JOURNAL OF DIFFERENTIAL EQUATIONS 13, 32–47 (1973)

Periodic Boundary Value Problems for Systems of Second Order Differential Equations

J. W. BEBERNES

Department of Mathematics, University of Colorado, Boulder, Colorado 80302, and Department of Mathematics, University of Utah, Salt Lake City, Utah 84112

AND

K. SCHMITT*

Department of Mathematics, University of Utah, Salt Lake City, Utah 84112

Received November 22, 1971

1. INTRODUCTION

Let I denote the compact interval $[0, 1]$, $\mathbf{R}^k$ k-dimensional Euclidean space with Euclidean norm $\|\cdot\|$, and let $f: I \times D \subset I \times \mathbf{R}^{2n} \to \mathbf{R}^n$ be continuous. In this paper, we consider the existence of periodic solutions for the system of second order equations

$$x'' = f(t, x, x'), \qquad (' = d/dt). \tag{1.1}$$

A solution $x(t)$ of (1.1) is called *1-periodic* (or periodic, since the period will be fixed throughout) in case

$$x(0) = x(1), \qquad x'(0) = x'(1). \tag{1.2}$$

More generally, if $G: I \times D \subset I \times \mathbf{R}^k \to \mathbf{R}^k$ is continuous, a solution $y(t)$ of

$$y' = G(t, y) \tag{1.3}$$

is periodic if $y(0) = y(1)$. In either case, a periodic solution in this sense may be extended so as to be a periodic solution in the usual sense in case the right side of the equation in either (1.1) or (1.3) is a 1-periodic function of t in the usual sense.

* Research of second author supported by U. S. Army Contract ARO-D-37-124-71-G146.

We establish essentially three different classes of results concerning the existence of periodic solutions for (1.1) utilizing a method based on a slight generalization (Theorem 2.1) of a result of Krasnosel'skii [6] together with a variation of the Borsuk–Antipodensatz (Theorem 2.3).

The method can be described as follows. Define a modification F of f which is bounded and continuous on $I \times \mathbf{R}^{2n}$ and agrees with f on $I \times D$. Apply Theorems 2.1 and 2.3 to the system

$$x'' = F(t, x, x') \tag{1.4}$$

to show that (1.4) has a periodic solution $x(t)$ by observing that the degree of a certain natural mapping associated with (1.4) is nonzero. In addition, there are conditions imposed on f which imply that $(x(t), x'(t)) \in D$ for all $t \in [0, 1]$ and hence $x(t)$ is a periodic solution of (1.1).

This approach has several advantages. The key theorems (of Section 2) are proven using only basic ideas from the theory of ordinary differential equations and degree theory. These basic results are in turn easy to apply to the problems considered and permit us to give a unified approach to several classes of problems, previously studied by Knobloch [4, 5], Mawhin [7], and Schmitt [8, 9]. With our approach we are able to obtain the results of [5] and [7] and several generalizations more directly, since we do not have to employ the functional analytic method of Cesari [1]. Further, we are able to show that the results in [9] remain valid without assuming one of the major hypotheses in [9].

2. A General Principle

Let I denote the compact interval $[0, 1]$, $\mathbf{R}$ the real line and $\mathbf{R}^m$ Euclidean m-space. Let Ω be a nonempty bounded open set in $\mathbf{R}^m$ and let $g: I \times \mathbf{R}^m \to \mathbf{R}^m$ be a continuous function having the following properties:

(a) $g(0, y) = y$,

(b) $g(1, y) = y$,

(c) for every t, $g(t, \cdot\,)$ is one–one,

(d) for every $y_0 \in \partial\Omega$ the function $y_0(t)$ defined by

$$y_0(t) = g(t, y_0)$$

has the property that $y_0(t)$ is differentiable at $t = 0$ and

$$g_t(0, y_0) = (d/dt)\, y_0(t)|_{t=0}$$

is a continuous function of y_0, $y_0 \in \partial\Omega$.

DEFINITION 1. For $y_0 \in \bar{\Omega}$, $\{g(t, y_0) | 0 < t \leqslant 1\}$ is called the set of recurrence points of y_0 with respect to the function g.

Let $G: I \times \mathbf{R}^m \to \mathbf{R}^m$ be continuous and consider the differential equation

$$y' = G(t, y), \qquad 0 \leqslant t \leqslant 1. \tag{2.1}$$

The following theorem, which is a generalization of a result of Krasnosel'skii [6, Theorem 6.1, p. 81] is the general principle which we shall employ throughout this paper.

THEOREM 2.1. *Assume that every solution of* (2.1) *emanating from* $\bar{\Omega}$ *exists on* [0, 1] *and that for every* $y_0 \in \partial\Omega$, *every solution* $y(t)$ *of* (2.1) *with* $y(0) = y_0$ *is such that* $y(t) \neq g(t, y_0) = y_0(t)$, $0 < t \leqslant 1$ (i.e., $\partial\Omega$ *consists of nonrecurrence points only*). *Then there exists a* 1*-periodic solution of* (2.1), *whenever*

$$\deg(-G(0, y_0) + g_t(0, y_0), \Omega, 0) \neq 0.$$

Remark. Theorem 2.1 implicitly assumes that the function $T(y_0) = -G(0, y_0) + g_t(0, y_0)$ does not vanish on $\partial\Omega$ in order that the topological degree $\deg(T(y_0), \Omega, 0)$ with respect to Ω and $0 \in \mathbf{R}^m$ be defined.

Proof of Theorem 2.1. We first prove the theorem under the additional assumption that for every $y_0 \in \bar{\Omega}$ there exists a unique solution $y(t)$ of (2.1) such that $y(0) = y_0$; this solution will be denoted by $y(t, y_0)$. We next define the vector field $U(t, y_0)$ on $\bar{\Omega}$ by

$$\begin{aligned} U(t, y_0) &= g(t, y_0) - y(t, y_0) \\ &= y_0(t) - y(t, y_0). \end{aligned}$$

Since $g(1, y_0) = y_0$, we shall have demonstrated the existence of a periodic solution once we have shown that the vector field $U(1, y_0)$ has a zero in $\bar{\Omega}$.

