Essentially compressible modules and rings

P.F. Smith\textsuperscript{a}, M.R. Vedadi\textsuperscript{b,c,*}

\textsuperscript{a} Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland, UK
\textsuperscript{b} Department of Mathematics, Isfahan University of Technology, Isfahan, Iran
\textsuperscript{c} Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, Iran

Received 16 July 2005
Available online 26 September 2005
Communicated by Kent R. Fuller

Abstract

Let $R$ be a ring with identity and let $M$ be a unitary right $R$-module. Then $M$ is essentially compressible provided $M$ embeds in every essential submodule of $M$. It is proved that every non-singular essentially compressible module $M$ is isomorphic to a submodule of a free module, and the converse holds in case $R$ is semiprime right Goldie. In case $R$ is a right FBN ring, $M$ is essentially compressible if and only if $M$ is subisomorphic to a direct sum of critical compressible modules. The ring $R$ is right essentially compressible if and only if there exist a positive integer $n$ and prime ideals $P_i$ ($1 \leq i \leq n$) such that $P_1 \cap \cdots \cap P_n = 0$ and the prime ring $R/P_i$ is right essentially compressible for each $1 \leq i \leq n$. It follows that a ring $R$ is semiprime right Goldie if and only if $R$ is a right essentially compressible ring with at least one uniform right ideal.

© 2005 Elsevier Inc. All rights reserved.

1. Essentially compressible modules

Rings will have units elements and modules will be unitary. The terminology not defined here may be found in [2,6]. Let $R$ be any ring and let $M$ be a right $R$-module. Following [8, 6.9.3], the right $R$-module $M$ is called compressible if for each non-zero submodule $N$ of $M$, there exists a monomorphism $\theta : M \to N$. For example, if $R$ is any domain then

\* Corresponding author.
E-mail addresses: pfs@maths.gla.ac.uk (P.F. Smith), mrvedadi@cc.iut.ac.ir (M.R. Vedadi).
every right ideal of $R$ is a compressible $R$-module. An $R$-module $M$ will be called \textit{essentially compressible} if, for each essential submodule $N$ of $M$, there exists a monomorphism $\theta : M \to N$. Clearly every compressible module is essentially compressible. However, every semisimple module is essentially compressible but need not be compressible. We shall say that an $R$-module $M$ is \textit{subisomorphic} to an $R$-module $M'$ if there exist $R$-monomorphisms $\alpha : M \to M'$ and $\beta : M' \to M$, and in this case we call the modules $M$ and $M'$ subisomorphic. We begin with the following proposition.

\textbf{Proposition 1.1.} The following statements are equivalent for a module $M$.

(a) $M$ is essentially compressible.

(b) $M$ is subisomorphic to an essentially compressible module.

(c) $M$ contains an essentially compressible submodule $N$ such that there exists a monomorphism $\varphi : M \to N$.

(d) There is an essential monomorphism $\psi : M \to M'$ for some essentially compressible module $M'$.

\textbf{Proof.} (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (d) are clear.

(b) $\Rightarrow$ (c). Suppose that there exist an essentially compressible module $M'$ and monomorphisms $\alpha : M \to M'$, $\beta : M' \to M$. Let $N = \beta(M')$. Then $N$ is an essentially compressible submodule of $M$ and $\beta \alpha : M \to N$ is a monomorphism.

(c) $\Rightarrow$ (a). Let $L$ be any essential submodule of $M$. Then $L \cap N$ is an essential submodule of $N$ and so there is a monomorphism $\theta : N \to L \cap N$. If $\iota : L \cap N \to L$ is the inclusion mapping then $\iota \theta \varphi : M \to L$ is a monomorphism. It follows that $M$ is essentially compressible.

(d) $\Rightarrow$ (b). Let $N = \psi(M)$. By our assumption, $M'$ can be embedded in $N$ and hence in $M$. Thus $M$ and $M'$ are subisomorphic.

\textbf{Proposition 1.2.} Every direct sum of essentially compressible modules is essentially compressible.

\textbf{Proof.} Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of essentially compressible modules $M_i$ ($i \in I$), for some (non-empty) index set $I$. Let $L$ be an essential submodule of $M$. Then, for each $i \in I$, $L \cap M_i$ is an essential submodule of $M_i$ and hence there exists a monomorphism $\theta_i : M_i \to L \cap M_i$. Clearly the mapping $\theta = \sum_{i \in I} \theta_i : M \to L$ is a monomorphism. \hfill \Box

\textbf{Proposition 1.3.} Being essentially compressible is a Morita invariant property.

\textbf{Proof.} In fact a module $M_R$ is essentially compressible if and only if for any $X \in \text{Mod-}R$ with an essential monomorphism $X_R \to M_R$, there is a monomorphism $M_R \to X_R$. Thus the result follows from the known fact that any category equivalence preserves (essential) monomorphisms (see, for example, [6, 18.26]). \hfill \Box
We recall some definitions and notations. Let $M$ be a right $R$-module and $X$ be any non-empty subset of $M$. We denote $\{r \in R : Xr = 0\}$ by $\text{ann}_R(X)$ (or by $r.\text{ann}_R(X)$ for more emphasis). A right $R$-module $N$ is said to be $M$-generated if there exists an epimorphism from a direct sum of copies of $M_R$ to $N_R$. We denote by $\sigma[M_R]$, the full subcategory of mod-$R$ whose objects are all right $R$-submodules of $M$-generated modules. If $N$ is an essential submodule of an injective module $E$ in the category $\sigma[M_R]$, then $E$ is called an $M$-injective hull of $N$ and is usually denoted by $\hat{N}$ (see [2, 1.9]). Finally, $M$ is said to be co-Hopfian if every injective endomorphism of $M_R$ is an isomorphism.

We collect further properties of essentially compressible modules in the following proposition.

**Proposition 1.4.** Let $M$ be a non-zero essentially compressible right $R$-module.

(a) If $N$ is either an essential submodule of $M_R$ or a submodule which is invariant under injective endomorphisms of $M_R$ then $N_R$ is also an essentially compressible module.

(b) If $N$ is a submodule of $M_R$ such that $\theta(N) + \theta^{-1}(N) \subseteq N$ for every monomorphism $\theta : M \to M$, then $M/N$ is an essentially compressible module.

(c) $M_R$ is semisimple co-Hopfian if and only if $M_R$ has a co-Hopfian essentially compressible submodule.

(d) $\text{ann}_R(M)$ is a semiprime ideal of $R$.

(e) $\hat{M}_R$ has no fully invariant essential submodule.

(f) If in addition $M_R$ is finitely generated, then $M_R$ does not contains an infinite direct sum of non-zero fully invariant submodules.

**Proof.** (a) Let $N$ be a submodule of $M$. If $N$ is essential then it is easy to check that $N_R$ is essentially compressible. Let $N$ be invariant under injective endomorphisms of $M_R$ and let $K$ be any essential submodule of $N_R$. There exists a submodule $N'$ of $M_R$ such that $K \cap N' = 0$ and $N \oplus N'$ is an essential submodule of $M_R$. Then $K \oplus N'$ is essential in $N \oplus N'$, and hence $K \oplus N'$ is essential in $M_R$. So by our assumption there is a monomorphism $f : M \to K \oplus N'$. Now $f(N) \subseteq N$ by our assumption, so $f(N) \cap N' = 0$. It follows that $f(N)$, and hence $N$, is embedded in $(K \oplus N')/N' \simeq K$. Therefore $N_R$ is essentially compressible.

(b) Let $L$ be any submodule of $M_R$, containing $N$, such that $L/N$ is an essential submodule of $M/N$. It is easy to check that $L$ is an essential submodule of $M_R$. By hypothesis, there exists a monomorphism $\varphi : M \to L$. Because $\varphi(N) + \varphi^{-1}(N) \subseteq N$, it follows that the induced mapping $\bar{\varphi} : M/N \to L/N$, defined by $\bar{\varphi}(m + N) = \varphi(m) + N$ for each $m \in M$, is a monomorphism. The result follows.

(c) The necessity is clear. Conversely, clearly any co-Hopfian essentially compressible module is semisimple. Thus, by (a), $M_R$ has an essential socle. It follows that $M_R$ is semisimple.

