Center-of-gravity fuzzy systems based on normal fuzzy implications✩

Xue-hai Yuanª,*, Zeng-liang Liuª, E. Stanley Leeª∗

ª Faculty of Electronic Information and Electrical Engineering, Dalian University of Technology, Dalian 116024, China
ª Instituto de Información Operativa, National Defence University of PLA, Beijing 100091, China
ª Department of Industrial and Manufacturing Systems Engineering, Kansas State University, Manhattan, KS 66506, USA

A R T I C L E  I N F O

Article history:
Received 25 February 2011
Accepted 23 March 2011

Keywords:
Fuzzy systems
Non-singleton fuzzifier
Adaptive universe
Normal implication
Center-of-gravity defuzzifier
Universal approximation

A B S T R A C T

In this paper, the construction and approximation problem of a single input and single output (SISO) fuzzy system with normal implication and center-of-gravity defuzzifier is discussed. First, the method of a non-singleton fuzzifier for the input variable and the concept of an adaptive universe for the output fuzzy set are proposed. Then, by using this method and this concept of an adaptive universe, SISO fuzzy systems based on center-of-gravity defuzzifier and normal implications such as the Kleene–Dienes implication or the Lukasiewicz implication are constructed. The constructed fuzzy systems have the general form

\[ S(x) = A^*_1(x) f(x_1) + A^* (x) + 1 f(x + 1), \]

with \( A^*_1(x) + A^* (x) + 1 = 1 \), and, furthermore, they are universal approximators. The sufficient conditions for the proposed fuzzy systems to be universal approximators are also obtained. To illustrate the universal property, an example is also given.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

Since Mamdani’s paper [1] on fuzzy logic controllers, various theories and applications have been introduced by different investigators [2–19] in this fuzzy control field. In most of these theories and applications, the construction of a fuzzy system and its capability of approximation are the main research objectives. One of the most important problems in fuzzy control is how to obtain a fuzzy system that approximates a desired control function up to a given accuracy.

It is well known that a fuzzy system is composed of four principal components: fuzzifier, fuzzy rule base, fuzzy inference engine, and defuzzifier [2]. In the construction of a fuzzy system, the most commonly used fuzzifier is the singleton fuzzifier and the fuzzy inference method, which depends on fuzzy implications, is the CRI method [20,21]. There are three defuzzifiers: the center-average defuzzifier, maximum defuzzifier and center-of-gravity defuzzifier [2]. Three types of fuzzy systems have been proposed, they are Mamdani, Takagi–Sugeno and Boolean fuzzy systems. In Mamdani and Takagi–Sugeno fuzzy systems, the center-average defuzzifier is generally used and implication operators are chosen as the t-norm operators, such as min(\(\wedge\)) (Mamdani implication) and product (\(\times\)) (Larsen implication) operators [3–13]. When normal fuzzy implications such as the Kleene–Dienes implication or the Lukasiewicz implication are used, the constructed fuzzy systems are not universal approximators [14,18]. In the Boolean fuzzy systems, the maximum defuzzifier is generally used and the implication operators are chosen as R-implication, S-implication or QL-implication [14–17].

To the best of our knowledge, there are no papers to study the constructions and approximation capability of fuzzy systems with a center-of-gravity defuzzifier. A fuzzy system with center-of-gravity defuzzifier is more complicated and needs to deal with complex integrals. However, in [18], Li discussed the probability significance of a fuzzy system with a...
center-of-gravity defuzzifier and showed that using the center-of-gravity defuzzifier in a fuzzy system is a reasonable and an optimal method in the sense of mean squares. Therefore, how to obtain a system with a center-of-gravity defuzzifier is an interesting and important problem.

In this paper, in order to construct a fuzzy system based on the center-of-gravity defuzzifier and normal fuzzy implication, the non-singleton fuzzifier for the input variable and the concept of adaptive universe for output are used. By using this non-singleton approach, fuzzy systems based on the Kleene–Dienes implication or the Łukasiewicz implication with the center-of-gravity defuzzifier are constructed. It should be noted that if a singleton fuzzifier is used in the above system, the resulting system would not be a universal approximator.

After this introduction, some preliminaries on fuzzy systems are given. Then in Section 3, the construction of fuzzy systems with a center-of-gravity defuzzifier is discussed. The universal approximations of fuzzy systems are studied in Section 4. One example is also given in Section 4. Finally, some conclusions are given in Section 5.

2. Preliminary

Consider the SISO fuzzy system. Let X and Y be the universe of input variable x and output variable y, respectively. In the construction of a fuzzy system, we need four steps as follows [2]:

Step 1. Singleton fuzzifier. Generally, the singleton fuzzifier of the input variable x is defined as

\[ A^i(x') = \begin{cases} 1, & x' = x \\ 0, & x' \neq x. \end{cases} \]

Step 2. Establishing the fuzzy inference rules. If x is in \( A_i \), then y is in \( B_i \) (i = 1, 2, . . . , n).

Step 3. Constructing fuzzy inference engine. We first select the fuzzy implication \( \theta \), then let \( R_i(x, y) = \theta(A_i(x), B_i(y)) \) and \( R = \bigcup_{i=1}^{n} R_i \). By use of CRI method and \( t \)-norm \( T \), we have that \( B^* = A^* \circ R \), where \( (A^* \circ R)(y) = \lor_{x \in X} A^*(x)TR(x, y) \).

Step 4. Defuzzifier. There are three defuzzifiers as follows:

1. Center-average defuzzifier. Let \( y_i \) be the peak point of \( B_i \), i.e., \( B(y_i) = 1 \), Then

\[ F(x) = \frac{\sum_{i=1}^{n} y_i B^*(y_i)}{\sum_{i=1}^{n} B^*(y_i)} \]

is called a Mamdani fuzzy system.


Let \( hgt(B^*) = \{ y \in Y | B^*(y) = \sup_{y' \in Y} B^*(y') \} \). Then \( G(x) \) is chosen as one element in \( hgt(B^*) \). It can be the maximum points, the minimum point or the average point of \( hgt(B^*) \), i.e., (a) \( G(x) = \inf hgt(B^*) \); (b) \( G(x) = \sup hgt(B^*) \); (c) \( G(x) = \frac{\int_{hgt(B^*)} y dy}{\int_{hgt(B^*)} dy} \).

Then \( G(x) \) is called a Boolean fuzzy system [16].

3. Center-of-gravity defuzzifier.

Let \( S(x) = \frac{\int_{B^*(y)} y dy}{\int_{B^*(y)} dy} \). Then \( S(x) \) is called the center-of-gravity fuzzy system.

Due to \( B^*(y) = \lor_{x \in X} (A^*(x)TR(x', y)) = R(x, y) \), so

\[ \bar{S}(x) = \frac{\int_{Y} y R(x, y) dy}{\int_{Y} R(x, y) dy}. \]  

In this paper, fuzzy implication operators are chosen as either Kleene–Dienes fuzzy implication or Łukasiewicz fuzzy implication, i.e., \( \theta_1(a, b) = (1 - a) \lor b \), \( \theta_2(a, b) = (1 - a + b) \land 1 \). Three \( t \)-norms are chosen as \( T_1(a, b) = a \land b \), \( T_2(a, b) = (a + b - 1) \lor 0 \) and \( T_3(a, b) = ab \).

3. The center-of-gravity fuzzy systems

3.1. Construction of the fuzzy system

Consider the following fuzzy inference rules:

If x is \( A_i \), then y is \( B_i \) (i = 1, 2, . . . , n).

In order to construct a fuzzy system based on fuzzy inference rules (2), we should consider the antecedents of fuzzy inference rules for support the degree of the input variable x, in other words,

(i) If \( A_i(x) = 0 \), then the inference rule “If x is \( A_i \), then y is \( B_i \)” does not support the input variable x. In this case, we should say that this inference rule does not take any effect on the construction of the fuzzy system.

(ii) If \( A_i(x) > 0 \), then the inference rule “If x is \( A_i \), then y is \( B_i \)” supports the input variable x. In this case, we should say that this inference rule takes effect on the construction of the fuzzy system.
Based on the view as above, a new method to construct fuzzy systems is proposed as follows:

1. Let $I(x) = \{i | i \in \{1, 2, \ldots, n\}, A_i(x) > 0\}$.

2. For the input variable $x$ and $i \in I(x)$, then

$$
A_i^*(x') = \begin{cases} 
A_i(x), & x' = x \\
0, & x' \neq x 
\end{cases}
$$

(3) is called a non-singleton fuzzifier of the input variable $x$.

3. Let $\theta$ be a fuzzy implication operator, $R_i(x, y) = \theta(A_i(x), B_i(y))$ and $B^* = \bigcup_{i=1}^{n} A_i^* \circ R_i$, i.e., $B^*(y) = \lor_{i \in I(x)} A_i^*(x) TR_i(x, y)$. Let $\text{supp } B_i = \{y \in Y | B_i(y) > 0\}$, then

$$
Y^*(x) = \bigcup_{i \in I(x)} \text{supp } B_i
$$

is called the adaptive universe for output fuzzy set $B^*$.

4. Let

$$
\mathcal{S}(x) = \frac{\int_{Y^*(x)} y B^*(y) dy}{\int_{Y^*(x)} B^*(y) dy}
$$

Then $\mathcal{S}(x)$ is called the center-of-gravity fuzzy system.

**Note 1.** The adaptive universe $Y^*(x)$ is dependent on the variable $x$, for example,

- If $I(x) = \{1, 2\}$, then $Y^*(x) = \text{supp } B_1 \cup \text{supp } B_2$;
- If $I(x) = \{3, 4\}$, then $Y^*(x) = \text{supp } B_3 \cup \text{supp } B_4$.

Therefore, $\mathcal{S}(x)$ is a center-of-gravity of fuzzy set $B^*$ on the adaptive universe $Y^*(x)$.

**Note 2.** If we chose a singleton fuzzifier and the normal fuzzy implications such as the Kleene–Dienes implication or the Lukasiewicz implication to construct a fuzzy system, then fuzzy systems constructed by a center-average defuzzifier and a center-of-gravity defuzzifier are not universal approximators [14,18]. However, by applying a non-singleton fuzzifier, we will construct fuzzy systems to approximate a desired control function up to a given level of accuracy.

By the use of the method in [14], we can present a lemma as follows.

**Lemma 1.** If $f : [a, b] \to \mathbb{R}$ be a continuous function and $[c, d] = f([a, b])$, then

1. There exist $\{x_i\}, \{y_i\}$ and fuzzy sets $\{A_i\}, \{B_i\}$ satisfying:
   - $a = x_1 < x_2 < \cdots < x_n = b, c = y_1 < y_2 < \cdots < y_{N+1} = d$.
   - $A_i(x_i) = 1, B_i(y_i) = 1$ (i = 1, 2, ..., n; k = 1, 2, ..., N + 1).
   - $A_i(x) = 0$ (j ≠ i, i + 1). When $y \in [y_k, y_{k+1})$, $B_i(y) = B_{k+1}(y)$ = 1, $B_j(y) = 0$ (j ≠ k, k + 1).