By hypothesis, $U(t, y_0) \neq 0$ when $t > 0$ and $y_0 \in \partial\Omega$. Hence,

$$\deg((1/t)U(t, y_0), \Omega, 0)$$

is defined. Further, our continuity and uniqueness assumptions imply that the vector fields $(1/t)U(t, y_0)$ and $(1/s)U(s, y_0)$, $0 < t$, $s \leqslant 1$ are homotopic which implies

$$\deg((1/t)U(t, y_0), \Omega, 0) = \text{constant}.$$

Letting $t \to 0_+$, we find that

$$\lim_{t \to 0+} (1/t)\, U(t, y_0) = -G(0, y_0) + g_t(0, y_0)$$

and therefore that

$$\begin{aligned}\deg(U(1, y_0), \Omega, 0) &= \deg((1/t)U(t, y_0), \Omega, 0)\\ &= \deg(-G(0, y_0) + g_t(0, y_0), \Omega, 0) \neq 0.\end{aligned}$$

This in turn implies the existence of at least one zero of $U(1, y_0)$ in Ω.

To prove the general case, we use an approximation procedure, i.e., we approximate G by a sequence of locally Lipschitz continuous functions which have the same properties as G, apply what has just been said to each of the approximations, and then complete the proof by means of a limiting argument. The reader will find that a slight modification of the arguments on pp. 81–83 of [6] will accomplish what has been described above.

In the first set of applications of Theorem 2.1, we make use of the special case where $g(t, y) = y$. We state this special case as

COROLLARY 2.2 (Krasnosel'skii). *Assume that solutions of* (2.1) *exist on* [0, 1] *and that for every* $y_0 \in \partial\Omega$ *every solution* $y(t)$ *of* (2.1) *with* $y(0) = y_0$ *is such that* $y(t) \neq y_0$, $0 < t \leqslant 1$. *Then there exists a* 1*-periodic solution of* (2.1) *whenever* $\deg(-G(0, y_0), \Omega, 0) \neq 0$.

Together with either Theorem 2.1 or Corollary 2.2, we shall need the following result which is a consequence of the Borsuk–Antipodensatz [6, p. 75].

DEFINITION 2. Let Ω be a convex bounded open set in $\mathbf{R}^m$. The (fixed point free) involution S of $\partial\Omega$ determined by $z \in \Omega$ is the mapping on $\partial\Omega$ onto $\partial\Omega$ which maps each $y \in \partial\Omega$ onto Sy by projecting y along the line joining y and z.

THEOREM 2.3. *Let* Ω *be a convex bounded open set in* $\mathbf{R}^m$, *let* $h: \bar{\Omega} \to \mathbf{R}^m$ *be continuous and such that* $h(y) \neq 0$ *for all* $y \in \partial\Omega$, *and let* S *be the fixed point free involution of* $\partial\Omega$ *determined by* $z \in \Omega$. *If* $h(y)$ *and* $h(Sy)$ *do not have the same direction for all* $y \in \partial\Omega$, *then*

$$\deg(h(y), \Omega, 0) \neq 0.$$

Proof of Theorem 2.3. Without loss of generality we may assume that the point $z = 0 \in \mathbf{R}^m$. Let $B \equiv \{y \mid \| y \| < \mathrm{a}\}$, where a is chosen small enough so that $B \subset \Omega$. We now retract Ω onto B along rays through $z = 0$. This homeomorphism is denoted by $H: \bar{\Omega} \to \bar{B}$.

Let $r \in \partial B$ and let $y = H^{-1}(r)$; then it is clear from the definition of S that $Sy = H^{-1}(-r)$. Hence, the involution S induces the antipodal map on ∂B. Let $k: \partial B \to \mathbf{R}^m$ be defined by $k = h \circ H^{-1}$. Then $k(r) = h(y)$ and $k(-r) = h(Sy)$, where $y = H^{-1}(r)$. Thus $k(r)$ and $k(-r)$ do not have the same direction and, furthermore, $k(r) \neq 0$ on ∂B. It follows from Theorem 5.13, p. 75, of [6] that

$$\deg(k, B, 0) \neq 0.$$

Using the product theorem of degree theory (see, e.g., [3, Theorem 7, p. 244]) we conclude that $\deg(h, \Omega, 0) \neq 0$.

3. Some Existence Theorems

In this section we show how Corollary 2.2 and Theorem 2.3 can be used to establish the existence of a periodic solution of the second order system

$$x'' = f(t, x, x') \tag{3.1}$$

by imposing various assumptions on f. Our first theorem of this section is a generalization of Theorem 1 of Knobloch [5] and of Theorem 3.1 of Schmitt [9]. We first establish a lemma.

Lemma 3.1. *If $f(t, x, x')$ is a continuous vector function on*

$$D = \{(t, x, x') | t \in I, \| x \| \leqslant R, x' \in \mathbf{R}^n\}$$

and if

(a) $x \cdot f + \| x' \|^2 \geqslant 0 \quad$ if $\quad x \cdot x' = 0, \| x \| = R,$ (3.2)

(b) $\| f \| \leqslant \varphi(\| x' \|),$ (3.3)

where φ is a positive continuous function on $[0, \infty)$ with $\int_0^\infty (s/\varphi(s))\, ds = \infty$ for all $(t, x, x') \in D$,

(c) *there exist $\alpha \geqslant 0$, $K \geqslant 0$ such that*

$$\| f \| \leqslant 2\alpha(x \cdot f + \| x' \|^2) + K \qquad \text{for all} \quad (t, x, x') \in D, \tag{3.4}$$

then there exists a continuous bounded function $F: I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ with $F(t, x, x') = f(t, x, x')$ on $E \equiv \{(t, x, x') | t \in I, \| x \| \leqslant R, \| x' \| \leqslant M\}$ satisfying

$$x \cdot F + \| x' \|^2 > 0 \qquad \textit{for} \quad x \cdot x' = 0, \| x \| > R, \tag{3.5}$$

$$\| F \| \leqslant \varphi(\| x' \|) \qquad \textit{for all} \quad (t, x, x') \in D, \tag{3.6}$$

$$\| F \| \leqslant 2\alpha(x \cdot F + \| x' \|^2) + K \qquad \textit{for all} \quad (t, x, x') \in D, \tag{3.7}$$

and

$$x \cdot F > 0 \qquad \textit{for} \quad \| x \| = R + 1,\ t \in I,\ x' \in \mathbf{R}^n. \tag{3.8}$$

Proof of Lemma 3.1. Define $F: I \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ by

$$F(t, x, x') = \delta_M(\| x' \|) f(t, x, x'), \qquad \| x \| \leqslant R$$

and

$$F(t, x, x') = \delta_R(\| x \|) F(t, Rx/\| x \|, x') + (1 - \delta_R(\| x \|)) x/\| x \|,$$

where $\delta_t(s)$ is a continuous function on $[0, \infty)$ with $\delta_t(s) = 1$ on $0 \leqslant s \leqslant t$ and $\delta_t(s) = 0$ for $s \geqslant t + 1$.