(d) Let $A = \text{ann}_R(M)$. Note that $A$ is a proper ideal of $R$. Let $B$ be any ideal of $R$ such that $B^2 \subseteq A$. Let $L = \{m \in M : mB = 0\}$. If $0 \neq m \in M$ then $mB^2 = 0$ so that there exists $n \in \{0, 1\}$ such that $mB^n \neq 0$ and $mB^{n+1} = 0$ (say $B^0 = R$) and in this case $0 \neq mB^n \subseteq mR \cap L$. It follows that $L$ is an essential submodule of $M_R$. By hypothesis, there exists
a monomorphism $\theta : M \to L$. Then $\theta(MB) = \theta(M)B \subseteq LB = 0$, so that $MB = 0$ and hence $B \subseteq A$. Thus $A$ is a semiprime ideal.

(e) Let $N$ be a fully invariant essential submodule of $M_R$. By hypothesis, there exist a submodule $L$ of $N$ and an isomorphism $\theta : L \to M$. Now $\theta$ can be extended to $\bar{\theta} \in \text{End}_R(M)$. Hence $M = \theta(L) = \bar{\theta}(L) \subseteq \bar{\theta}(N) \subseteq N$. It follows that $\hat{M} = \text{End}_R(M)M \subseteq \text{End}_R(\bar{M})N = N$.

(f) Let $N = N_1 \oplus N_2 \oplus \cdots$ be any direct sum of fully invariant submodules of $M_R$. It is well known that there exists a submodule $K$ of $M_R$ such that $N \cap K = 0$ and $N \oplus K$ is an essential submodule of $M_R$. By our assumption, there is a monomorphism $\varphi : M \to N \oplus K$. Since $M_R$ is finitely generated, we can assume that $\varphi(M) \subseteq N_1 \oplus \cdots \oplus N_t \oplus K$ for some positive integer $t$. It follows, by hypothesis, that $\varphi(N_{t+1} \oplus N_{t+2} \cdots) \subseteq \varphi(M) \cap (N_{t+1} \oplus N_{t+2} \cdots) = 0$. Thus $N_{t+1} \oplus N_{t+2} \cdots$ must be zero. □

**Corollary 1.5.** Let $M$ be an essentially compressible module over a commutative ring $R$. Then the uniform dimension of $S_M$ is finite where $S = \text{End}_R(M)$.

**Proof.** Let $N$ be an $S$-submodule of $M$ and $r \in R$. Define $\theta_r : M \to M$ by $\theta_r(m) = mr$. Because $R$ is commutative, $\theta_r \in S$. Thus $\theta_r(N) \subseteq N$. It follows that $N$ is a fully invariant submodule of $M_R$. The result is now clear by Proposition 1.4(f). □

Let $R$ be a ring. A non-zero right $R$-module $M$ is said to be prime if $\text{ann}_R(M) = \text{ann}_R(N)$ for all non-zero submodules $N$ of $M_R$ (similarly, prime left $R$-modules are defined). The next proposition gives some information about finitely generated essentially compressible modules and we use it in Theorem 4.12, to characterize semiprime right Goldie rings.

**Proposition 1.6.** Let $M_R$ be a finitely generated essentially compressible module with $S = \text{End}_R(M)$. If $S_M$ is a prime module then for each non-zero submodule $U$ of $M_R$, there exist a positive integer $n$ and $f_i \in \text{Hom}_R(U, M) (1 \leq i \leq n)$ such that $M$ can be embedded into $\sum_{i=1}^n f_i(U)$. If furthermore, $M_R$ is non-singular then $M_R$ has finite uniform dimension if and only if $M_R$ has a uniform submodule.

**Proof.** Let $U$ be any non-zero submodule of $M_R$. Let $N = \sum \{ f(U) : f : U_R \to M_R \}$. It is easy to check that $N$ is a non-zero fully invariant submodule of $M_R$. There exists a submodule $K$ of $M_R$ such that $N \oplus K$ is essential in $M_R$. By hypothesis, there is monomorphism $\theta : M \to N \oplus K$. Let $\varphi = \pi \theta$ where $\pi : N \oplus K \to K$ is the canonical projection. Now $\varphi(N) \subseteq N \cap K = 0$, and so by the prime condition on $S_M$, $\varphi$ is zero element of $S$. It follows that $\theta(M) \subseteq N$. Since $M_R$ is finitely generated, there exist a positive integer $n$ and $f_i \in \text{Hom}_R(U, M) (1 \leq i \leq n)$ such that $\theta(M) \subseteq f_1(U) + \cdots + f_n(U)$. The first statement is now clear. For the second part, let furthermore, $M_R$ be non-singular. If $U$ is uniform, define a mapping $\phi : U^{(n)} \to f_1(U) + \cdots + f_n(U)$ by $\phi(u_1, \ldots, u_n) = f_1(u_1) + \cdots + f_n(u_n)$ for all $u_i \in U (1 \leq i \leq n)$. Clearly $\phi$ is a homomorphism. Note that the module $U^{(n)}$ has uniform dimension $n$. Moreover, the $R$-module $U^{(n)}/\ker \phi$, is non-singular. By [2, 1.10 and 5.10(1)], $\sum_{i=1}^n f_i(U) \simeq U^{(n)}/\ker \phi$ has finite uniform dimension and hence so too
does $M_R$. It is well known that every module with finite uniform dimension has a uniform submodule. Thus $M_R$ has finite uniform dimension if and only if $M_R$ has a uniform submodule, as desired. □

The following lemma is also needed.

**Lemma 1.7.** Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform submodules $M_i$ ($i \in I$). Let $N$ be any non-zero submodule of $M$. Then there exist a subset $I'$ of $I$ and an essential monomorphism $\theta : N \rightarrow \bigoplus_{i \in I'} M_i$.

**Proof.** If $N \cap M_i \neq 0$ for all $i \in I$ then $N$ is an essential submodule of $M$ and there is nothing to prove. Suppose that $N \cap M_j = 0$ for some $j \in I$. By Zorn’s Lemma there exists a maximal subset $I''$ of $I$ such that $N \cap (\bigoplus_{i \in I''} M_i) = 0$. Note that $I''$ is a proper subset of $I$ and hence $I' = I \setminus I''$ is a non-empty set. Let $\pi : M \rightarrow \bigoplus_{i \in I'} M_i$ denote the canonical projection. Then $\pi |_N : N \rightarrow \bigoplus_{i \in I'} M_i$ is a monomorphism because $\ker(\pi |_N) = N \cap (\bigoplus_{i \in I''} M_i) = 0$. Let $k \in I'$. By the choice of $I''$, $N \cap \{M_k \oplus (\bigoplus_{i \in I''} M_i)\} \neq 0$ and hence $\pi(N) \cap M_k \neq 0$. It follows that $\pi(N) \cap M_i \neq 0$ for all $i \in I'$ so that $\pi(N)$ is an essential submodule of $\bigoplus_{i \in I'} M_i$. □

**Proposition 1.8.** Let $M$ be a direct sum of uniform compressible modules. Then any non-zero submodule of $M$ is an essentially compressible module.

**Proof.** By Lemma 1.7 and Propositions 1.2 and 1.4(a). □

Recall that the singular submodule $Z(M)$ of a right $R$-module $M$ is defined by $Z(M) = \{m \in M : mA = 0 \text{ for some essential right ideal } A \text{ of } R\}$. The module $M$ is called singular if $M = Z(M)$ and is called non-singular if $Z(M) = 0$.

The next result is about quasi-injective essentially compressible modules. We shall use it, in Theorem 5.9, to investigate rings $R$ for which the injective hull $E(R_R)$ is an essentially compressible right $R$-module.

**Proposition 1.9.** Let $\hat{M}_R$ be an essentially compressible module. Then:

(a) either $M_R$ is a semisimple module or $M_R$ has an infinite descending chain $B_1 \supset A_1 \supset B_2 \supset A_2 \supset \cdots$ such that each $A_i$ is an essential submodule of $B_i$ and each $B_i$ is a submodule of $M_R$ isomorphic to $\hat{M}_R$;
(b) if $M_R$ has DCC on direct summands then $M_R$ is a semisimple module; and
(c) $M = Z(\hat{M}) \oplus L$ for some essential compressible submodule $L$.