2. There exist fuzzy inference rules:

   If \( x \) is $A_i$, then $y \in C_i$ (i = 1, 2, ..., n)

satisfying

$$
\text{If } C_i = B_i, \text{ then } C_{i+1} \in \{B_{k-1}, B_k, B_{k+1}\}
$$

**Proof.**

We chose a positive integer $N$ and let $e = \frac{d-c}{N}$. Let

$$
y_1 = c, \quad y_2 = c + e, \quad \ldots, \quad y_k = c + (k-1)e, \quad \ldots, \quad y_{N+1} = d.
$$

$$
Y = \left[y_1, y_1 + \frac{e}{2}\right], \quad Y_2 = \left[y_2 - \frac{e}{2}, y_2 + \frac{e}{2}\right], \quad \ldots, \quad Y_{N+1} = \left[y_{N+1} - \frac{e}{2}, y_{N+1}\right]
$$

then $Y_i \cap Y_j = \emptyset$ (i ≠ j) and $[c, d] = Y_1 \cup Y_2 \cup \cdots \cup Y_{N+1}$.

Let $X_i = f^{-1}(Y_i)$ (i = 1, 2, ..., N + 1), then $[a, b] = X_1 \cup X_2 \cup \cdots \cup X_{N+1}$ and $X_i \cap X_j = \emptyset$ (i ≠ j). Then there exists $\delta > 0$ such that $|f(x') - f(x'')| < \frac{\varepsilon}{2}$ when $|x' - x''| < \delta$. Let $a = x_1 < x_2 < \cdots < x_n = b$ and $|x_{i+1} - x_i| < \delta$ (i = 1, 2, ..., n), then we have that $|f(x_{i+1}) - f(x_i)| < \frac{\varepsilon}{2}$. Let

$$
A_1(x) = \begin{cases}
\frac{x_2 - x}{x_2 - x_1}, & x \in [x_1, x_2] \\
0, & \text{else}
\end{cases}, \quad A_n(x) = \begin{cases}
\frac{x - x_n}{x_n - x_{n-1}}, & x \in [x_{n-1}, x_n] \\
0, & \text{else}
\end{cases}
$$

$$
B_1(y) = \begin{cases}
\frac{y_2 - y}{y_2 - y_1}, & y \in [y_1, y_2] \\
0, & \text{else}
\end{cases}, \quad B_{N+1}(y) = \begin{cases}
\frac{y - y_N}{y_{N+1} - y_N}, & y \in [y_N, y_{N+1}] \\
0, & \text{else}
\end{cases}
$$
For $i = 2, 3, \ldots, n - 1$, $k = 2, 3, \ldots, N$,

$$A_i(x) = \begin{cases} 
\frac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in [x_{i-1}, x_i] \\
\frac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in [x_i, x_{i+1}] \\
0, & \text{else},
\end{cases}$$

$$B_k(y) = \begin{cases} 
\frac{y - y_{k-1}}{y_k - y_{k-1}}, & y \in [y_{k-1}, y_k] \\
y_{k+1} - y, & y \in [y_k, y_{k+1}] \\
0, & \text{else}.
\end{cases}$$

Then we have that $A_i(x_i) = 1$ ($i = 1, 2, \ldots, n$), $B_k(y_k) = 1$ ($k = 1, 2, \ldots, N + 1$) and when $x \in [x_i, x_{i+1})$, $A_i(x) + A_{i+1}(x) = 1$, $A_i(x) = 0$ ($j \neq i, i + 1$).

When $y \in [y_k, y_{k+1})$, $B_k(y) + B_{k+1}(y) = 1$, $B_i(y) = 0$ ($j \neq k, k + 1$).

Then when $x_i \in X_k$, we let $C_i = B_k$ and construct the following inference rules:

If $x$ is $A_i$, then $y$ is $C_i$ ($i = 1, 2, \ldots, n$).

Then the inference rules $(\ast)$ satisfy the following conditions:

If $C_i = B_k$, then $C_{i+1} \in \{B_{k-1}, B_k, B_{k+1}\}$.

In fact, we have $f(x_i) \in Y_k$ and $y_k - \frac{\varepsilon}{2} \leq f(x_i) < y_k + \frac{\varepsilon}{2}$ since $x_i \in X_k$. Then $f(x_i) - \frac{\varepsilon}{2} < f(x_{i+1}) < f(x_i) + \frac{\varepsilon}{2}$ and consequently $y_{k+1} = y_k - e < f(x_{i+1}) < y_k + e = y_{k+1}$. It follows that $f(x_{i+1}) \in Y_{k-1} \cup Y_k \cup Y_{k+1}$. Then we have $x_{i+1} \in X_{k-1} \cup X_k \cup X_{k+1}$ and $C_{i+1} \in \{B_{k-1}, B_k, B_{k+1}\}$.

**Note 2.** The proof of Lemma 1 is actually from Theorem 3.4 in [14]. However, $\{Y_k\}$ in [14] does not satisfy the conditions: $Y_i \cap Y_j = \emptyset$ ($i \neq j$) and $X_i \cap X_j = \emptyset$ ($i \neq j$) since $\{Y_k\}$ in [14] has the form as $Y_1 = [y_1, y_1 + \frac{\varepsilon}{2}e], Y_2 = [y_2 - \frac{\varepsilon}{2}e, y_2 + \frac{\varepsilon}{2}e], \ldots, Y_{N+1} = [y_{N+1} - \frac{\varepsilon}{2}e, y_{N+1}]$.

Let $\theta$ be a fuzzy implication operator. Basing on the fuzzy inference rules (6) and (7), we have the fuzzy relations: $R_i(x, y) = \theta(A_i(x), C_i(y))$. Clearly, we have $I(x) = [i, i + 1]$ for any $x \in [x_i, x_{i+1})$. Assume that $C_i = B_k$, then we have from (3)-(4) and (6)-(7) that:

$$B^\ast(y) = A_i(x)T\theta(A_i(x), B_k(y)) \lor A_{i+1}(x)T\theta(A_{i+1}(x), C_{i+1}(y)), \quad (8)$$

$$Y^\ast(x) = \text{supp} C_i \cup \text{supp} C_{i+1} = \begin{cases} 
(y_{k-2}, y_{k+1}), & C_{i+1} = B_{k-1} \\
(y_{k-1}, y_{k+1}), & C_{i+1} = B_k \\
(y_{k+1}, y_{k+2}), & C_{i+1} = B_{k+1}.
\end{cases} \quad (9)$$

In the following discussions, we agree on $x \in [x_i, x_{i+1})$, $x_i = \frac{1}{2}(x_i + x_{i+1}), C_i = B_k$ in inference rules $(\ast)$ and

$$z_{i+1} = \begin{cases} 
y_{k-1}, & C_{i+1} = B_{k-1}, \\
y_k, & C_{i+1} = B_k, \\
y_{k+1}; & C_{i+1} = B_{k+1}.
\end{cases} \quad (10)$$

### 3.2. The fuzzy systems based on the Kleene–Dienes implication

We denote that

$$\begin{array}{c}
y_{k-2}^A = A_i(x)y_{k-1} + A_{i+1}(x)y_{k-2}, \\
y_{k-2}^\Delta = A_i(x)y_{k-2} + A_{i+1}(x)y_{k-1}; \\
y_{k-1}^A = A_i(x)y_{k-1} + A_{i+1}(x)y_{k-1}, \\
y_{k-1}^\Delta = A_i(x)y_{k-1} + A_{i+1}(x)y_{k-2}; \\
y_{k}^A = A_i(x)y_{k+1} + A_{i+1}(x)y_{k}, \\
y_{k}^\Delta = A_i(x)y_{k} + A_{i+1}(x)y_{k+1}; \\
y_{k+1}^A = A_i(x)y_{k+2} + A_{i+1}(x)y_{k+1}, \\
y_{k+1}^\Delta = A_i(x)y_{k+1} + A_{i+1}(x)y_{k+2};
\end{array} \quad (11)$$

Then

$$\begin{array}{c}
\text{when } y \in [y_{k-2}, y_{k-1}), A_i(x) \leq B_{k-1}(y) \iff y_{k-2}^A \leq y, A_i(x) \leq B_{k-1}(y) \iff y_{k-2}^\Delta \leq y; \\
\text{when } y \in [y_{k-1}, y_k), A_i(x) \leq B_k(y) \iff y_{k-1}^A \leq y, A_i(x) \leq B_k(y) \iff y_{k-1}^\Delta \leq y; \\
\text{when } y \in [y_k, y_{k+1}), A_i(x) \leq B_k(y) \iff y \leq y_k^A, A_i(x) \leq B_k(y) \iff y \leq y_k^\Delta; \\
\text{when } y \in [y_{k+1}, y_{k+2}), A_i(x) \leq B_{k+1}(y) \iff y \leq y_{k+1}^A, A_i(x) \leq B_{k+1}(y) \iff y \leq y_{k+1}^\Delta.
\end{array} \quad (12)$$

Then we have the following conclusion.
Theorem 1. (1) If $\theta = \theta_1$ and $T = T_1$, then $\tilde{S}(x) = \bar{S}_1(x) = E_i(x)y_k + E_{i+1}(x)z_{i+1}$ and $E_i(x) + E_{i+1}(x) = 1$, $\forall x \in (x_i, x_{i+1}]$, where

$$E_i(x) = \begin{cases} \frac{1 + A_i(x)}{2(1 + A_{i+1}(x))}, & x \in [x_i, \bar{x}_i) \\ \frac{3A_{i+1}(x)}{2(1 + A_i(x))}, & x \in [\bar{x}_i, x_{i+1}) \end{cases} \quad E_{i+1}(x) = \begin{cases} \frac{3A_{i+1}(x)}{2(1 + A_{i+1}(x))}, & x \in [x_i, \bar{x}_i) \\ \frac{1 + A_{i+1}(x)}{2(1 + A_i(x))}, & x \in [\bar{x}_i, x_{i+1}) \end{cases}$$

(13)

(2) If $\theta = \theta_1$ and $T = T_2$, then $\tilde{S}(x) = \bar{S}_2(x) = F_i(x)y_k + F_{i+1}(x)z_{i+1}$ and $F_i(x) + F_{i+1}(x) = 1$, $\forall x \in (x_i, x_{i+1}]$, where

$$F_i(x) = \frac{A_i^2(x)}{A_i^2(x) + A_{i+1}^2(x)}, \quad F_{i+1}(x) = \frac{A_{i+1}^2(x)}{A_i^2(x) + A_{i+1}^2(x)}$$

(14)

(3) If $\theta = \theta_2$ and $T = T_2$, then $\tilde{S}(x) = \bar{S}_3(x) = G_i(x)y_k + G_{i+1}(x)z_{i+1}$ and $G_i(x) + G_{i+1}(x) = 1$, $\forall x \in (x_i, x_{i+1}]$, where

$$G_i(x) = \frac{1}{2}A_i(x)(3 - 3A_i(x) + 2A_i^2(x))$$

$$G_{i+1}(x) = \frac{1}{2}A_{i+1}(x)(3 - 3A_{i+1}(x) + 2A_{i+1}^2(x))$$

(15)

Proof. (1) Let $T = T_1$.