It is straightforward to verify that F is a continuous bounded function on $I \times \mathbf{R}^n \times \mathbf{R}^n$ with $F(t, x, x') = f(t, x, x')$ on E satisfying (3.5)–(3.8).

THEOREM 3.2. *Let $f(t, x, x')$ be continuous on D and satisfy* (3.2)–(3.4), *then* (3.1) *has a periodic solution $x(t)$ with $\| x(t) \| \leqslant R$.*

Proof of Theorem 3.2. By (3.3) and (3.4) [2, Lemma 5.2, p. 429], there exists a positive constant M depending on α, K, φ, and R such that if $x(t)$ is a solution of (3.1) with $\| x(t) \| \leqslant R$ on $[0, 1]$, then $\| x'(t) \| \leqslant M$ on $[0, 1]$. Consider

$$x'' = F(t, x, x'), \tag{3.9}$$

where F is a modification of f relative to R and M as defined in proof of Lemma 3.1 which satisfies (3.5)–(3.8).

Let $S_1 = \{x \mid \| x \| < R + 1\}$, $S_2 = \{x' \mid \| x' \| < M + 2\}$, $\Omega = S_1 \times S_2$, $y = (x, x')$, $G(t, y) = (G_1(t, x, x'), G_2(t, x, x')) = (x', F(t, x, x'))$ and consider

$$y' = G(t, y), \tag{3.10}$$

the first-order system equivalent to (3.9).

We will apply Corollary 2.2 to (3.10) to prove that (3.10) has a periodic solution $y(t)$ on $[0, 1]$. It suffices for this to show that (i) all solutions of initial value problems for (3.10) exist on $[0, 1]$, (ii) every solution $y(t)$ of (3.10) with $y(0) = y_0 \in \partial\Omega$ is not in the set of recurrence points with respect to $g(t, y_0) = y_0$, and (iii) $\deg(-G(0, y), \Omega, 0) \neq 0$.

Since $F(t, x, x')$ is continuous and bounded on $[0, 1] \times \mathbf{R}^n \times \mathbf{R}^n$, it follows that $G(t, y)$ is continuous on $[0, 1] \times \mathbf{R}^{2n}$, that $\| G(t, y) \| \leqslant \| y \| + B$, and hence that solutions of initial value problems for (3.10) exist on $[0, 1]$. Thus, (i) is satisfied.

No solution $y(t) = (x(t), x'(t))$ of (3.10) with $y(0) = y_0 \in \partial\Omega$ is in the set of recurrence points with respect to $g(t, y_0) = y_0$. For if $y_0 \in \partial\Omega$, then $y_0 = (x_0, x_0') \in \partial S_1 \times S_2$ or $y_0 = (x_0, x_0') \in S_1 \times \partial S_2$. Let $(x_0, x_0') \in \partial S_1 \times S_2$ and assume $y(t) = (x(t), x'(t))$ is a solution of (3.10) with $(x(0), x'(0)) = (x_0, x_0')$ which is in the set of recurrence points for (3.10) relative to (x_0, x_0')

for some $t_1 \in (0, 1]$. Since $x(0) = x(t_1)$ and $x'(0) = x'(t_1)$, $u(t) = \frac{1}{2}\| x(t)\|^2$ has a maximum at $t_2 \in [0, t_1]$ with $u(t_2) \geqslant \frac{1}{2}(R+1)^2$, $u'(t_2) = 0$, and and $u''(t_2) \leqslant 0$. But then

$$u''(t_2) = (x(t_2) \cdot x'(t_2))' = x(t_2) \cdot F(t_2, x(t_2), x'(t_2)) + \| x'(t_2)\|^2 > 0$$

by (3.5) which is impossible. Let $y_0 = (x_0, x_0') \in S_1 \times \partial S_2$, then by the definition of F, any solution $y(t) = (x(t), x'(t))$ of (3.10) with $(x(0), x'(0)) = (x_0, x_0')$ satisfies $x''(t) = 0$ for all t in a right neighborhood of 0 as long as $\| x'(t)\| > M + 1$ and $\| x(t)\| \leqslant R$. This clearly implies that $y(t) = (x(t), x'(t))$ is not in the set of recurrence points with respect to $(x_0, x_0') \in S_1 \times \partial S_2$. If $\| x(t)\| > R$ for some t, then, by an argument similar to the one at the beginning of this paragraph, condition (3.5) prevents $y(t)$ from being in the set of recurrence points.

Define $k: \bar{\Omega} \to \mathbf{R}^{2n}$ by $k(y) = -G(0, y) = (-x', -F(0, x, x'))$. Then k is continuous on $\bar{\Omega}$ and is nondegenerate on $\partial\Omega$. For certainly $k(y) \neq 0$, whenever $x' \neq 0$. If $x' = 0$, then $k(y) = 0$ implies $-F(0, x, 0) = 0$ with $\| x \| = R + 1$. But by definition of the modification F, $F(0, x, 0) = x/\| x \| \neq 0$ for $\| x \| = R + 1$. Hence, $k(y) \neq 0$ for all $y \in \partial\Omega$ and $\deg(k, \Omega, 0)$ is defined.

By Theorem 2.3, we know that $\deg(k, \Omega, 0) \neq 0$ provided $k(y)$ and $k(Sy)$ do not have the same direction for any $y \in \partial\Omega$, where S is the involution of $\partial\Omega$ determined by $0 \in \Omega$. Note that the involution is $Sy = -y$. It is immediate that $k(y)$ and $k(-y)$ have different directions for all $z \in \partial\Omega$ with $x' \neq 0$. If $x' = 0$, then $x \cdot F(0, x, 0) > 0$ and $-x \cdot F(0, -x, 0) > 0$ for $\| x \| = R + 1$ by (3.8) which means that $k(y)$ and $k(-y)$ have different directions for $y \in \partial\Omega$ with $x' = 0$. In all cases, we have that $k(y)$ and $k(-y)$ have different directions. Thus, $\deg(k, \Omega, 0) \neq 0$.

By Corollary 2.2, (3.10) has a periodic solution $y(t)$ on $[0, 1]$ with $y(0) \in \Omega$. Hence, there is a solution $x(t)$ of (3.9) with $(x(0), x'(0)) = (x(1), x'(1)) \in \Omega$.