**Proof.** (a) Let $\hat{M}_R$ be essentially compressible and not semisimple, then $M_R$ has a submodule $B_1$ isomorphic to $\hat{M}_R$. Since $M_R$ is not semisimple, $B_1$ is not semisimple and so it has a proper essential submodule $A_1$. Again by the essentially compressible condition on $B_1$, $A_1$ has a submodule $B_2$ isomorphic to $B_1$. But since $A_1$ is not $M$-injective (otherwise $A_1 = B_1$), $B_2$ is a proper submodule of $A_1$. Proceed to obtain the infinite descending chain $B_1 \supset A_1 \supset B_2 \supset A_2 \supset \cdots$, as desired.
(b) It follows from (a).
(c) Let $Z = Z(M_R)$. Then there exists a submodule $K$ of $M_R$ such that $N = Z \oplus K$ is essential in $M_R$. By our assumption $N$ has a submodule $A$ isomorphic to $M_R$. Because $A$ is $M$-injective, there is a submodule $B$ of $N$ such that $N = A \oplus B$. Now, we have $Z = Z(N) = Z(A) \oplus Z(B)$. Consequently, we have $A \subseteq N = Z(A) \oplus C$ where $C = Z(B) \oplus K$. It follows that $A = Z(A) \oplus (C \cap A)$. Because $A \simeq M_R$, $M = Z \oplus L$ for some submodule $L$. Also $L \simeq M/Z$ is an essentially compressible right $R$-module by Proposition 1.4(b), as desired. □

The following theorem shows that the study of essentially compressible modules reduces to the study of such modules when they are either singular or non-singular.

**Theorem 1.10.** The following statements are equivalent for a module $M$.

(a) $M$ is essentially compressible.
(b) $M \simeq M_1 \oplus M_2$ where $M_1$ is a semisimple module and $M_2$ is an essentially compressible module with zero socle.
(c) $M$ is subisomorphic to $M_1 \oplus M_2$ where $M_1$ is a non-singular essentially compressible module and $M_2$ is a singular essentially compressible module.

**Proof.** (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a), by Propositions 1.1 and 1.2.
(a) $\Rightarrow$ (b). Let $M$ be a essentially compressible module and $S = \text{Soc}(M)$. Then there exists a submodule $K$ of $M$ such that $N = S \oplus K$ is an essential submodule. By hypothesis, there is a monomorphism $\theta : M \to N$. Let $U = \text{Soc}(\theta(M))$. Then $U \subseteq \text{Soc}(N) = S$. Because $S$ is semisimple, $S = U \oplus L$ for some submodule $L$ of $S$. Thus we have $U \subseteq \theta(M) \subseteq N = U \oplus L \oplus K$. It follows that $U$ is a direct summand of $\theta(M)$. Because $M$ is isomorphic to $\theta(M)$, there is submodule $N'$ of $M$ such that $M = S \oplus N'$. Now by Proposition 1.4(b) $N'$ is also essentially compressible, as desired.
(a) $\Rightarrow$ (c). Let $Z = Z(M)$ and $Z \oplus L$ be essential in $M$ for some submodule $L$ of $M$. The module $Z \oplus L$ is essentially compressible by Proposition 1.4(a) and is subisomorphic to $M$ because there is a monomorphism $\varphi : M \to Z \oplus L$. By Proposition 1.4, $Z$ and $L \simeq (Z \oplus L)/Z$ are essentially compressible modules. □

The following theorems gives some information about non-singular essentially compressible modules.

**Theorem 1.11.** For any ring $R$, every non-singular essentially compressible $R$-module is isomorphic to a submodule of a free $R$-module.

**Proof.** Let $M$ be any non-zero non-singular essentially compressible right $R$-module. By Zorn’s Lemma there exist an index set $I$ and non-zero elements $m_i \in M$ ($i \in I$) such that $\sum_{i \in I} m_i R$ is an essential submodule of $M_R$. Let $i \in I$. Let $A_i = rannR(m_i)$. Then $A_i$ is a right ideal of $R$ and hence there is a right ideal $B_i$ of $R$ such that $A_i \oplus B_i$ is an essential right ideal of $R$. Let $r \in R$ such that $m_ir \neq 0$. Then $rE \subseteq A_i \oplus B_i$ for some essential right ideal $E$ of $R$. Because $M$ is non-singular, we have $0 \neq m_irE \subseteq m_iB_i$. Thus $m_iB_i$ is as essential
submodule of $m_i R$. Note further that the mapping $\varphi_i : B_i \to m_i B_i$ defined by $\varphi_i(b) = m_i b$ ($b \in B_i$) is clearly an isomorphism. Finally note that $\bigoplus_{i \in I} m_i B_i$ is an essential submodule of $M$ and is isomorphic to the submodule $\bigoplus_{i \in I} B_i$ of the free $R$-module $R^{(I)}$. Because $M$ is essentially compressible, there exists a monomorphism $\theta : M \to R^{(I)}$. \[\square\]

A non-zero $R$-module $M$ is said to have enough uniforms if every non-zero submodule of $M$ contains a uniform submodule.

**Theorem 1.12.** The following statements are equivalent for a non-zero module $M$ over an arbitrary ring $R$.

(a) $M_R$ is non-singular, essentially compressible and has enough uniforms.

(b) $M_R$ is subisomorphic to a direct sum $\bigoplus_{i \in I} A_i$ of non-singular uniform right ideals $A_i$ ($i \in I$) of $R$ such that $A_i$ does not contain a non-zero nilpotent right ideal of $R$ for each $i \in I$.

(c) $M$ is non-singular and $M$ embeds in a direct sum of uniform compressible right $R$-modules.

**Proof.** (a) $\Rightarrow$ (b). By Zorn’s Lemma there exists a maximal collection of uniform cyclic submodules $U_i$ ($i \in I$) of $M_R$ such that $\sum_{i \in I} U_i$ is direct. It is easy to check that $\bigoplus_{i \in I} U_i$ is an essential submodule of $M_R$. Let $i \in I$ and let $U = U_i$. There is $x \in U$ such that $U = x R$. Let $C = r.\text{ann}_R(x)$. Note that $U$ is a non-singular $R$-module so that $C$ is not an essential right ideal of $R$. There exists a non-zero right ideal $A$ of $R$ such that $A \cap C = 0$. Note that $A \simeq x A$ and that $A$ is a non-singular uniform right ideal of $R$. Let $B$ a right ideal of $R$ such that $B^2 = 0$ and $B \subseteq A$. By Proposition 1.4(d), $MB = 0$ so that $xB = 0$ and we have $B \subseteq A \cap C = 0$. Thus $A$ does not contain a non-zero nilpotent right ideal of $R$. Consequently, for each $i \in I$ there is a non-singular uniform right ideal $A_i$ of $R$ such that $A_i$ does not contain a non-zero nilpotent right ideal of $R$ and $A_i$ is isomorphic to a non-zero submodule of $U_i$. It follows that $\bigoplus_{i \in I} A_i$ is isomorphic to an essential submodule of $M_R$. Because $M_R$ is essentially compressible, there exists a monomorphism $\theta : M \to \bigoplus_{i \in I} A_i$. This proves (b).

(b) $\Rightarrow$ (c). Suppose that (b) holds. Clearly $M_R$ is non-singular. It is enough to show that $A_i$ is a compressible right $R$-module for each $i \in I$. So suppose that $i \in I$ and let $A = A_i$. If $B$ is a non-zero submodule of $A$ then by our assumption, $BA \neq 0$. Then there exists $b \in B$ such that $bA \neq 0$. It follows that the mapping $\theta_b : A \to B$ defined by $\theta_b(a) = ba$ for all $a \in A$, is a non-zero homomorphism. Note that any non-zero homomorphism from a uniform module to a non-singular module is a monomorphism. This proves that $A$ is compressible, as desired.

