*, **35** (1983), 93–103.
19. I. Kaplansky, *Commutative Rings*, University of Chicago Press, Chicago, 1974.
20. F. Klein, Über die Transform siebenter Ordnung der elliptischen Funktionen, *Math. Ann.*, **14** (1878-79), 428–471.
21. M. L. Lopez, Residual finiteness of surface groups via tesselations, *Discrete Comput. Geom.*, **11** (1994), 201–211.
22. A. Lubotzky, R. Phillips and P. Sarnak, Ramanujan graphs, *Combinatorica*, **8** (1988), 261–277.
23. A. M. MacBeath, *Generators of Linear Fractional Groups*, Proc. Sympos. Pure Math., Vol. XII, American Mathematical Society, Providence, RI, 1969, pp. 14–32.
24. G. A. Miller, Groups defined by the orders of two generators and the order of their product, *Am. J. Math.*, **24** (1902), 96–106.
25. B. Mohar, Face-width of embedded graphs, *Math. Slovaca*, **47** (1997), 35–63.
26. R. Nedela and M. Škoviera, Exponents of orientable maps, *Proc. London Math. Soc. (3)*, **75** (1997), 1–31.
27. R. Nedela and M. Škoviera, Semiregular maps and regular graphs with specified girth, submitted.
28. N. Robertson and P. D. Seymour, Graph minors. VII. Disjoint paths on a surface, *J. Comb. Theory, Ser. B*, **45** (1988), 212–254.
29. N. Robertson and R. P. Vitray, *Representativity of surface embeddings*, Paths, Flows, and VLSI-Layout, B. Korte, L. Lovász, H. J. Prömel and A. Schrijver (eds), Springer-Verlag, Berlin, 1990, pp. 293–328.

30. P. G. Scott, Subgroups of surface groups are almost geometric, *J. London Math. Soc. (2)*, **17** (1978), 555–565.
31. C. Thomassen, Embeddings of graphs with no short noncontractible cycles, *J. Comb. Theory, Ser. B*, **48** (1990), 155–177.
32. A. Vince, Regular combinatorial maps, *J. Comb. Theory, Ser. B*, **35** (1983), 256–277.
33. A. Weiss, Girth of bipartite sextet graphs, *Combinatorica*, **4** (1984), 241–245.
34. A. T. White, *Graphs, Groups and Surfaces*, 2nd edn, North-Holland, Amsterdam, 1984.
35. S. Wilson, A construction from an existence proof, *Congr. Numer.*, **50** (1985), 25–30.
36. S. Wilson, Cantankerous maps and rotary embeddings of K_n, *J. Comb. Theory, Ser. B*, **47** (1989), 262–273.
37. S. Wilson, Applications and refinements of Vince's construction, *Geom. Ded.*, **48** (1993), 231–242.

Received 21 January 1999 and accepted 24 January 2000. Published electronically 1 December 2000

ROMAN NEDELA
Department of Mathematics,
Faculty of Finance, Matej Bel University,
974 00 Banská Bystrica,
Slovakia
AND

MARTIN ŠKOVIERA
Department of Computer Science,
Faculty of Mathematics and Physics, Comenius University,
842 48 Bratislava,
Slovakia