But (3.5) implies that $\| x(t)\| \leqslant R$ on $[0, 1]$, and (3.6), (3.7) imply $\| x'(t)\| \leqslant M$ on $[0, 1]$. This means that $(t, x(t), x'(t)) \in E$ on $[0, 1]$ and, since $F(t, x, x') = f(t, x, x')$ on E, $x(t)$ is a periodic solution of (3.1).

Remark. Theorem 3.2 is a generalization of Knobloch's Theorem 1 [5], p. 68] since there it is assumed that f satisfies a local Lipschitz condition with respect to x and x'. Our theorem also follows from Knobloch's Theorem 2 [5, p. 75] by taking $r = \| x \|^2 - R^2$; however, as emphasized in the introduction, we believe that our approach is somewhat simpler. It generalizes Theorem 3.1 of Schmitt [9] in that a weaker Nagumo condition is imposed and no assumption concerning uniqueness of boundary value problems is needed.

In $\mathbf{R}^n$, let $x \leqslant y$ if and only if $x_i \leqslant y_i$, $1 \leqslant i \leqslant n$, and $x < y$ if and only if $x_i < y_i$, $1 \leqslant i \leqslant n$. Also, let $[A, B] = \{x \in \mathbf{R}^n \mid A_i \leqslant x_i \leqslant B_i\}$.

THEOREM 3.3. *Assume there exist* $A, B, \varphi, \psi \in \mathbf{R}^n$ *with* $A < B$, $\varphi < 0 < \psi$ *such that* f *is continuous on* $[0, 1] \times [A, B] \times [\varphi, \psi]$ *and satisfies*

$$f_i(t, x_1, \ldots, x_{i-1}, A_i, x_{i+1}, \ldots, x_n, x_1', \ldots, x_{i-1}', 0, x_{i+1}', \ldots, x_n') \\ < 0 < f_i(t, x_1, \ldots, x_{i-1}, B_i, x_{i+1}, \ldots, x_n, x_1', \ldots, x_{i-1}', 0, x_{i+1}', \ldots, x_n') \tag{3.11}$$

for all $A_j \leqslant x_j \leqslant B_j$, $j \neq i$, *and all* $x' \in \{y \in \mathbf{R}^n \mid y_i = 0\}$ *for each* $i = 1,\ldots, n$;

$$f_i(t, x, x_1', \ldots, x_{i-1}', \varphi_i, x_{i+1}', \ldots, x_n') \quad \text{and} \quad f_i(t, x, x_1', \ldots, x_{i-1}', \psi_i, x_{i+1}', \ldots, x_n') \tag{3.12}$$

are nonzero for all $t \in [0, 1]$, $A \leqslant x \leqslant B$, *and all* $\varphi_j \leqslant x_j' \leqslant \psi_j$, $j \neq i$, $i = 1,\ldots, n$.

Then there exists a periodic solution $x(t)$ *of* (3.1) *with* $A \leqslant x(t) \leqslant B$ *and* $\varphi \leqslant x'(t) \leqslant \psi$ *on* $[0, 1]$.

Proof of Theorem 3.3. We proceed by defining a modification H of f in such a way that the differential system defined by H has a periodic solution by Corollary 2.2 and Theorem 2.3. We then show that this solution is actually a solution of (3.1).

Define a modification of f as follows. For each $i = 1,\ldots, n$, let

$$H_i^*(t, x, x') = \begin{cases} f_i(t, \bar{x}, x') + \dfrac{x_i - B_i}{1 + x_i^2}, & x_i > B_i, \\ f_i(t, \bar{x}, x'), & A_i \leqslant x_i \leqslant B_i, \\ f_i(t, \bar{x}, x') + \dfrac{x_i - A_i}{1 + x_i^2}, & x_i < A_i, \end{cases}$$

where $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is defined by

$$\bar{x}_j = \begin{cases} B_j, & x_j > B_j, \\ x_j, & A_j \leqslant x_j \leqslant B_j, \\ A_j, & x_j < A_j, \end{cases}$$

and $x' \in \mathbf{R}^n$. Then define for each i

$$H_i(t, x, x') = \begin{cases} H_i^*(t, x, \bar{x}'), & x_i' > \psi_i, \\ H_i^*(t, x, x'), & \varphi_i \leqslant x_i' \leqslant \psi_i, \\ H_i^*(t, x, \bar{x}'), & x_i < \varphi_i, \end{cases}$$

where $\bar{x}' = (\bar{x}_1',..., \bar{x}_n')$ is given by

$$\bar{x}_j' = \begin{cases} \psi_j, & x_j' > \psi_j, \\ x_j', & \varphi_j \leqslant x_j \leqslant \psi_j, \\ \varphi_j, & x_j' < \varphi_j, \end{cases}$$

and $t \in [0, 1]$, $x \in \mathbf{R}^n$.

The vector function $H = (H_1,..., H_n)$ so defined is continuous and bounded on $[0, 1] \times \mathbf{R}^n \times \mathbf{R}^n$. Consider

$$x'' = H(t, x, x') \tag{3.13}$$

and its equivalent first-order $2n$-dimensional formulation

$$y' = G(t, y), \tag{3.14}$$

where $y = (x, x')$, $G(t, y) = (G_1(t, y), G_2(t, y)) = (x', H(t, x, x'))$.

As in Theorem 3.1, we shall apply Corollary 2.2 to (3.14) on

$$\Omega = \operatorname{int}\{[A, B] \times [\varphi, \psi]\}.$$

Again all solutions of initial value problems for (3.14) exist on [0, 1]. It remains only to show that no solution $y(t)$ of (3.14) with $y(0) = y_0 \in \partial\Omega$ is in the set of recurrence points with respect to $g(t, y) = y$ and that $\deg(-G(0, y), \Omega, 0) \neq 0$.