(c) $\Rightarrow$ (a). If $m$ is any non-zero element of $M$ then $mR$ can be embedded in a finite direct sum of uniform modules and hence $mR$ contains a uniform submodule. Thus $M$ has enough uniforms. Finally, $M$ is essentially compressible by Proposition 1.8. \[\square\]

**Remark 1.13.** A non-zero right $R$-module $M$ is said to be critically compressible if it is compressible and, in addition, it cannot be embedded in any of its proper factor modules [9]. Clearly, non-singular uniform compressible modules are critically compressible.
In [9], Zelmanowitz studied right weakly primitive rings. They are rings which have faithful critically compressible right $R$-modules. Now, if $M_R$ is subisomorphic to a direct sum of uniform compressible right $R$-modules, as in Theorem 1.12, then $R/\text{ann}_R(M)$ is a subdirect product of right weakly primitive rings.

2. Essentially compressible modules over certain rings

The ring $R$ is said to be right semi-artinian if $R/A$ has a non-zero socle for every proper right ideal $A$ of $R$. It is easy to check that over a right semi-artinian ring every essentially compressible module is semisimple. In this section we investigate essentially compressible modules over right hereditary rings and right FBN rings. Also non-singular essentially compressible modules over semiprime right Goldie rings are characterized.

Recall that the ring $R$ is right (left) hereditary if every right (left) ideal of $R$ is projective, and $R$ is hereditary if it is right and left hereditary. Right hereditary rings $R$ are right non-singular rings (i.e. $Z(R_R) = 0$).

Proposition 2.1. Over a right hereditary ring $R$, a right $R$-module $M$ is essentially compressible if and only if $M = M_1 \oplus M_2$ is a direct sum of a singular essentially compressible submodule $M_1$ and a projective essentially compressible submodule $M_2$.

Proof. The sufficiency follows by Proposition 1.2. Conversely, suppose that $M$ is an essentially compressible right $R$-module. Let $Z = Z(M_R)$. By a well-known result $M/Z$ is non-singular (see, for example, [4, Proposition 3.29]). Also by Proposition 1.4(b) $M/Z$ is essentially compressible. Now Theorem 1.11 gives that $M/Z$ embeds in a free module. Since $R$ is right hereditary, it follows that $M/Z$ is projective. Thus $M = Z \oplus Z'$ for some projective essentially compressible submodule $Z'$ of $M$. Finally, by Proposition 1.4(a), $Z$ is a singular essentially compressible module. $\Box$

Proposition 2.2. Let $R$ be a ring with finite right uniform dimension. Then:

(a) a non-singular right $R$-module $M$ is essentially compressible if and only if $M_R$ embeds in a direct sum of uniform compressible right $R$-modules;
(b) every submodule of a non-singular essentially compressible right $R$-module is essentially compressible;
(c) if $M_R$ is a non-singular essentially compressible module then $R/\text{ann}_R(M)$ is a semiprime right Goldie ring.

Proof. (a) By Theorem 1.12, it is enough to prove that $M_R$ has enough uniforms. Let $0 \neq m \in M$. Then $mR \simeq R/A$ where $A = r.\text{ann}_R(m)$. By the non-singular condition on $mR$, $A$ is an essentially closed right ideal of $R$. Thus by [2, 1.10 and 5.10(1)], $mR$ has finite Goldie dimension and hence it has a uniform submodule. This shows that $M_R$ has enough uniforms.

(b) Clear by (a).
(c) Let $A = \text{ann}_R(M)$. Suppose that $rE \subseteq A$ for some $r \in R$ and essential right ideal $E$ of $R$. Then $MrE = 0$ and hence $Mr \subseteq Z(M) = 0$, i.e., $r \in A$. By [2, 5.10(1)], the ring $R/A$ is right non-singular and it has finite right uniform dimension. Moreover, $R/A$ is a semiprime ring by Proposition 1.4(d). Therefore $R/A$ is a semiprime right Goldie ring (see, for example [6, 11.13]). □

**Theorem 2.3.** Over a semiprime right Goldie ring $R$, a right $R$-module $M$ is non-singular and essentially compressible if and only if $M$ embeds in a free $R$-module.

**Proof.** The necessity follows by Theorem 1.11. Conversely, suppose that $M_R$ can be embedded in a free right $R$-module $F$. By [4, Corollary 5.4], $F$, and hence also $M_R$, is a non-singular module. There exist a positive integer $n$ and uniform right ideals $U_i$ ($1 \leq i \leq n$) of $R$ such that $U_1 \oplus \cdots \oplus U_n$ is an essential right ideal of $R$. Next, [4, Proposition 5.9] gives that $U_1 \oplus \cdots \oplus U_n$ contains a regular element $c$ of $R$ and hence $R \simeq cR \subseteq U_1 \oplus \cdots \oplus U_n$. Thus $F$, and hence $M_R$, embeds in a direct sum $\bigoplus_{i \in I} A_i$ of uniform right ideals $A_i$ ($i \in I$) of $R$. By Theorem 1.12, $M_R$ is essentially compressible. □

**Corollary 2.4.**

(a) Over a semiprime right Goldie ring every projective module is essentially compressible.

(b) Over a semiprime right and left Goldie ring every finitely generated non-singular module is essentially compressible.

**Proof.** By Theorem 2.3 and [4, Proposition 6.19]. □

**Remark 2.5.** In [5, 2.2.14], it is proved that if $R$ is a semiprime right Goldie ring then $M_R$ is a submodule of a finitely generated free right $R$-module if and only if $M_R$ is non-singular essentially compressible with finite uniform dimension. Now since finite uniform dimensional submodules of non-singular free right $R$-modules are indeed submodules of finitely generated free right $R$-modules (see, for example, the proof of (b) ⇒ (a) of [5, 2.2.14]), it is seen that [5, 2.2.14] follows from Theorem 2.3.

The *Krull dimension* of a module $M$, when defined, is denoted by $\text{K.dim}(M)$ and it is defined by a transfinite induction. First, $\text{K.dim}(M) = -1$ for $M = 0$. Second, consider an ordinal $\alpha \geq 0$; assuming that we have already defined which modules have Krull dimension $\beta$ for ordinals $\beta < \alpha$, we now say $\text{K.dim}(M) = \alpha$ if:

(a) we have not already defined $\text{K.dim}(M) = \beta$ for some $\beta < \alpha$;

(b) for every (countable) descending chain $M_0 \supseteq M_1 \supseteq \cdots$ of submodules of $M$, we have $\text{K.dim}(M_i/M_{i+1}) < \alpha$ for all but finitely many indices $i$.

A non-zero module $M$ is called $\alpha$-critical if $\text{K.dim}(M) = \alpha$ and $\text{K.dim}(M/N) < \alpha$ for all non-zero submodules $N$ of $M$. A module $M$ is called critical if it is $\alpha$-critical for some ordinal $\alpha$. It is well known that, critical modules are uniform, Noetherian modules have
Krull dimension and any non-zero module with Krull dimension has a non-zero critical submodule (see [4, Chapter 13]).

Recall that a prime ring $R$ is called right bounded if every essential right ideal of $R$ contains a non-zero two sided ideal. Following [4, p. 132], a ring $R$ is called a right FBN ring if $R$ is a right Noetherian ring such that $R/P$ is a right bounded ring for every prime ideal $P$ of $R$.

**Theorem 2.6.** Let $R$ be a right FBN ring. Then a right $R$-module $M$ is essentially compressible if and only if $M$ embeds in a direct sum of critical compressible right $R$-modules.