(i) When $C_{i+1} = B_{k+1}$, we have that $Y^*(x) = (y_{k-1}, y_{k+2})$. Then

(a) If $y \in (y_{k-1}, y_k)$, then $B^*(y) = A_i(x) \land (A_{i+1}(x) \lor B_k(x))$. It follows that

$$B^*(y) = \begin{cases} A_{i+1}(x), & y_{k-1} < y < y_{k-1} \land x \in [x_i, \bar{x}_i) \\ B_k(y), & y_{k-1} < y < y_{k-1} \land x \in [x_i, \bar{x}_i) \\ A_i(x), & y_{k-1} < y < y_{k-1} \land x \in [x_i, \bar{x}_i) \\ A_i(x), & x \in [x_i, x_{i+1}) \end{cases}$$

(b) If $y \in [y_k, y_{k+1}]$, then $B^*(y) = \begin{cases} A_i(x) \lor (A_{i+1}(x) \land B_k(y)), & x \in [x_i, \bar{x}_i] \\ A_{i+1}(x) \lor (A_{i+1}(x) \land B_k(y)), & x \in [\bar{x}_i, x_{i+1}) \end{cases}$. It follows that $B^*(y) = \begin{cases} A_i(x), & y < y_k \land B_k(y), & y_k < y \land y_k < y_k \land x \in [x_i, \bar{x}_i] \land y_k < y \land x \in [\bar{x}_i, x_{i+1}) \end{cases}$

(c) If $y \in (y_{k+1}, y_{k+2})$, then $B^*(y) = A_{i+1}(x) \land (A_i(x) \lor B_{k+1}(y))$. It follows that

$$B^*(y) = \begin{cases} A_{i+1}(x), & x \in [x_i, \bar{x}_i) \\ A_{i+1}(x), & y < y_{k+1} \land x \in [x_i, \bar{x}_i) \\ B_{k+1}(y), & y_{k+1} < y < y_{k+1} \land x \in [x_i, x_{i+1}) \\ A_i(x), & y_{k+1} < y < y_{k+1} \land x \in [x_i, x_{i+1}) \end{cases}$$

Thus, we have that for $x \in [x_i, \bar{x}_i]$,

$$\int_{y^*(x)} yB^*(y)dy = \int_{y_{k-1}}^{y_{k-1}} A_{i+1}(x)dy + \int_{y_{k-1}}^{y_{k-1}} B_k(y)dy + \int_{y_{k-1}}^{y_{k-1}} A_i(x)dy + \int_{y_{k-1}}^{y_{k-1}} B_k(y)dy + \int_{y_{k-1}}^{y_{k-1}} A_i(x)dy$$

$$= A_{i+1}(x)e + e \int_{A_{i+1}(x)}^{A_i(x)} tdt + 2A_i(x)A_{i+1}(x)e \int_{A_{i+1}(x)}^{A_i(x)} tdt + A_{i+1}(x)(1 + A_{i+1}(x))e$$

$$= (1 + A_{i+1}(x))e$$

$$\int_{y^*(x)} yB^*(y)dy = \int_{y_{k-1}}^{y_{k-1}} yA_{i+1}(x)dy + \int_{y_{k-1}}^{y_{k-1}} yB_k(y)dy + \int_{y_{k-1}}^{y_{k-1}} yA_i(x)dy + \int_{y_{k-1}}^{y_{k-1}} yB_k(y)dy + \int_{y_{k-1}}^{y_{k-1}} yA_i(x)dy$$

$$= \frac{1}{2}A_{i+1}(x)(y_{k-1} + y_{k-1})e + e \int_{A_{i+1}(x)}^{A_i(x)} (y_{k-1} + et)tdt + A_i(x)A_{i+1}(x)(y_{k-1} + y_{k-1})e$$

$$+ \int_{A_{i+1}(x)}^{A_i(x)} (y_{k-1} - et)tdt + \frac{1}{2}A_{i+1}(x)(1 + A_{i+1}(x))(y_{k+2} + y_{k-1})$$

$$= \left[ \frac{1}{2}(1 + A_i(x))y_k + \frac{3}{2}A_{i+1}(x)y_{k+1} \right]e.$$
For \( x \in [\bar{x}_i, x_{i+1}] \), we have that

\[
\int_{y^*(x)} B^*(y) \, dy = \int_{y_k}^{y_{k+1}} A_i(x) \, dy + \int_{y_k}^{y_{k+1}} B_{k+1}(y) \, dy + \int_{y_k}^{y_{k+1}} A_{i+1}(x) \, dy + \int_{y_k}^{y_{k+1}} B_{k+1}(y) \, dy + \int_{y_k}^{y_{k+1}} A_i(x) \, dy
\]

\[
\begin{align*}
&= A_i(x)(1 + A_i(x))e + e \int_{A_i(x)}^t dt + 2A_i(x)A_{i+1}(x)e + e \int_{A_i(x)}^t dt + A_i^2(x)e \\
&= (1 + A_i(x))e.
\end{align*}
\]

Then we have that

\[
\bar{s}(x) = \begin{cases} 
\frac{1 + A_i(x)}{2(1 + A_{i+1}(x))} y_k + \frac{3A_{i+1}(x)}{2(1 + A_{i+1}(x))} y_{k+1}, & x \in [x_i, \bar{x}_i) \\
\frac{A_{i+1}(x)}{3A_i(x)} y_k + \frac{2(1 + A_{i+1}(x))}{2(1 + A_i(x))} y_{k+1}, & x \in [\bar{x}_i, x_{i+1}). 
\end{cases}
\]

(ii) When \( C_{i+1} = B_{k-1} \), we have that \( Y^*(x) = (y_{k-2}, y_{k+1}) \). Then

(a) If \( y \in (y_{k-2}, y_{k-1}) \), then \( B^*(y) = A_{i+1}(x) \wedge (A_i(x) \vee B_{k-1}(y)) \). It follows that

\[
B^*(y) = \begin{cases} 
A_{i+1}(x), & x \in [x_i, \bar{x}_i) \\
A_i(x), & y < y_{k-2}, x \in [\bar{x}_i, x_{i+1}) \\
B_{k-1}(y), & y_{k-2} \leq y \leq y_{k-1}^*, x \in [\bar{x}_i, x_{i+1}) \\
A_{i+1}(x), & y > y_{k-2}, x \in [\bar{x}_i, x_{i+1}). 
\end{cases}
\]

(b) If \( y \in [y_{k-1}, y_k] \), then \( B^*(y) = A_{i+1}(x) \wedge (A_i(x) \vee B_k(y)) \). It follows that \( B^*(y) = \begin{cases} 
A_{i+1}(x), & y < y_{k-1}^* \\
B_k(y), & y_{k-1}^* \leq y \leq y_{k-1}^* \leq y \leq y_{k-1}, x \in [x_i, \bar{x}_i) \\
A_{i+1}(x), & y > y_{k-1}^* \\
A_{i+1}(x), & y > y_{k-1}^* \leq y \leq y_{k-1}, x \in [\bar{x}_i, x_{i+1}). 
\end{cases} \)

(c) If \( y \in (y_k, y_{k+1}) \), then \( B^*(y) = A_i(x) \wedge (A_{i+1}(x) \vee B_k(y)) \). It follows that

\[
B^*(y) = \begin{cases} 
A_i(x), & y < y_k^* \\
B_k(y), & y_k^* \leq y \leq y_k^*, x \in [x_i, \bar{x}_i) \\
A_{i+1}(x), & y > y_k^* \leq y \leq y_{k-1}, x \in [\bar{x}_i, x_{i+1}). 
\end{cases}
\]

Thus, for \( x \in [x_i, \bar{x}_i] \), we have that

\[
\int_{y^*(x)} B^*(y) \, dy = \int_{y_k}^{y_{k-1}} A_{i+1}(x) \, dy + \int_{y_k}^{y_{k-1}} B_{k-1}(y) \, dy + \int_{y_k}^{y_{k-1}} A_i(x) \, dy + \int_{y_k}^{y_{k-1}} B_{k-1}(y) \, dy + \int_{y_k}^{y_{k-1}} A_i(x) \, dy
\]

\[
\begin{align*}
&= A_{i+1}(x)(1 + A_{i+1}(x))e + e \int_{A_{i+1}(x)}^t dt + 2A_i(x)A_{i+1}(x)e + e \int_{A_{i+1}(x)}^t dt + A_{i+1}^2(x)e \\
&= (1 + A_{i+1}(x))e.
\end{align*}
\]
\[
\int_{Y^{*}(x)} yB^{*}(y)dy = \int_{y_{k-2}}^{y_{k-1}} yA_{i+1}(x)dy + \int_{y_{k-1}}^{y_{k}} yB_{k}(y)dy + \int_{y_{k}}^{y_{k+1}} yA_{i+1}(x)dy + \int_{y_{k}}^{y_{k+1}} yA_{i+1}(x)dy
\]

When \( x \in [\bar{x}_{i}, x_{i+1}] \), we have that

\[
\int_{Y^{*}(x)} B^{*}(y)dy = \int_{y_{k-2}}^{y_{k-1}} A_{i}(x)dy + \int_{y_{k-1}}^{y_{k}} B_{k-1}(y)dy + \int_{y_{k}}^{y_{k+1}} A_{i+1}(x)dy
\]

\[
= \left[ 1 + A_{i+1}(x) \right] (1 + A_{i+1}(x)) [y_{k-2} + y_{k-1}^{2}] d + \left[ y_{k-1} + et \right] dt + A_{i}(x)A_{i+1}(x) [y_{k}^{2} + y_{k-1}^{2}] d
\]

\[
= \left[ \frac{1}{2} (1 + A_{i}(x)) y_{k} + \frac{3}{2} A_{i+1}(x) y_{k-1} \right] e.
\]

(iii) When \( C_{i+1} = B_{k} \), we have that \( Y^{*}(x) = (y_{k-1}, y_{k+1}) \). Then

\[
B^{*}(y) = \begin{cases} A_{i}(x) \land (A_{i+1}(x) \lor B_{k}(y)) & x \in [x_{i}, \bar{x}_{i}] \\ A_{i+1}(x) \land (A_{i}(x) \lor B_{k}(y)) & x \in [\bar{x}_{i}, x_{i+1}] \end{cases}
\]

If \( y \in (y_{k-1}, y_{k}] \), then \( B^{*}(y) = \begin{cases} A_{i+1}(x) & y < y_{k}^{2} \\ B_{k}(y) & y_{k}^{2} \leq y \leq y_{k}^{2} \end{cases} \) when \( x \in [x_{i}, \bar{x}_{i}] \) and \( B^{*}(y) = \begin{cases} A_{i+1}(x) & y < y_{k}^{2} \\ B_{k}(y) & y_{k}^{2} \leq y \leq y_{k}^{2} \end{cases} \) when \( x \in [\bar{x}_{i}, x_{i+1}] \).

If \( y \in [y_{k}, y_{k+1}] \), then \( B^{*}(y) = \begin{cases} A_{i+1}(x) & y < y_{k}^{2} \\ B_{k}(y) & y_{k}^{2} \leq y \leq y_{k}^{2} \end{cases} \) when \( x \in [x_{i}, \bar{x}_{i}] \) and \( B^{*}(y) = \begin{cases} A_{i+1}(x) & y < y_{k}^{2} \\ B_{k}(y) & y_{k}^{2} \leq y \leq y_{k}^{2} \end{cases} \) when \( x \in [\bar{x}_{i}, x_{i+1}] \).