Let $y_0 \in \partial\Omega$. Then $y_0 = (x_0, x_0') \in \partial[A, B] \times [\varphi, \psi]$ or $y_0 \in [A, B] \times \partial[\varphi, \psi]$. If $(x, x') \in \partial[A, B] \times [\varphi, \psi]$, then $A \leqslant x \leqslant B$ with $x_i = A_i$ or B_i for some i and $\varphi \leqslant x \leqslant \psi$. To consider a specific case assume $x_i = B_i$ and that there is a solution $(x(t), x'(t))$ of (3.13) with $(x(0), x'(0)) = (x, x')$ and $(x(t_0), x'(t_0)) = (x, x')$ for some $t_0 \in (0, 1]$. This means, using (3.11), that there is an interval $[t_1, t_2] \subset [0, t_0]$ such that $x_i(t_j) = B_i$, $j = 1, 2$, and $x_i(t) > B_i$ on (t_1, t_2). The difference $x_i(t) - B_i$ therefore assumes a positive maximum at some point $s \in (t_1, t_2)$ with $x_i'(s) = 0$, $x_i''(s) \leqslant 0$. At s,

$$\begin{aligned} x_i''(s) &\geqslant H_i(s, x(s), x'(s)) \\ &\quad -H_i(s, x_1(s),..., x_{i-1}(s), B_i, x_{i+1}(s),..., x_n(s), x_1'(s),..., \\ &\qquad\qquad x_{i-1}'(s), 0, x_{i+1}'(s),..., x_n'(s)) \\ &= f_i(s, \bar{x}(s), \bar{x}'(s)) + \frac{x_i(s) - B_i}{1 + (x_i(s))^2} - f_i(s, \bar{x}(s), \bar{x}'(s)) \\ &> 0, \end{aligned}$$

which contradicts $x_i(t) - B_i$ having a positive maximum at s.

If $(x, x') \in [A, B] \times \partial[\varphi, \psi]$, then $A \leqslant x \leqslant B$, $\varphi \leqslant x' \leqslant \psi$ with $x_i' = \psi_i$ or φ_i for some i. Assume $x_i' = \varphi_i$ and that there is a solution $y(t)$ of (3.14) which is in the set of recurrence points of (x, x') with respect to $g(t, y) = y$. Then $y(0) = (x, x') = y(t_0)$ for some $t_0 \in (0, 1]$. Since, by (3.12), $H_i(0, x, x') = f_i(0, x, x')$ is nonzero, assume to be specific that $H_i(0, x, x') > 0$. This implies that $x_i'(t) > \varphi_i$ for all t in some right neighborhood of 0 and this in turn implies (since $y(t)$ is in the recurrent set for (x, x')) the existence of a t_1, $0 < t_1 \leqslant t_0$ such that $x_i'(t_1) = \varphi_i$ and $x_i'(t) > \varphi_i$ on $(0, t_1)$. If $A \leqslant x(t) \leqslant B$ on $[0, t_1]$, then $H_i(t, x(t), x'(t)) \ngtr 0$ on the φ_i face of $\partial([A, B] \times [\varphi, \psi])$ for all $t \in [0, t_1]$ which contradicts (3.12). If $x(t) \notin [A, B]$ for all $t \in [0, t_1]$, then $x_j(t) \notin [A_j, B_j]$ for some j and for all t in some subinterval of $[0, t_1]$. But by the definition of H, this too is impossible. We have thus shown that no solution $y(t)$ of (3.14) is the set of recurrence points for any $y_0 \in \partial\Omega$.

Define $k: \bar{\Omega} \to \mathbf{R}^{2n}$ by $k(y) = -G(0, y) = (-x', -H(0, x, x'))$. k is continuous on $\bar{\Omega}$ and nondegenerate on $\partial\Omega$ so $\deg(k, \Omega, 0)$ is defined.

The degree, $\deg(k, \Omega, 0)$, is nonzero provided $k(y)$ and $k(Sy)$ do not have the same direction for any $y \in \partial\Omega$ where S is the involution of $\partial\Omega$ determined by $(\frac{1}{2}(A + B), \frac{1}{2}(\varphi + \psi)) \in \Omega$. It is clear that k and $k \circ S$ have different directions for all $(x, x') \in \partial\Omega$ with $x' \neq 0$. If $x' = 0$, then (3.11) implies that k and $k \circ S$ have different directions.

By Corollary 2.2, (3.13) has a periodic solution $x(t)$ on $[0, 1]$ with

$$(x(0), x'(0)) = (x(1), x'(1)) \in [A, B] \times [\varphi, \psi].$$

But $A \leqslant x(t) \leqslant B$ on $[0, 1]$ by construction of H (using an argument identical to the one given showing that no solution is in the set of recurrence points for any $(x, x') \in \partial[A, B] \times [\varphi, \psi]$). But then (3.12) implies $x'(t) \in [\varphi, \psi]$ on $[0, 1]$. Hence, $x(t)$ is a periodic solution of (3.1) on $[0, 1]$ with $A \leqslant x(t) \leqslant B$.

By using an approximation argument, Theorem 3.3 can be improved. We state such an improvement and outline a proof.

THEOREM 3.4. *Assume the conditions of the previous theorem permitting* $A \leqslant B$ *and replacing* (3.11) *by*

$$\begin{gathered} f_i(t, x_1, \ldots, x_{i-1}, A_i, x_{i+1}, \ldots, x_n, x_1', \ldots, x_{i-1}', 0, x_{i+1}', \ldots, x_n') \\ \leqslant 0 \leqslant f_i(t, x_1, \ldots, x_{i-1}, B_i, x_{i+1}, \ldots, x_n, x_1', \ldots, x_{i-1}', 0, x_{i+1}', \ldots, x_n') \\ \textit{for all} \quad A_j \leqslant x_j \leqslant B_j, j \neq i, \quad \textit{and} \quad x' \in \{y \in R^n \mid y_i = 0\}. \end{gathered} \tag{3.15}$$

Then there is a periodic solution $x(t)$ *of* (3.1) *with* $A \leqslant x(t) \leqslant B$ *on* $[0, 1]$.

Proof of Theorem 3.4. Define the modification H of f exactly as in Theorem 3.3. Since, by (3.12), each f_i is nonzero on the φ_i and ψ_i-faces of $[A, B] \times [\varphi, \psi]$ for all $t \in [0, 1]$, $A \leqslant x \leqslant B$, $\varphi_j \leqslant x_j' \leqslant \psi_j$, $j \neq i$, there is an $\epsilon_0 >$ such that for each $i = 1, \ldots, n$, $f_i(t, x, x_1', \ldots, x_{i-1}', \varphi_i, x_{i+1}', \ldots, x_n')$ and $f_i(t, x, x_1', \ldots, x_{i-1}', \psi_i, x_{i+1}', \ldots, x_n')$ are nonzero for all $t \in [0, 1]$,

$A - \epsilon_0 e \leqslant x \leqslant B + \epsilon_0 e$ where $e = (1,..., 1) \in \mathbf{R}^n$ and $\varphi_j \leqslant x_j' \leqslant \psi_j$, $j \neq i$. Hence, there is an ϵ_1, $0 < \epsilon_1 \leqslant \epsilon_0$, such that

$$H_i(t, x, x_1',..., x_{i-1}', \varphi_i, x_{i+1}',..., x_n')$$

and $H_i(t, x, x_1',..., x_{i-1}', \psi_i, x_{i+1}',..., x_n')$ are nonzero for all $t \in [0, 1]$, $A - \epsilon_1 e \leqslant x \leqslant B + \epsilon_1$, e, and $\varphi_j \leqslant x_j' \leqslant \psi_j$, $j \neq 1,..., n$.