**Proof.** The sufficiency follows by Proposition 1.8. Conversely, suppose that $M$ is essentially compressible. Note that, by hypothesis, every cyclic right $R$-module is Noetherian and hence it has a critical submodule. Thus any non-zero submodule of $M_R$ contains a critical non-zero submodule. As in the proof of Theorem 1.12, there exist critical submodules $U_i$ $(i \in I)$ such that $\sum_{i \in I} U_i$ is direct and $\bigoplus_{i \in I} U_i$ is an essential submodule of $M$. Let $i \in I$ and $U = U_i$. Let $P = \{ r \in R : Vr = 0 \text{ for some non-zero submodule } V \text{ of } U \}$. Then $P$ is an ideal of $R$. Because $P$ is a finitely generated right ideal of $R$ there exists a non-zero submodule $U'$ of $U$ such that $P = \text{ann}_R(U')$ and hence $P$ is a prime ideal of $R$. By [4, Corollary 8.3], $U'$ is a non-singular right $(R/P)$-module. By [4, Lemma 6.17] $U'$ has an $R/P$-submodule isomorphic to a non-zero right ideal of $R/P$. Thus $U'$ contains a non-zero compressible $R/P$-submodule $Y$. Clearly $Y$ is a compressible right $R$-module. Consequently, for each $i \in I$, $U_i$ contains a non-zero compressible submodule $X_i$. Note that $X_i$ is also a critical submodule of $M$. Moreover $X = \bigoplus_{i \in I} X_i$ is an essential submodule of $M$. Thus there exists a monomorphism $\theta : M \to X$. \qed

In particular, Theorem 2.6 applies to commutative Noetherian rings. In fact, the proof of Theorem 2.6 shows that the following result holds.

**Theorem 2.7.** Let $R$ be a commutative Noetherian ring. Then an $R$-module $M$ is essentially compressible if and only if $M$ embeds in a direct sum of $R$-modules of the form $R/P$, where $P$ is a prime ideal of $R$.

### 3. Essentially compressible classes

A class $C$ of modules is called essentially compressible if all its members are essentially compressible. In this section we study essentially compressibility of the classes: cyclic $R$-modules, co-cyclic $R$-modules and injective $R$-modules. Let $M$ be a non-zero right $R$-module. $M$ is called a co-semisimple module (or a V-module) if every simple module in $\sigma[M_R]$ is $M$-injective (see [2, 2.13] for more information). The ring $R$ is called a right V-ring if the right $R$-module $R$ is co-semisimple. It is well known that a commutative ring $R$ is a (right) V-ring if and only if $R$ is von Neumann regular (see, for example, [6, 3.73, p. 97]). A right $R$-module is called co-cyclic provided it contains an essential simple submodule.
Proposition 3.1. Let $R$ be any ring. Then the following statements are equivalent for a module $M_R$.

(a) $M_R$ is co-semisimple.
(b) In $\sigma[M_R]$, every co-cyclic right $R$-module is essentially compressible.
(c) In $\sigma[M_R]$, every cyclic co-cyclic right $R$-module is essentially compressible.

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are clear.

(c) $\Rightarrow$ (a). Let $S$ be a simple module in $\sigma[M_R]$ with $M$-injective hull $\hat{S}$. Let $0 \neq x \in \hat{S}$. Then $xR$ is a cyclic co-cyclic module with essential socle $S$. By (c), there exists a monomorphism $\theta : xR \rightarrow S$. It follows that $xR$ is simple and hence $xR = S$. Thus $\hat{S} = S$ and $S$ is $M$-injective. □

Proposition 3.2. The following statements are equivalent for a ring $R$.

(a) $R$ is a right $V$-ring.
(b) Every co-cyclic right $R$-module is essentially compressible.
(c) Every cyclic co-cyclic right $R$-module is essentially compressible.

Moreover, if $R$ is a hereditary Noetherian ring, (a)–(c) are equivalent to:

(d) Every non-zero finitely generated right $R$-module is essentially compressible.

Proof. (a) $\iff$ (b) $\iff$ (c). Apply Proposition 3.1 to $M = R$.

(d) $\Rightarrow$ (c). This is clear.

Now suppose that $R$ is a hereditary Noetherian ring.

(a) $\Rightarrow$ (d). Let $M_R$ be finitely generated and let $Z = Z(M_R)$. By [8, Proposition 5.4.5] $Z$ has finitely generated essential socle $S$. By (a), $S$ is injective and hence $S = Z$ and $M = S \oplus T$ for some submodule $T$. Note that $T$ is finitely generated and non-singular. Moreover, $R$ is a semiprime ring by [2, 2.13]. Now, by Corollary 2.4(b), $T$ is essentially compressible. Thus $M_R$ is essentially compressible by Proposition 1.2. □

A right $R$-module $M$ is called locally Noetherian if any finitely generated submodule of $M_R$ is Noetherian.

Proposition 3.3. Let $M$ be a right $R$-module. If every cyclic right $R$-module $X \in \sigma[M_R]$ is essentially compressible, then $M_R$ is locally Noetherian co-semisimple.

Proof. By Proposition 3.1, $M_R$ is co-semisimple. Let $N$ be any semisimple module in $\sigma[M_R]$ and let $0 \neq x \in \hat{N}$. Note that, by hypothesis, $xR$ is an essentially compressible module with essential socle $xR \cap N$. Thus there exists a monomorphism $\theta : xR \rightarrow xR \cap N$. It follows that $xR$ is semisimple and hence $x \in N$. Thus every semisimple module is injective. This in turn implies that every direct sum of $M$-injective hulls of simple modules (in $\sigma[M_R]$) is $M$-injective so that $M_R$ is locally Noetherian by [2, 2.5]. □
Corollary 3.4. A commutative ring \( R \) is semiprime artinian if and only if every cyclic \( R \)-module is essentially compressible.

Corollary 3.5. Let \( R \) be a ring Morita equivalent to a commutative ring. Then \( R \) is semiprime artinian if and only if every finitely generated right (or left) \( R \)-module is essentially compressible.

Proof. By Corollary 3.4 and Proposition 1.3. \( \square \)

In [7, Theorem 4.2], it is proved that every finitely generated right \( R \)-module is essentially compressible if and only if for every right \( R \)-module \( M \) the following implication holds: (i) if \( N \) is a finitely generated submodule of \( \text{E}(MR) \) then there exists a submodule \( X \) of \( \text{E}(MR) \) such that \( N \subseteq X \cong MR \).

In Proposition 3.7, we give a characterization of a ring \( R \) over which all finitely generated right \( R \)-modules are essentially compressible.

Let \( R \) be any ring, let \( y \in R \), and let \( I \) be a right ideal of \( R \). We will denote \( \{a \in R : ya \in I\} \) by \((y:I)_R\).

Let \( I \) and \( J \) be right ideals in a ring \( R \) such that \( I \subseteq J \). We say that \( I \) and \( J \) are \((\ast)\)-related if \( J \cap (xR + I) = I \) implies \( x \in I \) for all \( x \in R \).

Lemma 3.6. Let \( R \) be a ring and let \( I \) be a right ideal of \( R \). Then the following statements are equivalent.

(a) \((R/I)_R \) is essentially compressible.

(b) For any \((\ast)\)-related ideals \( I \subseteq J \) of \( R \), there is \( y \in J \) such that \((y:I)_R \) = \( I \).

Proof. It follows from the facts that \( R/I \) can be embedded into \( J/I \) as a right \( R \)-module if and only if there is \( y \in J \) such that \((y:I)_R \) = \( I \), and \( J/I \) is an essential \( R \)-submodule of \((R/I)_R \) if and only if \( I \) and \( J \) are \((\ast)\)-related. \( \square \)

Combining Lemma 3.6 and Proposition 1.3 with the fact that under the standard Morita equivalence of \( R \) with a matrix ring \( S = \text{Mat}_{n \times n}(R) \), \( n \)-generated right \( R \)-modules correspond to cyclic right \( S \)-modules, we obtain the following result.

Proposition 3.7. For a ring \( R \), the following statements are equivalent.

(a) All non-zero finitely generated right \( R \)-modules are essentially compressible.

(b) For any \( n \geq 1 \) and any \((\ast)\) related right ideals \( I \subseteq J \) of \( S = \text{Mat}_{n \times n}(R) \), there is \( y \in J \) such that \((y:I)_R \) = \( I \).

We are going now to characterize rings \( R \) for which the class \( \text{Mod-R} \) of all right \( R \)-modules is essentially compressible.

Theorem 3.8. The following statements are equivalent for a right \( R \)-module \( M \).
(a) \( M_R \) is semisimple.
(b) The class \( \sigma[M_R] \) is essentially compressible.
(c) In the category \( \sigma[M_R] \), every injective module is essentially compressible.
(d) \( \hat{K} \) can be embedded in \( K \) for all \( K \in \sigma[M_R] \).
(e) For all \( M_1 \) and \( M_2 \) in \( \sigma[M_R] \), \( M_1 \simeq M_2 \) implies that \( M_1 \) is subisomorphic to \( M_2 \).
(f) In \( \sigma[M_R] \), every cyclic right \( R \)-module and every uniform right \( R \)-module is essentially compressible.