Thus, for \( x \in [x_{i}, \bar{x}_{i}] \), we have that:

\[
\int_{Y^{*}(x)} B^{*}(y)dy = \int_{y_{k-1}}^{y_{k}} A_{i}(x)dy + \int_{y_{k}}^{y_{k+1}} B_{k}(y)dy + \int_{y_{k}}^{y_{k+1}} A_{i+1}(x)dy
\]

\[
= e.
\]

\[
\int_{Y^{*}(x)} yB^{*}(y)dy = \int_{y_{k-1}}^{y_{k}} yA_{i}(x)dy + \int_{y_{k}}^{y_{k+1}} yB_{k}(y)dy + \int_{y_{k}}^{y_{k+1}} yA_{i+1}(x)dy
\]

\[
= y_{k} e.
\]

Similarly, when \( x \in [\bar{x}_{i}, x_{i+1}] \), we have \( \int_{y_{k}}^{y_{k+1}} B^{*}(y)dy = e, \int_{y_{k}}^{y_{k+1}} yB^{*}(y)dy = y_{k} e. \)

Then we have that \( \tilde{S}(x) = y_{k} = E_{i}(x) y_{k} + E_{i+1}(x) y_{k} \).

(ii) Let \( T = T_{2} \), then

(i) If \( C_{i+1} = B_{k+1} \), then \( Y^{*}(x) = (y_{k-1}, y_{k+2}) \). Then we have that

(1) \( B^{*}(y) = \begin{cases} A_{i}(x) + B_{k}(y) - 1 & y \geq y_{k}^{2} \\ 0 & y < y_{k}^{2} \end{cases} \) if \( y \in (y_{k-1}, y_{k}) \) and consequently

\[
\int_{y_{k-1}}^{y_{k}} B^{*}(y)dy = \int_{y_{k}^{2}}^{y_{k}} (A_{i}(x) + B_{k}(y) - 1)dy = \frac{1}{2} A_{i}^{2}(x)e.
\]

(2) \( B^{*}(y) = \begin{cases} A_{i}(x) + B_{k}(y) - 1 & y \leq y_{k}^{2} \\ A_{i+1}(x) + B_{k+1}(y) - 1 & y > y_{k}^{2} \end{cases} \) if \( y \in [y_{k}, y_{k+1}] \) and consequently

\[
\int_{y_{k}}^{y_{k+1}} B^{*}(y)dy = \int_{y_{k}}^{y_{k}} (A_{i}(x) + B_{k}(y) - 1)dy + \int_{y_{k}}^{y_{k+1}} (A_{i+1}(x) + B_{k+1}(y) - 1)dy
\]

\[
= \frac{1}{2} (A_{i}^{2}(x) + A_{i+1}^{2}(x)) e.
\]
(3) \(B^*(y) = \begin{cases} A_{i+1}(x) + B_{k+1}(y) - 1, & y \leq y^2_{k+1} \\
0, & y > y^2_{k+1} \end{cases}\)

if \(y \in (y_{k+1}, y_{k+2})\) and consequently \(\int_{y_{k+1}}^{y_{k+2}} B^*(y) \, dy = \int_{y_{k+1}}^{y_{k+2}} (A_{i+1}(x) + B_{k+1}(y) - 1) \, dy = \frac{1}{2} A^2_{i+1}(x) e.

Thus \(\int_{y^*(x)}^{y}(B^*(y) \, dy = \frac{1}{2} A_{i+1}(x) (1 + A^2_{i+1}(x)) + \frac{1}{2} (1 - A_{i+1}(x) A_{i+1}(x)) e + \frac{1}{2} A_{i+1}(x) (1 + A^2_{i+1}(x)) e = e.

Since

\[
\int_{y^*(x)}^{y} y B^*(y) \, dy = \int_{y_{k-1}}^{y_{k-1}} y A_{i+1}(x) A_{i+1}(x) dy + \int_{y_{k-1}}^{y_k} y A_{i}(x) B_k(y) dy + \int_{y_k}^{y} y A_{i}(x) B_k(y) dy
\]

\[
+ \int_{y_k}^{y_{k+1}} y A_{i+1}(x) B_{k+1}(y) dy + \int_{y_{k+1}}^{y^2_{k+1}} y A_{i+1}(x) B_{k+1}(y) dy + \int_{y^2_{k+1}}^{y} y A_{i}(x) A_{i+1}(x) dy
\]

\[
= \frac{1}{2} A_{i}(x) A^2_{i+1}(x) (y_{k-1} + y^2_{k-1}) e + A_{i}(x) e \int_{A_{i}(x)}^{1} (y_{k-1} + et) \, dt
\]

\[
+ A_{i+1}(x) e \int_{A_{i}(x)}^{1} (y_{k+2} - et) \, dt + \frac{1}{2} A^2_{i}(x) A_{i+1}(x) (y^2_{k+1} + y_{k+2}) e
\]
Then we have

\[ G(\theta) = G_1(\theta) = G_1(\theta) \gamma_k + G_{i+1}(\theta) \gamma_{k+1} \]

and \( G(\theta) + G_{i+1}(\theta) = 1 \), so we have from (5) that \( \tilde{S}(\theta) = G(\theta) \gamma_k + G_{i+1}(\theta) \gamma_{k+1} \).

Similarly,

(ii) When \( C_{i+1} = B_k \), we can obtain that \( \tilde{S}(\theta) = G(\theta) \gamma_k + G_{i+1}(\theta) \gamma_{k+1} \).

(iii) When \( C_{i+1} = B_k \), we can obtain that \( \tilde{S}(\theta) = y_k = G(\theta) \gamma_k + G_{i+1}(\theta) \gamma_{k+1} \).

\[ \square \]

3.3. The fuzzy systems based on the Lukasiewicz fuzzy implication

We denote

\[
\begin{align*}
\dot{y}_{k-2} &= (A_{i+1}(\theta) - A_i(\theta)) y_{k-2} + 2A_i(\theta) y_{k-2} \\
\dot{y}_{k-1} &= (A_i(\theta) - A_{i+1}(\theta)) y_{k-1} + 2A_{i+1}(\theta) y_{k-1} \\
\dot{y}_k &= (A_i(\theta) - A_{i+1}(\theta)) y_k + 2A_{i+1}(\theta) y_{k+1} \\
\dot{y}_{k+1} &= (A_{i+1}(\theta) - A_i(\theta)) y_{k+1} + 2A_i(\theta) y_{k+2}
\end{align*}
\]

Then

\[ \begin{align*}
\text{When } y \in [y_{k-2}, y_{k-1}], & \ A_{i+1}(\theta) \leq A_i(\theta) + B_{k-1}(y) \Leftrightarrow \dot{y}_{k-2} \leq y \\
\text{When } y \in [y_{k-1}, y_k], & \ A_{i+1}(\theta) \leq A_i(\theta) + B_{k-1}(y) \Leftrightarrow y \leq y_{k-1} \\
\text{When } y \in [y_k, y_{k-1}], & \ A_i(\theta) \leq A_{i+1}(\theta) + B_k(y) \Leftrightarrow y \leq y_k
\end{align*} \]

Then we have

**Theorem 2.** (1) If \( \theta = \theta_1 \) and \( T = T_1 \), then \( \tilde{S}(\theta) = \tilde{S}_4(\theta) = H_i(\theta) \gamma_k + H_{i+1}(\theta) \gamma_{i+1} \), where

\[
H_i(\theta) = \begin{cases} 
\frac{2 + 3A_{i+1}(\theta) - 8A_i^2(\theta)}{2A_i(\theta)(1 + 4A_{i+1}(\theta))}, & x \in [x_i, \bar{x}_i] \\
\frac{3A_{i+1}(\theta)}{2A_i(\theta)(1 + 4A_{i+1}(\theta))}, & x \in [\bar{x}_i, x_{i+1}]
\end{cases}
\]

\[ \text{and } H_i(\theta) + H_{i+1}(\theta) = 1. \]

(2) If \( \theta = \theta_2 \) and \( T = T_2 \), then \( \tilde{S}(\theta) = \tilde{S}_5(\theta) = L_i(\theta) \gamma_k + L_{i+1}(\theta) \gamma_{i+1} \), where

\[
L_i(\theta) = \frac{A_i(\theta)(3 - A_i(\theta))}{2(1 + A_i(\theta)L_{i+1}(\theta))}, \quad L_{i+1}(\theta) = \frac{A_{i+1}(\theta)(3 - A_{i+1}(\theta))}{2(1 + A_i(\theta)L_{i+1}(\theta))}
\]

and \( L_i(\theta) + L_{i+1}(\theta) = 1 \).

(3) If \( \theta = \theta_3 \) and \( T = T_3 \), then \( \tilde{S}(\theta) = \tilde{S}_6(\theta) = M_i(\theta) \gamma_k + M_{i+1}(\theta) \gamma_{i+1} \), where

\[
M_i(\theta) = \begin{cases} 
\frac{6 - 9A_{i+1}(\theta) - 18A_i^2(\theta) + 28A_i^3(\theta) - 37A_{i+1}^3(\theta) + 11A_{i+1}^4(\theta)}{3A_i(\theta)(1 - 3A_i(\theta) + 11A_i^2(\theta) - 7A_i^3(\theta))}, & x \in [x_i, \bar{x}_i] \\
\frac{9A_{i+1} - 18A_i^2(\theta) + 3A_i^3(\theta) + 16A_i^4(\theta) - 11A_{i+1}^3(\theta)}{3A_{i+1}(\theta)(1 - 3A_{i+1}(\theta) + 11A_i^2(\theta) - 7A_i^3(\theta))}, & x \in [\bar{x}_i, x_{i+1}]
\end{cases}
\]
\[ M_{i+1}(x) = \begin{cases} 
\frac{9A_{i+1}(x) - 18A_{i+1}^2(x) + 3A_{i+1}^3(x) + 16A_{i+1}^4(x) - 11A_{i+1}^5(x)}{3A(x)(1 - 3A(x) + 11A_{i+1}^2(x) - 7A_{i+1}^3(x))}, & x \in [x_i, \bar{x}_i] \\
\frac{6 - 9A(x) - 18A_{i+1}^2(x) + 48A_{i+1}^3(x) - 37A_{i+1}^4(x) + 11A_{i+1}^5(x)}{3A_{i+1}(x)(1 - 3A_{i+1}(x) + 11A_{i+1}^2(x) - 7A_{i+1}^3(x))}, & x \in [\bar{x}_i, x_{i+1}) 
\end{cases} \tag{20b} \]

and \( M_i(x) + M_{i+1}(x) = 1 \).