Define $\alpha = A - \epsilon_1 e$ and $\beta = B + \epsilon_1 e$, then for each $i = 1, 2,..., n$, $H_i(t, x_1,..., x_{i-1}, \alpha_i, x_{i+1},..., x_n, x_1',..., x_{i-1}', 0, x_{i+1}',..., x_n') = f_i(t, \bar{x}_1,..., \bar{x}_{i-1}, A_i, \bar{x}_{i+1},..., \bar{x}_n, x_1',..., x_{i-1}', 0, x_{i+1}',..., x_n') - [\epsilon_1/1 + (A_i - \epsilon_1)^2] < 0 < H_i(t, x_1,..., x_{i-1}, \beta_i, x_{i+1},..., x_n, x_1',..., x_{i-1}', 0, x_{i+1}',..., x_n')$ for all $t \in [0, 1]$, $\alpha_j \leqslant x_j \leqslant \beta_j$, $j \neq i$, and all $x' \in \{\in \mathbf{R}^n \mid y_i = 0\}$.

Let $\Omega = [\alpha, \beta] \times [\varphi, \psi]$. The argument proceeds as in Theorem 3.3 to get a periodic solution $x(t)$ of (3.13) on [0, 1] with $(x(t), x'(t)) \in \Omega$ for all $t \in [0, 1]$.

If $x(t)$ is a periodic solution of (3.13) then $A \leqslant x(t) \leqslant B$. For if $x(t) \notin [A, B]$, then assume $x(t_0) \nleqslant B$ for some $t_0 \in [0, 1]$ which implies $x_i(t_0) > B_i$ for some i. Then $x_i - B_i$ has a positive maximum at some $t^* \in [0, 1]$ with $x_i'(t^*) = 0$, $x_i''(t^*) \leqslant 0$. But the definition of H and (3.15) imply that $x_i''(t^*) > 0$.

Hence, $(x(t), x'(t)) \in [A, B] \times [\varphi, \psi]$ and $x(t)$ is a periodic solution of (3.1) on [0, 1].

The proof of the next theorem is analogous to the proofs of Theorems 3.2 and 3.3.

THEOREM 3.5. *Assume there exist* $A, B \in \mathbf{R}^n$ *with* $A \leqslant B$ *such that* f *is continuous on* $[0, 1] \times [A, B] \times \mathbf{R}^n$ *and satisfies* (3.15) *and a Nagumo condition given by*

$$\begin{aligned}&\| f \| \leqslant \varphi(\| x' \|) \text{ where } \varphi \text{ is a positive nondecreasing}\\ &\text{continuous function on } [0, \infty) \text{ with } \textstyle\int_0^\infty s/\varphi(s)\, ds = +\infty \qquad (3.3')\\ &\text{for all } (t, x, x') \in [0, 1] \times [A, B] \times \mathbf{R}^n, \text{ and}\end{aligned}$$

$$\begin{aligned}&\text{there exist } \alpha \geqslant 0,\ k \geqslant 0 \text{ such that}\\ &\| f \| \leqslant 2\alpha(x \cdot f + \| x' \|^2) + K \qquad (3.4')\\ &\text{for all } (t, x, x') \in [0, 1] \times [A, B] \times \mathbf{R}^n.\end{aligned}$$

Then there exists a periodic solution $x(t)$ *of* (3.1) *with* $A \leqslant x(t) \leqslant B$ *on* [0, 1].

Remark. We observe that Theorem 3.3 may also be proved by applying Theorem 3.5 to $f(t, x, \bar{x}')$, where $\bar{x}'$ is defined as in the proof of Theorem 3.3. The conditions of Theorems 3.4 and 3.5 may be considered so as to yield still more general results. For example, consider system (3.1) where $f = (\bar{f}, f^*)$, $\bar{f}: I \times \mathbf{R}^{2n} \to \mathbf{R}^k$, $f^*: I \times \mathbf{R}^{2n} \to \mathbf{R}^{n-k}$, $k < n$, and assume that either (3.11) or (3.15) are satisfied. In addition, assume that $\bar{f}_i$ satisfies (3.12) for $i = 1,\ldots, k$ uniformly with respect to the variables $(x'_{k+1},\ldots, x_n')$ whereas f^* satisfies (3.3′) and (3.4′) with x' replaced by $(x'_{k+1},\ldots, x_n')$ but uniformly with respect to $(x_1',\ldots, x_k')$. It then follows from the proofs of the preceding theorems that (3.1) will have a periodic solution. Using these ideas we may obtain as a corollary to Theorems 3.4 and 3.5 a generalization of Theorem 2 of Mawhin [7] for the system

$$x_i'' = x_i f_i(t, x, x') + g_i(t, x, x'), \qquad i = 1,\ldots, n. \tag{3.16}$$

COROLLARY 3.6. *For $i = 1,\ldots, n$, let $f_i, g_i : I \times \mathbf{R}^{2n} \to \mathbf{R}$ be continuous and let the following conditions be satisfied*:

(i) $f_i \geqslant 0$,

(ii) *there exists a constant C_i such that $|g_i(t, x, y)| \leqslant C_i f_i(t, x, y)$ for all (t, x, y), $y_i = 0$,*

(iii) *$|g_i(t, x, y)| \to \infty$ and $f_i(t, x, y)/|g_i(t, x, y)| \to 0$ as $|y_i| \to \infty$ uniformly on compact (t, x) sets and uniformly with respect to y_j, $j \neq i$, $i = 1,\ldots, n_0$, $0 \leqslant n_0 \leqslant n$,*

(iv) *$|f_i(t, x, y)|/\|y''\|^2 \to 0$, $|g_i(t, x, y)|/\|y'\|^2 \to 0$, $i = n_0 + 1,\ldots, n$, as $\|y''\| \to \infty$ uniformly on compact (t, x, y')-sets where $y' = (y_1,\ldots, y_{n_0})$, $y'' = (y_{n_0+1},\ldots, y_n)$.*

Then (3.16) *has a periodic solution $x(t)$ such that $|x_i(t)| \leqslant C_i$.*

Proof of Corollary 3.6. Conditions (i) and (ii) imply that condition (3.15) of Theorem 3.4 is satisfied with $B = (C_1,\ldots, C_n)$ and $A = -B$. On the other hand, (iii) implies that for f_i, $1 \leqslant i \leqslant n_0$, there exist constants φ_i, ψ_i so that condition (3.12) is satisfied. Condition (iv) implies that $f^* = (f_{n_0+1},\ldots, f_n)$ satisfies (3.3′) and (3.4′). By the remark above, we obtain the desired conclusion.