**Proof.** We need prove only (d) \( \Rightarrow \) (b) and (f) \( \Rightarrow \) (a). Thus suppose that (d) holds. Let \( 0 \neq N \in \sigma[M_R] \) and let \( K \) be an essential submodule of \( N \). Then \( \hat{K} = \hat{N} \). By (d), \( \hat{K} \), and hence \( N \), can be embedded in \( K \). This shows that \( N \) is essentially compressible. Thus (b) holds.

(f) \( \Rightarrow \) (a). By Proposition 3.3 \( M_R \) is locally Noetherian. On the other hand, if \( U \) is a uniform right \( R \)-module in \( \sigma[M_R] \), by (f), \( \hat{U} \) is essentially compressible and so there exists a monomorphism \( \theta : \hat{U} \to U \). Now \( \theta(\hat{U}) \) is an \( M \)-injective submodule of \( U \) so that it is a non-zero direct summand of \( U \) and we have \( \theta(\hat{U}) = U \). Consequently, in \( \sigma[M_R] \), every uniform module is \( M \)-injective. It follows that, in \( \sigma[M_R] \), every uniform module is simple and \( M \)-injective. Therefore, in \( \sigma[M_R] \), every module with finite uniform dimension is semisimple. The last statement implies that \( M_R \) has an essential socle because \( M_R \) is locally Noetherian. But the socle of any locally Noetherian module \( M_R \) is a direct summand of \( M_R \). Consequently, \( \text{Soc}(M_R) = M \).

**Corollary 3.9.** The following statements are equivalent for a ring \( R \).

(a) \( R \) is semiprime artinian.
(b) Every right \( R \)-module is essentially compressible.
(c) Every injective right \( R \)-module is essentially compressible.
(e) Every cyclic right \( R \)-module and every uniform right \( R \)-module is essentially compressible.

4. Essentially compressible rings

Let \( R \) be a ring. We say that \( R \) is a right essentially compressible ring if the right \( R \)-module \( R \) is essentially compressible. In this section we investigate right essentially compressible rings. It is well known that any semiprime right Goldie ring is right essentially compressible. Clearly any domain is a right essentially compressible ring but it is not necessarily right Goldie. In Theorem 4.12, we shall show that any right essentially compressible ring with a uniform right ideal is semiprime right Goldie.

For any non-empty subset \( S \) of a ring \( R \) we shall denote the right annihilator of \( S \) in \( R \) by \( r.\text{ann}_R(S) \) and the left annihilator of \( S \) in \( R \) by \( l.\text{ann}_R(S) \). In particular if \( a \) is any element of \( R \) and \( S = \{a\} \) then \( r.\text{ann}_R(S) \) will be denoted simply by \( r.\text{ann}(a) \). An element \( c \) in \( R \) is called right regular if \( r.\text{ann}_R(c) = 0 \). For any ideal \( I \) of \( R \) the set of elements \( c \) in \( R \) such that \( c + I \) is a right regular element of the ring \( R/I \) will be denoted by \( C'(I) \).
Lemma 4.1. A ring \( R \) is right essentially compressible if and only if every essential right ideal contains a right regular element of \( R \).

Proof. This is routine. \( \square \)

Corollary 4.2. Let \( R \) be a semiprime ring which satisfies ACC and DCC on right annihilators. Then every free right (or left) \( R \)-module is essentially compressible.

Proof. By [1, Theorem 1.19] and Lemma 4.1 the ring \( R \) is right and left essentially compressible. The result follows by Proposition 1.2. \( \square \)

Compare the next result with Corollary 2.4(b).

Corollary 4.3. Let \( R \) be a semiprime left Goldie ring. Then every free right \( R \)-module is essentially compressible.

Proof. By Corollary 4.2 and [4, Lemma 5.8]. \( \square \)

The following proposition is a consequence of parts (a) and (d) of Proposition 1.4 and Proposition 1.2.

Proposition 4.4. Let \( R \) be a right essentially compressible ring.

(a) Every ideal of \( R \) is an essentially compressible right \( R \)-module.
(b) Every essential submodule of a free right \( R \)-module is an essential compressible \( R \)-module.
(c) If \( A \) is any ideal of \( R \) with \( A^2 \neq A \) then \( R/A^2 \) is not an essentially compressible right \( R \)-module.

Proposition 4.5. Every essentially compressible ring is a right non-singular semiprime ring.

Proof. Let \( R \) be an essentially compressible ring. The semiprimeness of \( R \) is a consequence of Proposition 1.4(d). By Theorem 1.10 there exists a monomorphism \( \theta : R \to Z \oplus A \) for some singular right \( R \)-module \( Z \) and non-singular right \( R \)-module \( A \). Let \( \theta(1) = c = a + z \) for some \( a \in A \) and \( z \in Z \). Note that \( 0 = r.\text{ann}_{R}(c) = r.\text{ann}_{R}(a) \cap r.\text{ann}_{R}(z) \) and \( r.\text{ann}_{R}(z) \) is an essential right ideal of \( R \). Thus \( r.\text{ann}_{R}(a) = 0 \). But \( R \simeq aR \subseteq A \), hence \( Z(R_R) = 0 \). \( \square \)

Proposition 4.6. Every essentially compressible ring does not contains an infinite direct sum of non-zero ideals.

Proof. By Proposition 1.4(f). \( \square \)

Corollary 4.7. A commutative ring \( R \) is essentially compressible if and only if \( R \) is a semiprime Goldie ring.
Proof. By Propositions 4.5 and 4.6. □

We observe next that any right essentially compressible ring is a direct sum of a semi-
prime artinian ring and a right essentially compressible ring with zero right socle. This is
a consequence of Theorem 1.10 which should be compared with [1, Theorem 1.23]. Note
that if $R$ is a right essentially compressible ring then $R$ is semiprime (Proposition 4.4) and
hence the right and left socle of $R$ coincide (see, for example, [1, p. 18]) and in this case
we shall refer simply to the socle of $R$.

Theorem 4.8. Let $S$ denote the socle of a right essentially compressible ring $R$. Then
$S = eR$ for some central idempotent element $e$ of $R$. Consequently, the ring $R$ is a direct
sum of a semiprime artinian ring and a right essentially compressible ring with zero right
socle.

Proof. By Theorem 1.10, $S$ is a direct summand of $R$ and so there is an idempotent $e$ in $R$ such that $S = eR$. Since $S$ is an ideal, $(1 - e)Re \subseteq (1 - e)eR = 0$ and hence $[eR(1 - e)R]^2 = 0$, so that $eR(1 - e) = 0$ by Proposition 4.4. Thus $er = ere = re$ for all $r \in R$, i.e. $e$ is central. For the last statement consider that $R/S$ is an essentially compressible right $R$-module by Proposition 1.4(b), and so it is essentially compressible as a right $R/S$-module. It follows that $(1 - e)R$ is a right essentially compressible ring. □

Lemma 4.9. Let $R$ be a semiprime ring and $U$ be an ideal of $R$. Then $rann_R U = lann_R U$.

Proof. This is clear. □

In view of Lemma 4.9, we denote, by $ann_R U$, a right (or left) annihilator of an ideal $U$ in a semiprime ring $R$. An ideal $I$ of an arbitrary ring $R$ will be called uniform if $I \neq 0$ and $A \cap B \neq 0$ for all ideals $A, B$ of $R$ such that $A \subseteq I$ and $B \subseteq I$.

The next result shows that the study of right essentially compressible rings reduces to the study of such rings when they are prime.

Theorem 4.10. A ring $R$ is right essentially compressible if and only if there exist a positive integer $n$ and prime ideals $P_i$ $(1 \leq i \leq n)$ of $R$ such that $P_1 \cap \cdots \cap P_n = 0$ and the ring $R/P_i$ is right essentially compressible for each $1 \leq i \leq n$.