**Proof.** Let \( T = T_1 \), then

(i) If \( C_{i+1} = B_{k+1} \), then \( Y^*(x) = (y_{k-1}, y_{k+2}) \). Then

(1) If \( y \in (y_{k-1}, y_k) \), then we have that \( B^*(y) = (A(x) \land (A_{i+1}(x) + B_k(y))) \) and

\[
B^*(y) = \begin{cases} 
A_{i+1}(x) + B_k(y), & y \leq y'_{k-1}, x \in [x_i, \bar{x}_i] \\
A_i(x), & y > y'_{k-1}, x \in [x_i, \bar{x}_i) \\
A_i(x), & x \in [\bar{x}_i, x_{i+1}). 
\end{cases}
\]

Thus, for \( x \in [x_i, \bar{x}_i] \), we have that

\[
\int_{y_{k-1}}^{y_k} B^*(y)dy = \int_{y_{k-1}}^{y_k} (A_{i+1}(x) + B_k(y))dy + \int_{y_k}^{y_{k+1}} A_i(x)dy = \frac{1}{2}(A_i(x) - A_{i+1}(x) + 4A_i(x)A_{i+1}(x))e.
\]

For \( x \in [\bar{x}_i, x_{i+1}) \), we have \( \int_{y_{k-1}}^{y_k} B^*(y)dy = A_i(x)e. \)

(2) If \( y \in [y_k, y_{k+1}) \), then we have

\[
B^*(y) = \begin{cases} 
A_i(x), & y \leq y'_{k}, x \in [x_i, \bar{x}_i] \\
A_{i+1}(x) + B_k(y), & y > y'_{k}, x \in [x_i, \bar{x}_i] \\
A_i(x) + B_{k+1}(y), & y \leq y'_{k+1}, x \in [\bar{x}_i, x_{i+1}) \\
A_i(x), & y > y'_{k+1}, x \in [\bar{x}_i, x_{i+1}). 
\end{cases}
\]

Thus, for \( x \in [x_i, \bar{x}_i] \), we have that

\[
\int_{y_k}^{y_{k+1}} B^*(y)dy = \int_{y_k}^{y_k} A_i(x)dy + \int_{y_k}^{y_{k+1}} (A_{i+1}(x) + B_k(y))dy = \frac{1}{2}(A_i(x) - A_{i+1}(x) + 4A_i(x)A_{i+1}(x))e.
\]

For \( x \in [\bar{x}_i, x_{i+1}) \), we have that

\[
\int_{y_k}^{y_{k+1}} B^*(y)dy = \int_{y_k}^{y_k} A_i(x)dy + \int_{y_k}^{y_{k+1}} (A_i(x) + B_{k+1}(y))dy = \frac{1}{2}(A_i(x) - A_{i+1}(x) + 4A_i(x)A_{i+1}(x))e.
\]

(3) When \( y \in (y_{k+1}, y_{k+2}) \), we have that:

\[
B^*(y) = \begin{cases} 
A_{i+1}(x), & x \in [x_i, \bar{x}_i] \\
A_{i+1}(x), & y \leq y'_{k+1}, x \in [\bar{x}_i, x_{i+1}) \\
A_i(x) + B_{k+1}(y), & y > y'_{k+1}, x \in [\bar{x}_i, x_{i+1}). 
\end{cases}
\]

Thus, for \( x \in [x_i, \bar{x}_i] \), we have \( \int_{y_{k+1}}^{y_{k+2}} B^*(y)dy = \int_{y_{k+1}}^{y_{k+2}} A_{i+1}(x)dy = A_{i+1}(x)e. \)

For \( x \in [\bar{x}_i, x_{i+1}) \), we have

\[
\int_{y_{k+1}}^{y_{k+2}} B^*(y)dy = \int_{y_{k+1}}^{y_{k+1}} A_{i+1}(x)dy + \int_{y_{k+1}}^{y_{k+2}} (A_i(x) + B_{k+1}(y))dy = \frac{1}{2}(A_i(x) - A_{i+1}(x) + 4A_i(x)A_{i+1}(x))e.
\]

Then, when \( x \in [x_i, \bar{x}_i] \), we have that

\[
\int_{y^*(x)} B^*(y)dy = \frac{1}{2}(A_i(x) - A_{i+1}(x) + 4A_i(x)A_{i+1}(x))e + \frac{1}{2}(A_i(x) - A_{i+1}(x) + 4A_i(x)A_{i+1}(x))e + A_{i+1}(x)e \\
= A_i(x)(1 + 4A_{i+1}(x))e.
\]
\[
\int_{Y(x)} yB^*(y)dy = \int_{y_{k-1}}^{y_k} y(A_{k+1}(x) + B_k(y))dy + \int_{y_{k+1}}^{y_k} yA_i(x)dy \\
+ \int_{y_{k-1}}^{y_k} yA_i(x)dy + \int_{y_{k+1}}^{y_k} y(A_{k+1}(x) + B_k(y))dy + \int_{y_{k+1}}^{y_{k+2}} yA_{i+1}(x)dy \\
= \frac{1}{2}A_{i+1}(x)(A_i(x) - A_{i+1}(x))\left(y_{k-1} + y_{k+1}\right)e + e\int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_{k-1} + et)dt \\
+ A_i(x)A_{i+1}(x)(y_k + y_{k+1})e + \int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_k + y_{k+1}) \\
+ \frac{1}{2}A_{i+1}(x)(A_i(x) - A_{i+1}(x))\left(y_{k+1} + y_{k+1}\right)e + e\int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_{k+1} - et)dt \\
+ \frac{1}{2}A_{i+1}(x)(y_{k+2} + y_{k+1})e.
\]

Since \(A_i(x) + A_{i+1}(x) = 1\), \(y_{k-1} = 2y_k - y_{k+1}\), \(y_{k+2} = 2y_k - y_k\), it follows that

\[
\int_{Y(x)} yB^*(y)dy = \left[\frac{1}{2}(2 + 3A_{i+1}(x) - 8A_{i+1}^2(x))y_k + \frac{3}{2}A_{i+1}(x)y_{k+1}\right]e.
\]

When \(x \in [\tilde{x}_i, x_{i+1}]\), we have that

\[
\int_{Y(x)} B^*(y)dy = A_i(x)e + \frac{1}{2}(A_{i+1}(x) - A_i(x) + 4A_i(x)A_{i+1}(x))e + \frac{1}{2}(A_{i+1}(x) - A_i(x) + 4A_i(x)A_{i+1}(x))e \\
= A_{i+1}(x)(1 + 4A_i(x))e.
\]

\[
\int_{Y(x)} yB^*(y)dy = \int_{y_{k-1}}^{y_k} yA_i(x)dy + \int_{y_{k+1}}^{y_k} y(A_i(x) + B_k(y))dy + \int_{y_{k+1}}^{y_{k+1}} yA_i(x)dy \\
+ \int_{y_{k+1}}^{y_k} yA_i(x)dy + \int_{y_{k+2}}^{y_k} y(A_i(x) + B_k(y))dy \\
= \frac{1}{2}A_i(x)(y_k + y_{k+1})e + \frac{1}{2}A_i(x)(A_{i+1}(x) - A_i(x))(y_k + y_{k+1})e + e\int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_k + et)dt \\
+ A_i(x)A_{i+1}(x)(y_{k+1} + y_{k+1})e + \int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_{k+2} + y_{k+1})e + e\int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_{k+2} - et)dt \\
= \left[\frac{3}{2}A_i(x)y_k + \frac{1}{2}(2 + 3A_{i+1}(x) - 8A_{i+1}^2(x)y_{k+1}\right]e.
\]

By (5), we have that

\[
\mathcal{S}(x) = \begin{cases} 
\frac{2 + 3A_{i+1}(x) - 8A_{i+1}^2(x)}{2A_i(x)(1 + 4A_i(x))}y_k + \frac{3A_{i+1}(x)}{2A_i(x)(1 + 4A_i(x))}y_{k+1}, & x \in [x_i, \tilde{x}_i] \\
\frac{2A_i(x)(1 + 4A_i(x))y_k + 2 + 3A_{i+1}(x) - 8A_{i+1}^2(x)}{2A_{i+1}(x)(1 + 4A_i(x))}y_{k+1}, & x \in [\tilde{x}_i, x_{i+1}]
\end{cases}
\]

Clearly, \(H_i(x) + H_{i+1}(x) = 1\).

(iii) Similarly, when \(C_{i+1} = B_k\), we have \(\mathcal{S}(x) = H_i(x)y_k + H_{i+1}(x)y_{k+1}\).

(iii) If \(C_{i+1} = B_k\), then \(Y^*(x) = (y_{k-1}, y_{k+1})\) and when \(y \in (y_{k-1}, y_{k})\), \(B^*(y) = \begin{cases} 
A_{i+1}(x), & y \geq y_{k-1} \\
A_{i+1}(x) + B_k(y), & y < y_{k-1}
\end{cases}\) and when \(y \in [y_k, y_{k+1}]\).

\[
B^*(y) = \begin{cases} 
A_{i+1}(x), & y \leq y_k \\
A_{i+1}(x) + B_k(y), & y > y_k
\end{cases}
\]

Thus

\[
\int_{Y(x)} B^*(y)dy = \int_{y_{k-1}}^{y_k} (A_{i+1}(x) + B_k(y))dy + \int_{y_{k-1}}^{y_k} A_i(x)dy + \int_{y_{k-1}}^{y_k} A_i(x)dy + \int_{y_{k-1}}^{y_{k+1}} (A_{i+1}(x) + B_k(y))dy \\
= A_{i+1}(x)(A_i(x) - A_{i+1}(x))e + e\int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_{k-1} + et)dt \\
+ A_i(x)A_{i+1}(x)(y_k + y_{k+1})e + e\int_{A_i(x) - A_{i+1}(x)}^{A_i(x) - A_{i+1}(x)} (y_{k+1} - et)dt \\
= (A_i(x) - A_{i+1}(x) + 4A_i(x)A_{i+1}(x))e.
\]
\[ \int_{Y^*(x)} y B^*(y) dy = \int_{yk-1}^{yk} y (A_{i+1}(x) + B_k(y)) dy + \int_{yk}^{yk+1} y A_i(x) dy + \int_{yk+1}^{yk+2} y (A_{i+1}(x) + B_k(y)) dy \]

\[ = \frac{1}{2} A_{i+1}(x) (A_i(x) - A_{i+1}(x)) (yk-1 + y'_k - 1) e + e \int_{0}^{A_i(x) - A_{i+1}(x)} (yk-1 + et) dt + 2A_i(x) A_{i+1}(x) (yk+1 + y'_k - 1) e + \frac{1}{2} A_{i+1}(x) (A_i(x) - A_{i+1}(x)) (yk+1 + y'_k) e \]

\[ + e \int_{0}^{A_i(x) - A_{i+1}(x)} (yk+1 - et) dt \]

\[ = (A_i(x) - A_{i+1}(x)) + 4A_i(x) A_{i+1}(x) y_k e. \]

It follows that \( y_k = L_i(y) y_k + L_{i+1}(y) y_k. \)

(II) Let \( T = T_2, \) then

(i) When \( C_{i+1} = B_{k+1}, Y^*(x) = (yk-1, yk+2). \) Then

(1) If \( y \in (yk-1, y_k), \) then we have that \( A_{i+1}(x) + B_k(y) \leq 1 \iff y \leq y_{k-1}^* \) and consequently

\[ B^*(y) = \begin{cases} A_i(y), & y \geq y_k^* \\
B_k(y), & y < y_k^*. \end{cases} \]

(2) If \( y \in [yk, y_{k+1}], \) then we have that

\[ A_{i+1}(x) + B_k(y) \leq 1 \iff y \leq y_{k+1}^*, B_k(y) \leq A_{i+1}(x) \iff y \geq y_{k+1}^*. \]

It follows that \( B^*(y) = \begin{cases} A_i(y), & y < y_k^* \\
B_k(y), & y \geq y_k^* \end{cases} \) when \( x \in [x_i, \tilde{x}_i] \) and \( B^*(y) = \begin{cases} A_i(y), & y \leq y_k^* \\
B_{k+1}(y), & y \geq y_{k+1}^* \end{cases} \) when \( x \in [\tilde{x}_i, x_{i+1}] \).