Remark. Corollary 3.6 is more general than Theorem 2 of [7] in that f_i need not be strictly positive. This allows us to further generalize a result of Corduneanu [7, *p*. 528] for the system

$$x'' = f(t, x). \tag{3.17}$$

COROLLARY 3.7. *Let $f: I \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and let $\partial f_i/\partial x_i$ be continuous and nonnegative. Further assume that*

$$|f_i(t, x_1,\ldots, x_{i-1}, 0, x_{i+1},\ldots, x_n)| \leqslant C_i(\partial f_i/\partial x_i)(t, x),$$

where C_i is a positive constant. Then there exists a periodic solution $x(t)$ of (3.17) *with $|x_i(t)| \leqslant C_i$.*

Proof of Corollary 3.7. Write (3.17) as

$$x_i'' = x_i \int_0^1 \frac{\partial f_i}{\partial x_i}(t, x_1, ..., x_{i-1}, sx_i, x_{i+1}, ..., x_n)\, ds$$
$$+ f_i(t, x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_n) \tag{3.18}$$

and then apply the previous corollary.

Remark. Theorem 3.5 may also be obtained from Knobloch's Theorem 3 [5, p. 79] by taking the functions $p(t, x)$ of the form $A_i - x_i$, $x_i - B_i$, $i = 1, ..., n$, $N = 2n$. A similar remark applies to Theorem 4.1.

4. Another Existence Theorem

In this section, we present an application of Theorems 2.1 and 2.3 in the case where the function $g(t, y)$ of Theorem 2.1 is not necessarily the identity map for each t. The result obtained here is essentially a generalization of Theorems 4.2 and 4.3 of Schmitt [9]; in fact, we show that Theorems 4.2 and 4.3 of [9] remain valid without the uniqueness assumption on solutions of certain two point boundary value problems which was made in [9] although condition (4.2) of our Theorem 4.1 is somewhat more restrictive. In order to avoid notational difficulties, we shall assume in this section that f is independent of x'. This will shorten the proof somewhat; however, the result remains valid if f depends on x' provided we assume conditions concerning the x' dependence of f analogous to those in the previous section. It is clear from the discussion in Section 3 how one must proceed to treat this more general situation.

Consider the n-dimensional second order system

$$x'' = f(t, x). \tag{4.1}$$

Theorem 4.1. *Assume that* $\alpha, \beta: I \to \mathbf{R}^n$, $\alpha(t) \leqslant \beta(t)$, *are twice continuously differentiable functions, that* f *is continuous on*

$$\{(t, x) \mid t \in [0, 1], \alpha(t) \leqslant x \leqslant \beta(t)\},$$

and that

$$\alpha(0) = \alpha(1), \beta(0) = \beta(1), \alpha'(0) = \alpha'(1), \beta'(0) = \beta'(1), \tag{4.2}$$

$$\alpha_i''(t) \geqslant f_i(t, x_1, ..., x_{i-1}, \alpha_i(t), x_{i+1}, ..., x_n),$$

and (4.3)

$$\beta_i''(t) \leqslant f_i(t, x_1, ..., x_{i-1}, \beta_i(t), x_{i+1}, ..., x_n),$$

$$\textit{for} \quad \alpha_j(t) \leqslant x_j \leqslant \beta_j(t), \quad j \neq i, \quad i = 1, ..., n.$$

Then (4.1) *has a periodic solution* $x(t)$ *with* $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ *on* [0, 1].

Proof of Theorem 4.1. Let $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$, where

$$\bar{x}_i = \begin{cases} \beta_i(t), & x_i > \beta_i(t), \\ x_i, & \alpha_i(t) \leqslant x_i \leqslant \beta_i(t), \\ \alpha_i(t), & x_i < \alpha_i(t), \end{cases}$$

and define $F(t, x)$ by

$$F_i(t, x) = \begin{cases} f_i(t, \bar{x}) + (x_i - \beta_i(t))/(1 + x_i^2), & x_i > \beta_i(t), \\ f_i(t, \bar{x}), \quad \alpha_i(t) \leqslant x_i \leqslant \beta_i(t), \\ f_i(t, \bar{x}) + (x_i - \alpha_i(t))/(1 + x_i^2), & x_i < \alpha_i(t), \end{cases}$$

for all $t \in [0, 1]$, $x \in \mathbf{R}^n$.

It follows from the definition of F and condition (4.3) that if $x(t)$ is a periodic solution of

$$x'' = F(t, x), \tag{4.4}$$

then $\alpha(t) \leqslant x(t) \leqslant \beta(t)$ on [0, 1] and hence that $x(t)$ is a periodic solution of (4.1) (see proof of Theorem 3.4). Because of this reason, it suffices to consider (4.4).

Let $\epsilon > 0$ be given and let $A(t)$, $B(t)$ be defined by $A(t) = \alpha(t) - \epsilon e$ $B(t) = \beta(t) + \epsilon e$, where $e = (1,..., 1) \in \mathbf{R}^n$. Then, by (4.3), A and B satisfy

$$\begin{aligned} A_i''(t) &> F_i(t, x_1, ..., x_{i-1}, A_i, x_{i+1}, ..., x_n), \\ B_i''(t) &< F_i(t, x_1, ..., x_{i-1}, B_i, x_{i+1}, ..., x_n), \end{aligned} \tag{4.5}$$

if $A_j(t) \leqslant x_j \leqslant B_j(t)$, $j \neq i$, $i = 1,..., n$.

Let $x(t)$ be any solution of (4.4) with $x(0) \in [A(0), B(0)]$. Then

$$x(t) = x(0) + x'(0)t + (F_i(\xi_i, x(\xi_i)))(t^2/2), \quad 0 < \xi < t, \tag{4.6}$$

and therefore,

$$\| x(t)\| \geqslant \| x'(0)\| t - \| x(0)\| - \| (F_i(\xi_i, x(\xi_i)))\| (t^2/2). \tag{4.7}$$

Since F is bounded, it follows from (4.7) and (4.6) and an elementary indirect argument that there exists $t_0 \in (0, 1]$ and a constant $N = N(t_0)$, $N > \| A'(t)\|, \| B'(t)\|$ such that $x(t_0) \notin [A(t_0), B(t_0)]$, whenever $\| x'(0)\| \geqslant N$ and $x(t) \neq x(0)$, $0 < t \leqslant t_0$. Furthermore, it follows from the definition of F and (4.5) that if $x(t)$ is a solution of (4.4) such that $x(t_1) \in [A(t_1), B(t_1)]$, $x(t_2) \notin [A(t_2), B(t_2)]$ for some $t_1, t_2, 0 \leqslant t_1, t_2, 0 \leqslant t_1 < t_2 < 1$, then $x(t) \notin [A(t), B(t)]$ for any $t \geqslant t_2$.