Proof. Let $R$ be right essentially compressible. Then by Propositions 4.4 and 4.6 and [6, 11.41 and 11.43, p. 336], $R$ has a finite number of minimal prime ideals $P_1, \ldots, P_n$ such that $P_i = ann_R U_i$ for some uniform ideal $U_i$ of $R$ for each $1 \leq i \leq n$ and $P_1 \cap \cdots \cap P_n = 0$. Let $i \in \{1, \ldots, n\}$, let $P = P_i$ and let $U = U_i$. Let $E$ be a right ideal of $R$ such that $P \subseteq E$ and $E/P$ is an essential right ideal of the ring $R/P$. Then $E$ is an essential right ideal of $R$. By Lemma 4.1, $E$ contains a right regular element $c$ of $R$. Let $r \in R$ such that $cr \in P$. Then $crU = 0$ and hence $rU = 0$. This implies that $r \in P$. Thus $E/P$ contains the right regular element $c + P$ of $R/P$. By Lemma 4.1, the ring $R/P$ is right essentially compressible.
Conversely, suppose that \( P_i \cap \cdots \cap P_n = 0 \) for some positive integer \( n \) and prime ideals \( P_i \) \((1 \leq i \leq n)\) such that \( R/P_i \) is a right essentially compressible ring for each \( 1 \leq i \leq n \). Without loss of generality we can suppose that \( P_i \nsubseteq P_j \) for all \( 1 \leq i \neq j \leq n \). Let \( F \) be any essential right ideal of \( R \). Let \( A \) be a right ideal of \( R \) such that \((F \cap P_2 \cap \cdots \cap P_n) \cap A \subseteq P_1\). Then \( F \cap (P_2 \cap \cdots \cap P_n \cap A) \subseteq P_1 \cap \cdots \cap P_n = 0 \), so that \( P_2 \cap \cdots \cap P_n \cap A \subseteq P_1 \) \(\textit{and hence} \ A \subseteq P_1 \). It follows that the right ideal \([(F \cap P_2 \cap \cdots \cap P_n) + P_1]/P_1 \) of the ring \( R/P_1 \) is essential. By Lemma 4.1, there exist \( d_1 \in (F \cap P_2 \cdots \cap P_n) \cap C'(P_1)\). Similarly, for each \( 2 \leq i \leq n \) there exist \( d_i \in (F \cap P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n) \cap C'(P_1)\). Let \( d = d_1 + \cdots + d_n \in F \). It is easy to check that if \( ds = 0 \) for some \( s \in R \) then \( d_i s \in P_i \) \(\textit{and hence} \ s \in P_i \ (1 \leq i \leq n) \), so that \( s \in P_1 \cap \cdots \cap P_n \) and \( s = 0 \). Thus \( d \) is a right regular element of \( R \). By Lemma 4.1, \( R \) is right essentially compressible. \(\square\)

The next result is presumably well known but we include it for completeness.

**Lemma 4.11.** Let \( U \) be a uniform right ideal in the ring \( R \) and let there be prime ideals \( P_i \ (1 \leq i \leq n) \) of \( R \) such that \( P_i \nsubseteq P_j \) for all \( 1 \leq i \neq j \leq n \) and \( P_1 \cap \cdots \cap P_n = 0 \). Then \((U + P_i)/P_i\) is a uniform right ideal of \( R/P_i \) for each \( 1 \leq i \leq n \).

**Proof.** Assume that \( P = P_1 \) and \( A, B \) are right ideals of \( R \) such that \( P \subseteq A \subseteq (U + P) \), \( P \subseteq B \subseteq (U + P) \) and \( A \cap B \subseteq P \). Let \( A' = A \cap U \), \( B' = B \cap U \). We have \( A' \cap (P_2 \cap \cdots \cap P_n \cap B') \subseteq P_1 \cap \cdots \cap P_n = 0 \). Hence by the uniform condition on \( U \), \( A' = 0 \) or \( (P_2 \cap \cdots \cap P_n \cap B') = 0 \). Now if \( A' = 0 \) then \( U \cap P \subseteq A' = 0 \) and so \((U + P)/P \simeq U \) is a uniform right \( R \)-module as well as a uniform \( R/P \)-module, and if \( (P_2 \cap \cdots \cap P_n \cap B') = 0 \) then \( B' \subseteq P \). This in turn implies that \( B \subseteq P \), because if \( u + p = b \in B \) where \( u \in U \) and \( p \in P \), then \( u = (b - p) \in B \cap U = B' \subseteq P \) \(\textit{and hence} \ b = (u + p) \in P \). Consequently, \((U + P)/P \) is a uniform right ideal of \( R/P \). Similarly, \((U + P_i)/P_i \) is an uniform right ideal of \( R/P_i \) for each \( 2 \leq i \leq n \). \(\square\)

**Theorem 4.12.** The following statements are equivalent for a ring \( R \).

(a) \( R \) is a semiprime right Goldie ring.
(b) \( R \) is a right essentially compressible ring with at least one uniform right ideal.

**Proof.** (a) \(\Rightarrow\) (b). By Corollary 2.4(a).

(b) \(\Rightarrow\) (a). By Proposition 4.5, \( R \) is a right non-singular semiprime ring, so it is enough to show that the right uniform dimension of \( R \) is finite. For a moment assume that \( R \) is prime, then by Proposition 1.6 \( R \) has finite uniform dimension. Thus, by Theorem 4.10 and Lemma 4.11, we can conclude that the uniform dimension \( R \) is finite. \(\square\)

5. Ring extensions of essentially compressible rings

In this section we study some ring extensions of a right essentially compressible ring. We begin investigating polynomial extensions.
Proposition 5.1. If $R$ is a right essentially compressible ring then so also is the polynomial ring $R[x]$.

Proof. Let $A$ be an essential right ideal of $R[x]$ and let $L(A) = \{L(f) : f \in A\}$ where $L(f)$ = the leading coefficient of $f$. Then $L(A)$ is an essential right ideal of $R$. To see this, let non-zero $a \in R$ then there is $g \in R[x]$ (with minimal degree) such that $0 \neq ag \in A$. Then $0 \neq L(ag) \in L(A) \cap aR[x]$, proving that $L(A)$ is essential. Now by our assumption and Lemma 4.4, $A$ has an element $f$ such that $L(f)$ is right regular. It is easy to verify that $f$ must be a right regular element of $R[x]$. Thus $R[x]$ is a right essentially compressible ring by Lemma 4.1.

Now we investigate conditions under which the converse of Proposition 5.1 is true. It is well known that a semiprime right Goldie ring is a right non-singular semiprime ring with finite right uniform dimension. This in turn implies that $R$ is a semiprime right Goldie ring if and only if $R[x]$ is also (see [6, Theorem 6.65 and Example 7.14e]). Also if $I$ is a uniform right ideal in $R$ then $I[x]$ is a uniform right ideal in $R[x]$ (see [6, Lemma 6.68]). Now in view of Corollaries 4.7 and 2.4(a) and Theorem 4.12, we can conclude that if either $R$ is commutative or $R$ has a uniform right ideal then the converse of Proposition 5.1 is true. □

The following result is also needed and it is obtained from the proof of [3, Proposition 5.1].

Proposition 5.2. Suppose that $R$ is an algebra over an uncountable field $K$ with $\dim_K R < |K|$. Then $p(x) \in R[x]$ is right regular if and only if there exists $\lambda$ in $K$ such that $p(\lambda)$ is a right regular element of $R$.

Proposition 5.3. Suppose that $R$ is an algebra over an uncountable field $K$ with $\dim_K R < |K|$. If $R[x]$ is a right essentially compressible ring then so too is $R$.

Proof. Let $I$ be an essential right ideal of $R$. Then $I[x]$ is an essential right ideal of $R[x]$. To see this let $0 \neq \sum a_ix^i = p(x) \in R[x]$, then since $I^n$ is an essential $R$-submodule of $R^n$ for any $n \geq 1$, there exists $r \in R$ such that $a_ir \in I$ for all $i$, and not all $a_ir = 0$. Consequently, $0 \neq p(x)r \in I[x]$. It follows that $I[x]$ is essential. Now by Lemma 4.1, there exists a right regular $p(x) \in I[x]$. So by Proposition 5.2, there exists $\lambda$ in $K$ such that $p(\lambda)$ is a right regular element of $R$. Clearly $p(\lambda) \in I$ and so $R$ is a right essentially compressible ring by Lemma 4.1. □

We are now going to study matrix extensions of a right essentially compressible ring.