(3) If \( y \in (yk+1, y_{k+2}), \) then we have that \( A_{i+1}(x) + B_{k+1}(y) \leq 1 \iff y \geq y_{k+1}^* \) and consequently

\[ B^*(y) = \begin{cases} A_{i+1}(x), & y \leq y_k^* \\
B_{k+1}(y), & y > y_k^* \end{cases} \]

Then

\[ \int_{yk}^{yk} B^*(y) dy = \int_{yk-1}^{yk} B_k(y) dy + \int_{yk}^{yk+1} A_i(x) dy = \frac{1}{2} A_i(x) (1 + A_{i+1}(x)) e. \]

\[ \int_{yk+1}^{yk+2} B^*(y) dy = \int_{yk+1}^{yk+2} A_{i+1}(x) dy + \int_{yk+1}^{yk+2} B_{k+1}(y) dy = \frac{1}{2} A_{i+1}(x) (1 + A_i(x)) e. \]

and when \( x \in [x_i, \tilde{x}_i], \) we have that

\[ \int_{yk}^{yk+1} B^*(y) dy = \int_{yk}^{yk+1} B_k(y) dy + \int_{yk}^{yk+1} A_{i+1}(x) dy \]

\[ = \int_{yk}^{yk+1} y B_k(y) dy + \int_{yk}^{yk+1} y A_i(x) dy + \int_{yk}^{yk+1} y B_k(y) dy + \int_{yk}^{yk+1} y A_i(x) dy + \int_{yk}^{yk+1} y B_{k+1}(y) dy \]

\[ = \int_{yk}^{yk+1} (yk+1 + et) dt + 2A_i(x) A_{i+1}(x) y_k e + e \int_{A_i(x)}^{A_{i+1}(x)} (yk+1 - et) dt \]

\[ + \frac{1}{2} A_{i+1}(x) [1 + 2A_i(x)] y_{k+1} + (A_{i+1}(x) - A_i(x)) y_k e + e \int_{0}^{A_i(x)} (yk-1 + et) dt \]

\[ = e \int_{yk}^{yk+1} (yk+1 + y_{k+2}) dt + e \int_{yk}^{yk+2} (yk+1 + et) dt + 2A_i(x) A_{i+1}(x) y_k e \]

\[ + \frac{1}{2} A_{i+1}(x) [1 + 2A_i(x)] y_{k+1} + (A_{i+1}(x) - A_i(x)) y_k e \]

\[ = \left[ \frac{1}{2} A_i(x) (3 - A_i(x)) y_k + \frac{1}{2} A_{i+1}(x) (3 - A_{i+1}(x)) y_{k+1} \right] e. \]

Similarly, when \( x \in [\tilde{x}_i, x_{i+1}], \) we have

\[ \int_{yk}^{yk+1} B^*(y) dy = \int_{yk}^{yk+1} A_i(x) dy + \int_{yk}^{yk+1} B_{k+1}(y) dy + \int_{yk}^{yk+1} A_{i+1}(x) dy = \frac{1}{2} e, \]
\[ \int_{Y^*(x)} yB^*(y) \, dy = \int_{y_{k-1}^*}^{y_k^*} yB_k(y) \, dy + \int_{y_{k-1}^*}^{y_k^*} yA_i(x) \, dy + \int_{y_{k-1}^*}^{y_k^*} yB_{k+1}(y) \, dy + \int_{y_{k-1}^*}^{y_k^*} yA_{i+1}(x) \, dy + \int_{y_{k-1}^*}^{y_k^*} yB_{k+1}(y) \, dy \]

\[ = e \int_0^{A_{i+1}(x)} (y_{k-1} + \epsilon) \, dt + \frac{1}{2} A_i(x) (y_{k-1} + y_k^* - e) + e \int_{A_{i+1}(x)}^{A_{i+1}(x)} (y_{k+1} + \epsilon + e) \, dt \]

\[ + A_i(x) A_{i+1}(x) (y_k^* + y_{k+1}^*) + e \int_0^{A_{i+1}(x)} (y_{k+2} - e) \, dt \]

\[ = e \int_0^{A_{i+1}(x)} (y_{k-1} + y_{k+2}) \, dt + e \int_{A_{i+1}(x)}^{A_{i+1}(x)} (y_{k} + y_{k+2}) \, dt \]

\[ + \frac{1}{2} A_i(x) [(1 + 2A_{i+1}(x)) y_k + (A_i(x) - A_{i+1}(x)) y_{k+1}] e + 2A_i(x) A_{i+1}(x) y_{k+1} e \]

\[ = \left[ \frac{1}{2} A_i(x)(3 - A_i(x)) y_k + \frac{1}{2} A_{i+1}(x)(3 - A_{i+1}(x)) y_{k+1} \right] e. \]

Thus

\[ \int_{Y^*(x)} B^*(y) \, dy = \frac{1}{2} A_i(x)(1 + A_{i+1}(x)) e + \frac{1}{2} A_{i+1}(x)(1 + A_i(x)) e + \frac{1}{2} e \]

\[ = (1 + A_i(x) A_{i+1}(x)) e \]

and we have from (5) that

\[ \bar{s}(x) = \frac{A_i(x)(3 - A_i(x))}{2(1 + A_i(x) A_{i+1}(x))} y_k + \frac{A_{i+1}(x)(3 - A_{i+1}(x))}{2(1 + A_i(x) A_{i+1}(x))} y_{k+1} = L_i(x) y_k + L_{i+1}(x) y_{k+1}. \]

Clearly, \( L_i(x) + L_{i+1}(x) = 1. \)

(ii) When \( C_{i+1} = B_{k-1}, \) \( y^*(x) = (y_{k-2}, y_{k+1}). \) Then it follows that \( B^*(y) = \begin{cases} A_{i+1}(x), & y \leq y_k^* \\ B_k(y), & y_k^* - 1 \leq y \leq y_{k-1}^* \\ A_i(x), & y_{k-1}^* < y \end{cases} \)

when \( y \in (y_k, y_{k+1}), \) and when \( y \in [y_{k-1}, y_k], \) we have that

\[ B^*(y) = \begin{cases} A_{i+1}(x), & y < y_{k-1}^* \\ B_k(y), & y_{k-1}^* - 1 \leq y \leq y_{k-1}^* \\ A_i(x), & y_{k-1}^* < y \end{cases} \]

Then for \( x \in [x_i, x_i], \) we have that

\[ \int_{Y^*(x)} B^*(y) \, dy = \int_{y_{k-2}^*}^{y_{k-1}^*} B_{k-1}(y) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} A_{i+1}(x) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} B_k(y) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} A_i(x) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} B_k(y) \, dy \]

\[ = e \int_0^{A_{i+1}(x)} t \, dt + A_{i+1}(x) e + e \int_{A_{i+1}(x)}^{A_{i+1}(x)} t \, dt + 2A_i(x) A_{i+1}(x) e \int_0^{A_{i+1}(x)} \, dt \]

\[ = (1 + A_i(x) A_{i+1}(x)) e. \]

\[ \int_{Y^*(x)} yB^*(y) \, dy = \int_{y_{k-2}^*}^{y_{k-1}^*} yB_{k-1}(y) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} yA_{i+1}(x) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} yB_k(y) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} yA_i(x) \, dy + \int_{y_{k-1}^*}^{y_{k-1}^*} yB_k(y) \, dy \]

\[ = e \int_0^{A_{i+1}(x)} (y_{k-2} + \epsilon) \, dt + \frac{1}{2} A_{i+1}(x) [(1 + 2A_i(x)) y_{k-1} + (A_{i+1}(x) - A_i(x)) y_k] e \]

\[ + e \int_{A_{i+1}(x)}^{A_{i+1}(x)} (y_{k+1} - \epsilon) \, dt \]

\[ = \left[ \frac{1}{2} A_i(x)(3 - A_i(x)) y_k + \frac{1}{2} A_{i+1}(x)(3 - A_{i+1}(x)) y_{k+1} \right] e. \]
For \( x \in [\bar{x}_i, x_{i+1}) \), we have that
\[
\int_{y^+(x)} B^*(y)dy = \int_{y_{k-2}}^{y_{k-1}} B_{k-1}(y)dy + \int_{y_{k-2}}^{y_{k-1}} A_{i+1}(x)dy + \int_{y_{k-2}}^{y_{k-1}} B_k(y)dy + \int_{y_{k-1}}^{y_{k+1}} A_i(x)dy + \int_{y_{k+1}}^{y_{k+1}} B_{k+1}(y)dy
\]
\[
= e \int_{0}^{A_{i+1}(x)} tdt + 2A_i(x)A_{i+1}(x)e + \int_{A_{i+1}(x)}^{A_i(x)} tdt + A_i(x)e + e \int_{0}^{A_i(x)} tdt
\]
\[
= (1 + A_i(x)A_{i+1}(x))e.
\]
\[
\int_{y^+(x)} yB^*(y)dy = \int_{y_{k-2}}^{y_{k-1}} yB_{k-1}(y)dy + \int_{y_{k-2}}^{y_{k-1}} yA_{i+1}(x)dy + \int_{y_{k-2}}^{y_{k-1}} yB_k(y)dy + \int_{y_{k-1}}^{y_{k+1}} yA_i(x)dy + \int_{y_{k+1}}^{y_{k+1}} yB_{k+1}(y)dy
\]
\[
= e \int_{0}^{A_{i+1}(x)} (y_{k-2} + et)tdt + 2A_i(x)A_{i+1}(x)y_{k-1}e + e \int_{A_{i+1}(x)}^{A_i(x)} (y_k - et)tdt
\]
\[
+ \frac{1}{2}A_i(x)[(1 + 2A_{i+1}(x))y_k + (A_i(x) - A_{i+1}(x))y_{k-1}]e + e \int_{0}^{A_i(x)} (y_{k+1} - et)tdt
\]
\[
= \left[ \frac{1}{2}A_i(x)(3 - A_i(x))y_k + \frac{1}{2}A_{i+1}(x)(3 - A_{i+1}(x))y_{k-1} \right]e.
\]
It follows that \( \tilde{S}(x) = L_i(x)y_k + L_{i+1}(x)y_{k-1} \).

(iii) Similarly, when \( c_{i+1} = b_{k+1} \), we have \( \tilde{S}(x) = y_k = L_i(x)y_k + L_{i+1}(x)y_{k-1} \).