Let $\Omega_i = \{(x_i, x_i') \mid A_i(0) < x_i < B_i(0), -N < x_i' < N\}$ and $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$. Define $g_i : I \times \partial\Omega_i \to \mathbf{R}^2$ by

$$g_i(t, x_i, x_i') = \begin{cases} (\lambda A_i(t) + (1 - \lambda) B_i(t), \pm N), \\ \quad \text{if } x_i = \lambda A_i(0) + (1 - \lambda) B_i(0) \\ \quad \text{and } x_i' = \pm N \\ (C_i(t), \mu(\pm N) + (1 - \mu) C_i'(t)), \text{ if } x_i = C_i(0) \\ x_i' = \mu(\pm N) + (1 - \mu) C_i'(0) \\ C_i = A_i, B_i \end{cases} \tag{4.8}$$

and extend g_i to $I \times \bar{\Omega}_i$ so that, for every $(x_i, x_i') \in \bar{\Omega}_i$, g_i is differentiable with respect to t at $t = 0$, and so that this derivative is continuous on $\bar{\Omega}_i$. This is easily accomplished by observing the definition of g_i given in (4.8). (We emphasize that in (4.8) six separate cases are notationally combined for each i.)

Next define $g: I \times \bar{\Omega} \to \mathbf{R}^{2n}$ by

$$g(t, x, x') = (g_1(t, x_1, x_1'), \ldots, g_n(t, x_n, x_n')).$$

To show that the hypotheses of Theorem 2.1 are satisfied, let

$$y = (x_1, x_1', \ldots, x_n, x_n')$$

and

$$G(t, y) = (x_1', F_1(t, x), x_2', F_2(t, x), \ldots, x_n', F_n(t, x)). \tag{4.9}$$

It follows from the definition of g and the discussion preceding it that every solution $y(t)$ of

$$y' = G(t, y) \tag{4.10}$$

with $y(0) = y_0 \in \partial\Omega$ has the property that $y(t) \neq g(t, y_0)$, $0 < t \leqslant 1$, and that $y(t)$ exists on $[0, 1]$.

We next need to show that $\deg(-G(0, y) + g_t(0, y), \Omega, 0)$ is nonzero. Let $z_i \in \Omega_i$ be the point of intersection of the straight lines joining the points $(B_i(0), B_i'(0))$, $(A_i(0), A_i'(0))$ and $(A_i(0), -N)$, $(B_1(0), N)$, respectively, and set $z = (z_1, \ldots, z_n)$. Let S denote the involution of $\partial\Omega$ defined by that point.

It will be clear from what is to follow that $k(y) = -G(0, y) + g_t(0, y)$ is nondegenerate on $\partial\Omega$ and hence that $\deg(k(y), \Omega, 0)$ is defined. In turn, we know by Theorem 2.3 that the degree is nonzero provided that, for every $y \in \partial\Omega$, $k(y)$ and $k(Sy)$ do not have the same direction. If $y \in \partial\Omega$, then there exists i, $1 \leqslant i \leqslant n$, such that $y_i \in \partial\Omega_i$. If we show that $k_i(y)$ is not zero and does not have the same direction as $k_i(Sy)$, the proof will be complete. Computing $k_i(y)$, we obtain

$$k_i(y) = \left(\pm N + \frac{B_i(0) - x_i}{B_i(0) - A_i(0)} A_i'(0) + \frac{x_i - A_i(0)}{B_i(0) - A_i(0)} B_i'(0), -F_i(0, x)\right)$$
$$\text{if } x_i' = \pm N, \qquad A_i(0) \leqslant x_i \leqslant B_i(0). \quad (4.11)$$

$$k_i(y) = \left(C_i'(0) - x_i', \frac{x_i' + N}{C_i'(0) + N} C_i''(0) - F_i(0, x_1, ..., x_{i-1}, C_i(0), x_{i+1}, ..., x_n)\right)$$
$$\text{if } x_i = C_i(0), \qquad -N \leqslant x_i' \leqslant C_i'(0), \qquad C_i = A_i \text{ or } B_i. \quad (4.12)$$

$$k_i(y) = \left(C_i'(0) - x_i', \frac{N - x_i'}{N - C_i'(0)} C_i''(0) - F_i(0, x_1, ..., x_{i-1}, C_i(0), x_{i+1}, ..., x_n)\right)$$
$$\text{if } x_i = C_i(0), \qquad C_i'(0) \leqslant x_i' \leqslant N, \qquad C_i = A_i \text{ or } B_i. \quad (4.13)$$

The verification that $k_i(y)$ and $k_i(Sy)$ do not have the same direction now follows from (4.11)–(4.13), (4.5), and the choice of N. This completes the proof.

References

1. L. Cesari, Functional analysis and periodic solutions of nonlinear differential equations, *Contrib. Differential Equations* 1 (1963), 149–167.
2. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
3. E. Heinz, An elementary analytic theory of the degree of mapping in n-dimensional space, *J. Math. Mech.* 8 (1959), 231–247.
4. H. W. Knobloch, Eine neue Methode zur Approximation periodischer Lösungen nichtlinearer Differential-gleichungen zweiter Ordnung, *Math. Z.* 82 (1963), 177–197.
5. H. W. Knobloch, On the existence of periodic solutions of second order vector differential equations, *J. Differential Equations* 9 (1971), 67–85.
6. M. A. Krasnosel'skii, "Translation Along Trajectories of Differential Equations," Transl. of Russian Monograph Ser., Vol. 19, Amer. Math. Soc., Providence, RI, 1968.
7. J. Mawhin, Existence of periodic solutions for higher order differential systems that are not of class D, *J. Differential Equations* 8 (1970), 523–530.
8. K. Schmitt, Periodic solutions of nonlinear second order differential equations, *Math. Z.* 98 (1967), 200–207.
9. K. Schmitt, Periodic solutions of systems of second order differential equations, *J. Differential Equations* 11 (1972), 180–192.