Lemma 5.4. Let $M$ be a non-zero $R$-module such that $\text{Hom}_R(M, N)$ is non-zero for all non-zero submodule $N$ of $M$ and let $S = \text{End}_R(M)$.

(a) If $J$ is an essential right ideal of $S$ then $JM$ is an essential submodule of $M_R$.
(b) If $M_R$ is quasi-projective and $N$ is an essential submodule of $M_R$ then $\text{Hom}_R(M, N)$ is an essential right ideal of $S$.
Proof. (a) Let $J$ be an essential right ideal of $S$ and $JM \cap N = 0$ for some non-zero submodule $N$ of $M$. By our assumption there is a non-zero $f : M \to N$. Consequently $JM \cap f(M) = 0$. It follows that $J \cap fS = 0$, a contradiction.

(b) Let $N$ be an essential submodule of $MR$ and $0 \neq g \in S$ and $I = \text{Hom}_R(M, N)$. Then $0 \neq \text{Hom}_R(M, N \cap g(M)) \subseteq I \cap \text{Hom}_R(M, g(M))$. On the other hand, $\text{Hom}_R(M, gS) = gS$ because $MR$ is quasi-projective and $gS$ is a finitely generated right ideal of $S$ [2, 3.4]. Thus $I \cap (gS) \neq 0$. □

Proposition 5.5. Suppose that $MR$ is quasi-projective and $\text{Hom}_R(M, N)$ is non-zero for all non-zero submodule $N$ of $M$ and let $S = \text{End}_R(M)$.

(a) If $MR$ is finitely generated then $S$ is a right essentially compressible ring.

(b) If $S$ is a right essentially compressible ring then $MR$ is essentially compressible.

Proof. (a) Let $I$ be an essential right ideal of $S$. Then by Lemma 5.4(a), $IM$ is an essential submodule of $MR$ and hence there exists a monomorphism $f \in \text{Hom}_R(M, IM) = I$. Clearly $f$ is a right regular element in $S$. Thus $S$ is a right essentially compressible ring by Lemma 4.1.

(b) Let $N$ be an essential submodule of $MR$. Then by Lemma 5.4(b), $I = \text{Hom}_R(M, N)$ is an essential right ideal of $S$ and hence by the essentially compressible condition on $S$, $I$ contains a right regular element $f$ of $S$. Now $0 = \text{r.am}(f) = \text{Hom}_R(M, \ker f)$ and so by our assumption, $\ker f = 0$. Consequently, $MR$ is embedded in $N$, proving that $MR$ is essentially compressible. □

Remark 5.6. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}$. Then $M$ is a generator in Mod-$\mathbb{Z}$ and by Proposition 1.2, it is essentially compressible but $\text{End}_\mathbb{Z}(M)$ is not even a semiprime ring.

Corollary 5.7. Let $n \geq 1$. If $R$ is a right essentially compressible ring, then so too is the matrix ring $\text{Mat}_{n \times n}(R)$.

Proof. Apply Proposition 5.5(a) for $M = R^n_R$.

If $R$ is a right essentially compressible ring then it is right non-singular (Proposition 4.5). It is well known that if $R$ is any right non-singular ring then the injective hull $E(R_R)$ has a unique ring structure compatible with its right $R$-module structure (see, for example, [4, Proposition 4.26]). The following result, in this case, implies that $E(R_R)$ is also a right essentially compressible ring. □

Theorem 5.8. Let $R$ be a right essentially compressible subring of a ring $S$ such that $R$ is an essential submodule of the right $R$-module $S$. Then $S$ is a right essentially compressible ring.

Proof. Let $A$ be any essential right ideal of the ring $S$. Let $B$ be a right ideal of $R$ such that $A \cap B = 0$. Let $a \in A \cap BS$. Then $a = b_1s_1 + \cdots + b_ns_n$ for some positive integer $n$ and element $b_i \in B$, $s_i \in S$ ($1 \leq i \leq n$). For each $1 \leq i \leq n$ there exists an essential right ideal $E_i$ of $R$ such that $s_iE_i \subseteq R$. Let $E = E_1 \cap \cdots \cap E_n$. Then $aE \subseteq (b_1s_1 + \cdots + b_ns_n)E \subseteq$
\( A \cap B = 0 \). But \( R \) is right non-singular by Proposition 4.4 and hence \( S \) is a non-singular right \( R \)-module. It follows that \( a = 0 \). Hence \( A \cap BS = 0 \), so that \( BS = 0 \) and hence \( B = 0 \). Thus \( A \cap R \) is an essential right ideal of \( R \). By Lemma 4.1, there exists a right regular element \( c \) of \( R \) such that \( c \in A \). Because \( R \) is essential in the \( R \)-module \( S \), it is easy to check that \( c \) is a right regular element of \( S \). By Lemma 4.1 it follows that the ring \( S \) is right essentially compressible.

Theorem 5.8 gives a method of producing examples of right essentially compressible rings. For example, let \( R \) be a domain which is not right Ore and let \( Q \) denote the maximal right quotient ring of \( R \). By \([6, 13.15(5), p. 359]\), \( Q \) is a right self-injective von Neumann regular ring and, by Theorem 5.8, \( Q \) is right essentially compressible. However, \( Q \) is not semiprime artinian because \( R \) is not right Ore.

Finally, we investigate rings \( R \) for which the injective envelope \( E(R_R) \) is an essentially compressible right \( R \)-module.

A right \( R \)-module \( M \) is said to Dedekind finite if the condition \( M \cong M \oplus N \) in \( \text{Mod-}R \), implies that \( N = 0 \). The ring \( R \) is called directly finite if the right \( R \)-module \( R \) is Dedekind finite or equivalently for all \( x, y \) in \( R \), \( xy = 1 \) implies that \( yx = 1 \) (see \([6, p. 5]\)).

**Theorem 5.9.** The following statements are equivalent for a ring \( R \).

(a) \( R \) is a semiprime artinian ring.
(b) \( R \) is a directly finite ring such that \( E(R_R) \) is an essentially compressible right \( R \)-module.

**Proof.** (a) \( \Rightarrow \) (b). This is well known.

(b) \( \Rightarrow \) (a). Let \( Q = E(R_R) \). By Proposition 1.9(b), it is enough to prove that \( Q \) is a semiprime artinian ring, because then \( R_R \) has DCC on direct summands (in fact \( R \) has finite right uniform dimension, see for example, \([4, \text{ex. 4X, ex. 4W}]\)) and so \( R_R \) is semi-simple. By Theorem 5.8, \( Q \) is a right essentially compressible ring. Thus \( Q \) is a right non-singular ring by Proposition 4.5. It follows that \( Q \) is a right self-injective ring \([6, 13.1(5)]\). Note that injective Dedekind finite modules are co-Hopfian. Thus, by Proposition 1.4(c), it is enough to prove that \( Q \) is a Dedekind finite \( Q \)-module. Now by our assumption, for some idempotent \( e \) in \( R \), we have \( Q = eR \) which is a direct summand of \( R \). Suppose that \( Q = X \oplus Y \) for some non-zero \( Q \)-modules \( X, Y \) such that the \( Q \)-modules \( X \) and \( Q \) are isomorphic. Thus, as \( R \)-modules, \( Q = X \oplus Y \) and \( X \) and \( Q \) are isomorphic. Consequently, \( eR \) is \( R \)-isomorphic to \( eR \oplus Y \) and hence \( R = eR \oplus (e - 1)R \) is \( R \)-isomorphic to \( R \oplus Y \), a contradiction because \( R \) is directly finite. Hence \( Q \) is a Dedekind finite \( Q \)-module, as required. \( \square \)

**Acknowledgment**

The research of the second author was in part supported by a grant from IPM (No. 84160043).
References