(iii) Let \( T = T_3 \), then

(i) When \( c_{i+1} = b_{k+1}, y^+(x) = (y_{k-1}, y_{k+2}) \). Then

(1) When \( y \in (y_{k-1}, y_k) \), we have that
\[
B^*(y) = \begin{cases} A_i(x)(A_{i+1}(x) + B_k(y)), & y \leq y_{k-1} \\ A_i(x), & y > y_{k-1} \end{cases}
\]
and
\[
\int_{y_{k-1}}^{y_{k+1}} B^*(y)dy = \int_{y_{k-1}}^{y_{k-1}} A_i(x)(A_{i+1}(x) + B_k(y))dy + \int_{y_{k-1}}^{y_{k+1}} A_i(x)dy
\]
\[
= \frac{1}{2}A_i(x)[1 + A_{i+1}(x) + A_i(x)A_{i+1}(x)]e.
\]

(2) When \( y \in (y_{k+1}, y_{k+2}) \), we have that
\[
B^*(y) = \begin{cases} A_{i+1}(x), & y < y_{k+1} \\ A_{i+1}(x)(A_i(x) + B_{k+1}(y)), & y \geq y_{k+1} \end{cases}
\]
and
\[
\int_{y_{k+1}}^{y_{k+2}} B^*(y)dy = \int_{y_{k+1}}^{y_{k+1}} A_{i+1}(x)dy + \int_{y_{k+1}}^{y_{k+2}} A_{i+1}(x)(A_i(x) + B_{k+1}(y))dy
\]
\[
= \frac{1}{2}A_{i+1}(x)[1 + A_i(x) + A_i(x)A_{i+1}(x)]e.
\]

(3) When \( y \in [y_k, y_{k+1}) \), we have that
\[
A_{i+1}(x) + B_k(y) \leq 1 \iff A_i(x) + B_{k+1}(y) \geq 1 \iff y^\Delta \leq y_k,
\]
and
\[
B^*(y) = \begin{cases} A_i(x)[(A_{i+1}(x) + B_k(y)) \lor 1] \lor A_{i+1}(x)[(A_i(x) + B_{k+1}(y)) \land 1] \\ A_i(x) \lor A_{i+1}(x)(A_i(x) + B_{k+1}(y)), & y < y^\Delta \\ A_{i+1}(x) \lor A_i(x)(A_{i+1}(x) + B_k(y)), & y \geq y^\Delta \end{cases}
\]
Since
\[
A_i(x) \geq A_{i+1}(x)(A_i(x) + B_{k+1}(y)) \iff y \leq \frac{\Delta_i^2(x)}{A_{i+1}(x)}y_{k+1} + \left(1 - \frac{\Delta_i^2(x)}{A_{i+1}(x)}\right)y_k \Delta y^\Delta
\]
\[
A_i(x) \geq A_i(x)(A_{i+1}(x) + B_k(y)) \iff y \geq \left(1 - \frac{\Delta_i^2(x)}{A_i(x)}\right)y_{k+1} + \frac{\Delta_i^2(x)}{A_i(x)}y_k \Delta y^\Delta.
\]
It follows that for \( x \in [x_i, \bar{x}_i] \),

\[
\int_{y_k}^{y_{k+1}} B^*(y) dy = \int_{y_k}^{y_k^1} A_i(x) dy + \int_{y_k^1}^{y_k^2} A_i(x) (A_{i+1}(x) + B_k(y)) dy + \int_{y_k^2}^{y_{k+1}} A_{i+1}(x) dy
\]

\[
= A_i(x)A_{i+1}(x)e + A_i(x)A_{i+1}(x) \left( \frac{A_{i+1}(x) - A_i(x)}{A_i(x)} \right) e + A_i(x)e \int_{y_k^1}^{y_{k+1}} \frac{A_i(x)}{A_{i+1}(x)} \Delta x \ dt + A_i(x)A_{i+1}(x) e
\]

\[
= \frac{1 - 4A_i(x)A_{i+1}(x)}{2A_i(x)} e
\]

and for \( x \in [\bar{x}_i, x_{i+1}) \),

\[
\int_{y_k}^{y_{k+1}} B^*(y) dy = \int_{y_k}^{y_k^1} A_i(x) dy + \int_{y_k^1}^{y_k^2} A_i(x) (A_{i+1}(x) + B_k(y)) dy + \int_{y_k^2}^{y_{k+1}} A_{i+1}(x) dy
\]

\[
= A_i(x)A_{i+1}(x)e + A_i(x)A_{i+1}(x) \left( \frac{A_{i+1}(x) - A_i(x)}{A_i(x)} \right) e + A_i(x)e \int_{y_k^1}^{y_{k+1}} \frac{A_i(x)}{A_{i+1}(x)} \Delta x \ dt + A_i(x)A_{i+1}(x) e
\]

\[
= \frac{1 - 4A_i^2(x)A_{i+1}(x)}{2A_{i+1}(x)} e.
\]

Thus

\[
\int_{y^* (x)}^{y^* (x)} B^*(y) dy = \begin{cases} 
\frac{1}{2A_i(x)}[1 - 3A_i(x) + 11A_i^2(x) - 7A_i^3(x)]e, & x \in [x_i, \bar{x}_i], \\
\frac{1}{2A_{i+1}(x)}[1 - 3A_{i+1}(x) + 11A_{i+1}^2(x) - 7A_{i+1}^3(x)]e, & x \in [\bar{x}_i, x_{i+1}).
\end{cases}
\]

Similarly, for \( x \in [x_i, \bar{x}_i] \), we have that

\[
\int_{y^* (x)}^{y^* (x)} yB^*(y) dy = \int_{y_{k-1}}^{y_k} yA_i(x) (A_{i+1}(x) + B_k(y)) dy + \int_{y_k}^{y_{k+1}} yA_i(x) dy + \int_{y_{k+1}}^{y_{k+2}} yA_{i+1}(x) dy
\]

\[
+ \int_{y_{k+1}}^{y_{k+2}} yA_{i+1}(x)dy + \int_{y_k}^{y_k^1} yA_i(x)dy + \int_{y_k^1}^{y_{k+1}} yA_{i+1}(x)dy
\]

\[
= (C_i(x)y_k + C_{i+1}(x)y_{k+1}) e
\]

where

\[
C_i(x) = \frac{1}{6A_i^2(x)} [6 - 9A_{i+1}(x) - 18A_{i+1}^2(x) + 48A_{i+1}^3(x) - 37A_{i+1}^4(x) + 11A_{i+1}^5(x)],
\]

\[
C_{i+1}(x) = \frac{1}{6A_{i+1}(x)} [9A_{i+1}(x) - 18A_{i+1}^2(x) + 3A_{i+1}^3(x) + 16A_{i+1}^4(x) - 11A_{i+1}^5(x)]
\]

and \( C_i(x) + C_{i+1}(x) = \frac{1}{2A_i(x)} [1 - 3A_i(x) + 11A_i^2(x) - 7A_i^3(x)] \).

For \( x \in [\bar{x}_i, x_{i+1}) \), we have that

\[
\int_{y^* (x)}^{y^* (x)} yB^*(y) dy = \int_{y_{k-1}}^{y_k} yA_i(x) (A_{i+1}(x) + B_k(y)) dy + \int_{y_k}^{y_{k+1}} yA_i(x) dy + \int_{y_{k+1}}^{y_{k+2}} yA_{i+1}(x) dy
\]

\[
+ \int_{y_{k+1}}^{y_{k+2}} yA_{i+1}(x)dy + \int_{y_k}^{y_k^1} yA_i(x)dy + \int_{y_k^1}^{y_{k+1}} yA_{i+1}(x)dy
\]

\[
= (D_i(x)y_k + D_{i+1}(x)y_{k+1}) e
\]
where, $D_i(x) = \frac{1}{6A_{i+1}^1(x)}[9A_i(x) - 18A_i^2(x) + 3A_i^3(x) + 16A_i^4(x) - 11A_i^5(x)],$

$D_{i+1}(x) = \frac{1}{6A_{i+1}^2(x)}[6 - 9A_i(x) - 18A_i^2(x) + 48A_i^3(x) - 37A_i^4(x) + 11A_i^5(x)]$

and $D_i(x) + D_{i+1}(x) = \frac{1}{6A_{i+1}^1(x)}[1 - 3A_{i+1}(x) + 11A_{i+1}^2(x) - 7A_{i+1}^3(x)].$

It follows that $\tilde{S}(x) = M_i(x)y_k + M_{i+1}(x)y_{k+1},$ and $M_i(x) + M_{i+1}(x) = 1.$

(ii) Similarly, when $C_{i+1} = B_{k-1},$ we have that $\tilde{S}(x) = M_i(x)y_k + M_{i+1}(x)y_{k-1},$ and when $C_{i+1} = B_k,$ we have that $\tilde{S}(x) = y_k = M_i(x)y_k + M_{i+1}(x)y_{k-1}.$

4. Universal approximations of fuzzy systems

Fuzzy systems in Theorems 1 and 2 have the following forms:

$\tilde{S}(x) = A_i^*(x)y_k + A_{i+1}^*(x)z_{i+1},$ when $x \in [x_i, x_{i+1}].$

where $A_i^*(x) + A_{i+1}^*(x) = 1.$ For $x \in [x_i, x_{i+1}],$ we set

$S_1(x) = E_i(x)f(x_k) + E_{i+1}(x)f(x_{i+1}); \quad S_2(x) = F_i(x)f(x_k) + F_{i+1}(x)f(x_{i+1});$
The simulation curves of $S_3$ to $\sin(x)$ when $n = 32$
The error curve when $n = 32$

![Fig. 3. The simulation curves of $S_3$ to $\sin(x)$ when $n = 32$](image1)

$S_3(x) = G_i(x)f(x_i) + G_{i+1}(x)f(x_{i+1});$
$S_4(x) = H_i(x)f(x_i) + H_{i+1}(x)f(x_{i+1});$
$S_5(x) = L_i(x)f(x_i) + L_{i+1}(x)f(x_{i+1});$
$S_6(x) = M_i(x)f(x_i) + M_{i+1}(x)f(x_{i+1}).$

Then $S_i(x)$ ($i = 1, 2, \ldots, 6$) have the following forms:

$S(x) = A_i^+(x)f(x_i) + A_{i+1}^+(x)f(x_{i+1})$ and $S(x_i) = f(x_i), S(x_{i+1}) = f(x_{i+1}).$

Then we have the following conclusion.

**Theorem 3.** If $f : [a, b] \to \mathbb{R}$ is a continuous function, then $\forall \varepsilon > 0$, $\exists \bar{S}(x) \in \{S_i(x)\}_{i = 1, \ldots, 6}$ and $S(x) \in \{S_i(x)\}_{i = 1, \ldots, 6}$ such that

1. $\|\bar{S} - f\|_{\infty} < \varepsilon$;
2. $\|\bar{S} - f\|_{\infty} < \frac{\varepsilon}{2}$;
3. $\|\bar{S} - S\|_{\infty} < \frac{\varepsilon}{2}$.

**Proof.** (1) We only consider the case of $z_{i+1} = y_{k+1}$; the others are similar.

From Lemma 1, we choose the positive integer number $N$ such that $e = \frac{d\varepsilon}{N} < \varepsilon$. Then when $x \in [x_i, x_{i+1})$, we have that

$$|\bar{S}(x) - f(x)| \leq A_i^+(x)|f(x) - y_i| + A_{i+1}^+(x)|f(x) - y_{k+1}|$$

$$\leq A_i^+[f(x) - f(x_i)] + |f(x_i) - y_k| + A_{i+1}^+[f(x) - f(x_{i+1})] + |f(x_{i+1}) - y_{k+1}|.$$
Since \( |f(x) - f(x_i)| < \frac{\varepsilon}{2}, |f(x) - f(x_{i+1})| < \frac{\varepsilon}{2}, |f(x_i) - y_k| \leq \frac{\varepsilon}{2}, |f(x_{i+1}) - y_{k+1}| \leq \frac{\varepsilon}{2} \), it follows that \( |\tilde{S}(x) - f(x)| \leq A_i^*(x)[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}] + A_{i+1}^*(x)[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}] = e(A_i^*(x) + A_{i+1}^*(x)) = e \) and consequently \( \|\tilde{S} - f\|_{\infty} = \sup_{x \in [a, b]} |\tilde{S}(x) - f(x)| \leq e < \varepsilon \).

(2) Since
\[
|S(x) - f(x)| \leq A_i^*(x)|f(x) - f(x_i)| + A_{i+1}^*(x)|f(x) - f(x_{i+1})| \\
\leq \frac{e}{2}(A_i^*(x) + A_{i+1}^*(x)) = \frac{e}{2},
\]
so \( \|S - f\|_{\infty} = \sup_{x \in [a, b]} |S(x) - f(x)| \leq \frac{\varepsilon}{2} < \frac{\varepsilon}{2} \).

(3) Since
\[
|\tilde{S}(x) - S(x)| \leq A_i^*(x)|f(x_i) - y_k| + A_{i+1}^*(x)|f(x_{i+1}) - z_{i+1}| \\
\leq \frac{e}{2}(A_i^*(x) + A_{i+1}^*(x)) = \frac{e}{2},
\]
so \( \|\tilde{S} - S\|_{\infty} = \sup_{x \in [a, b]} |\tilde{S}(x) - S(x)| \leq \frac{\varepsilon}{2} < \frac{\varepsilon}{2} \). \qed
Lemma 2 ([4]). Let $F(x) = A_i(x)f(x_i) + A_{i+1}(x)f(x_{i+1})$. If $f(x)$ is continuously bidifferentiable over $[a, b]$, then $\|F - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2$, where $h = \max_{1 \leq i \leq n} |x_{i+1} - x_i|$.

Theorem 4. Let $h = \max_{1 \leq i \leq n} |x_{i+1} - x_i|$. If function $f : [a, b] \to \mathbb{R}$ is continuously bidifferentiable, then

\begin{align*}
(1) \quad & \|S_1 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{6} \|f'\|_\infty h; \\
(2) \quad & \|S_2 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{4} \|f'\|_\infty h; \\
(3) \quad & \|S_3 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{8} \|f'\|_\infty h; \\
(4) \quad & \|S_4 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{2} \|f'\|_\infty h; \\
(5) \quad & \|S_5 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{8} \|f'\|_\infty h; \\
(6) \quad & \|S_6 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{2} \|f'\|_\infty h.
\end{align*}

Proof. (1) When $x \in [x_i, x_{i+1}]$,

$$|E_i(x) - A_i(x)| = \left| \frac{1 + A_i(x)}{2(1 + A_{i+1}(x))} - A_i(x) \right| = \frac{A_i(x)(1 - 2A_i(x))}{2(1 + A_{i+1}(x))} \leq \frac{A_i(x)}{2(1 + A_{i+1}(x))} \leq \frac{1}{6}.$$ 

Let $\varphi(t) = \frac{1}{1+t^2} (t \in [0, \frac{1}{2}])$, then $\varphi'(t) = \frac{-1}{(1+t^2)^2} > 0$. Then max $\varphi(t) = \frac{1}{2}$ and consequently $|E_i(x) - A_i(x)| \leq \frac{A_i(x)}{2(1 + A_{i+1}(x))} \leq \frac{1}{8}.$

When $x \in [x_i, x_{i+1}]$,

$$|E_i(x) - A_i(x)| = \left| \frac{3A_i(x)}{2(1 + A_{i+1}(x))} - A_i(x) \right| = \frac{A_i(x)(1 - 2A_i(x))}{2(1 + A_{i+1}(x))} \leq \frac{A_i(x)}{2(1 + A_{i+1}(x))} \leq \frac{1}{6}.$$ 

By $E_i(x) - A_i(x) = -(E_{i+1}(x) - A_{i+1}(x))$, we have

$$S_1(x) - f(x) = (E_i(x) - A_i(x))f(x_i) + (E_{i+1}(x) - A_{i+1}(x))f(x_{i+1})$$

and

$$|S_1(x) - f(x)| = |E_i(x) - A_i(x)| \|f(x_{i+1}) - f(x_i)\| \leq \frac{1}{6} \|f'\|_\infty |x_{i+1} - x_i| \leq \frac{1}{6} \|f'\|_\infty h.$$

Thus $\|S_1 - f\|_\infty = \sup_{x \in [a, b]} \|S_1(x) - f(x)\| \leq \frac{1}{6} \|f'\|_\infty h$.

From Lemma 2, we have that $\|S_1 - f\|_\infty \leq \|F - f\|_\infty + \|S_1 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{6} \|f'\|_\infty h$.

(2) By

$$|F_i(x) - A_i(x)| = A_i(x)A_{i+1}(x) \left| \frac{A_i^2(x) - A_{i+1}^2(x)}{A_i^2(x) + A_{i+1}^2(x)} \right| \leq A_i(x)A_{i+1}(x) \leq \frac{1}{4},$$

we have that $\|S_2 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{4} \|f'\|_\infty h$.

(3)

By $|G_i(x) - A_i(x)| \leq \frac{1}{2} A_i(x)A_{i+1}(x) \|1 - 2A_i(x)\| \leq \frac{1}{2} A_i(x)A_{i+1}(x) \leq \frac{1}{8}$,

we have that $\|S_3 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{8} \|f'\|_\infty h$.

(4) When $x \in [x_i, x_{i+1}]$,

$$|H_i(x) - A_i(x)| = \frac{A_i(x)}{2A_{i+1}(x)} \left| \frac{1 - 6A_i(x) + 8A_i^2(x)}{1 + 4A_i(x)} \right| \leq \frac{1}{2} \left| \frac{1 - 2A_i(x)}{1 + 4A_i(x)} \right|.$$

Since $A_i(x) \in [0, \frac{1}{2}]$, so $1 - \varphi(t) = \begin{cases} \frac{2(1 - 4t)}{1 + 4t}, & t \in [0, \frac{1}{4}] \\ \frac{2 - 2t + 8t^2}{1 + 4t}, & t \in (\frac{1}{4}, \frac{1}{2}] \end{cases}$, then $0 \leq \varphi(t) \leq 1$.

Since $A_i(x) \in [0, \frac{1}{2}]$, so $\left( \frac{1 - 2A_i(x)}{1 + 4A_i(x)} \right) \leq 1$, i.e., $|H_i(x) - A_i(x)| \leq \frac{1}{2}$.

Similarly, when $x \in [x_i, x_{i+1}]$, we have $|H_{i+1}(x) - A_{i+1}(x)| \leq \frac{1}{2}$. It follows that

$$\|S_4 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{2} \|f'\|_\infty h.$$

(5) By $|L_i(x) - A_i(x)| = \frac{1}{2} A_i(x)A_{i+1}(x) \left| \frac{1 - 2A_{i+1}(x)}{1 + 4A_{i+1}(x)} \right| \leq \frac{1}{2} A_i(x)A_{i+1}(x) \leq \frac{1}{8}$, we have that

$$\|S_4 - f\|_\infty \leq \frac{1}{8} \|f''\|_\infty h^2 + \frac{1}{8} \|f'\|_\infty h.$$
(6) When \( x \in [x_i, x_{i+1}) \),

\[
M_i(x) - A_i(x) = \frac{1}{3} A_{i+1}(x) \frac{3 - 18 A_i(x) + 39 A_i^2(x) - 35 A_i^3(x) + 10 A_i^4(x)}{2 + 2 A_i(x) - 10 A_i^2(x) + 7 A_i^3(x)}.
\]

Let \( \varphi(t) = \frac{3 - 18 t + 39 t^2 - 35 t^3 + 10 t^4}{2 + 2 t - 10 t^2 + 7 t^3} \), \( t \in [0, 1/2] \), then

\[
2 + 2t - 10t^2 + 7t^3 = 2(1 - 3t^2) + 2t(1 - 2t) + 7t^3 > 0,
\]

\[
3 - 18t + 39t^2 - 35t^3 + 10t^4 = (1 - 2t)[3(1 - 2t)(1 - t)^2 + t^3] \geq 0.
\]

\[
2 - \varphi(t) = \frac{t(42 - 78t + 91t^2 - 20t^3)}{2(2 + 2t - 10t^2 + 7t^3)} \geq 0.
\]

Thus \( 0 \leq \varphi(t) \leq \frac{3}{2} \) and \( |M_i(x) - A_i(x)| \leq \frac{1}{2} A_i(x) \leq \frac{1}{2} \). It follows that

\[
\|S_6 - f\| \leq \frac{1}{8} \|f''\| \|h^2 + \frac{1}{2} \|f'''\|h.\]

\( \square \)

**Note 3.** (1) **Theorem 3** has shown that fuzzy systems \( \tilde{S}_i(x) \) and \( S_i(x) \) are the universal approximators to \( f(x) \). (2) **Theorem 4** has shown that the fuzzy system \( S_i(x) \) has first order approximation accuracy to \( f(x) \). (3) **Theorem 4** has also given the sufficient conditions for the fuzzy systems \( (S_i) \) as universal approximators.

**Example 1.** Let \( f(x) = \sin x \), \( [a, b] = [-3, 3] \), then \( \|f''\| \|h\| = \|f''\| \|h\| = 1 \). Let \( \varepsilon = 0.1 \),

\[
h = 3(3 - 3\varepsilon) = \frac{6}{5}.
\]

By applying **Theorem 4**, we have that

If \( \frac{1}{8} \|f''\| \|h^2 + \frac{1}{2} \|f'''\|h < 0.1 \), then \( n \geq 14 \). If \( \frac{1}{8} \|f''\| \|h^2 + \frac{1}{2} \|f'''\|h \leq 0.1 \), then \( n \geq 18 \).

Then, when \( n = 9 \), \( \|S_1 - f\| \leq 0.1 \) \( (i = 3, 5) \); when \( n = 14 \), \( \|S_1 - f\| \leq 0.1 \).

When \( n = 18 \), \( \|S_1 - f\| \leq 0.1 \); when \( n = 32 \), \( \|S_1 - f\| \leq 0.1 \) \( (i = 2, 6) \).

The simulation curves of \( S_i(x) \) (\( i = 1, 2, \ldots, 6 \)) to \( f(x) \) and their error curves are shown in Figs. 1–6 when \( n = 32 \), respectively.

### 5. Conclusions

The method of a non-singleton fuzzifier for the input variable and the concept of the adaptive universe for the output fuzzy set are proposed. Based on this fuzzifier and the center-of-gravity defuzzifier, SISO fuzzy systems based on the normal implications such as the Kleene–Dienes implication or the Lukasiewicz implication were obtained. It is pointed out that these fuzzy systems are universal approximators. This work has shown that we can construct fuzzy systems to approximate a desired control function up to a given level of accuracy by applying the normal implications and the center-of-gravity defuzzifier.

### References


