Application of prismatic constructions to gauge field theory

B. Akyar

Department of Mathematics, Dokuz Eylul University, TR-35160 Izmir, Turkey

ARTICLE INFO

Article history:
Received 8 September 2010
Received in revised form 25 January 2012
Accepted 28 February 2012

MSC:
18G30
57R20
57R05
53C05
32C81

Keywords:
Simplicial set
Characteristic class
Chern–Simons theory
Prismatic subdivision
Deligne cohomology

ABSTRACT

We give a variational formula for the Chern–Simons invariants for a given bundle on a simplicial set with a connection using prismatic subdivision.

© 2012 Elsevier B.V. All rights reserved.

1. Introduction

Prismatic sets were introduced by Dupont and Ljungmann [8] in order to have the construction of an integration map in smooth Deligne cohomology. The author worked with the prismatic subdivision of a simplicial set \( S \) in connection with Lattice Gauge Theory [1]. In this paper, we give formulas for the difference of the Chern–Simons classes for a given bundle \( F \to |S| \) with a connection \( \omega \) using prismatic subdivision, where \( |S| \) is the geometric realization of \( S \). We do the variation of the space of connections for Chern–Simons classes using prismatic constructions. The reason for using prismatic construction is that we work on locally trivial bundles where the combinatorial methods for calculating invariants depend on the use of simplicial complexes. In general, a fibre is not a simplicial complex but one can have its natural decomposition into prisms. The development of the theory of bundles, trivializations and transition functions in the setting of simplicial sets leads us to Chern–Weil and Chern–Simons theory. We recall the prismatic constructions, simplicial bundle and transition functions from Akyar and Dupont [3]. The transition functions lead us to define the classifying map which plays an important role for the variational formula. We give the connection in the bundle over the prismatic subdivision of a simplicial set \( S \) using the classifying map. In this work we give the evaluation of the Chern–Simons form for the prismatic subdivision and the variational formula for the Chern–Simons classes for a given bundle \( F \to |S| \) with a connection \( \omega \). This is our main result (Theorem 3.8) and this variation plays an important role in topological quantum field theory (see Witten [16]). Fixing one of the connections in Theorem 3.8 and considering the evaluation of the Chern–Simons form on each simplex in the prismatic subdivision can be used as the Lagrangian of a quantum field theory. We give an example at the end of this
section. Furthermore we interpret the Chern–Simons functional in terms of simplicial forms and Deligne cohomology for a general case.

2. Characteristic classes and Chern–Simons theory

In this section, we refer to Dupont [6] for Chern–Weil Theory for a differentiable principal $G$-bundle $E \to M$, where $G$ is a Lie group with its Lie algebra $\mathfrak{g}$ and $M$ is a differentiable manifold. Let $V$ be a finite dimensional real vector space and denote $S^k(V^*)$ by the vector space of symmetric multilinear real valued functions in $k$ variables for $k \geq 1$. The set of differential forms on $M$ with values in $\mathfrak{g}$ is denoted by $\Omega^*(M, \mathfrak{g})$.

**Definition 2.1.** 1) A linear map $P \in S^k(V^*)$, $P : V^\otimes k \to \mathbb{R}$, is called invariant under the symmetric group action induced by the adjoint representation. The set of invariant polynomials in $S^k(\mathfrak{g}^*)$ is denoted by $I^*(G)$, where $I^*(G) = \bigoplus_{k=0}^{\infty} I^k(G)$ is a subring of $S^*(\mathfrak{g}^*)$.

2) The corresponding cohomology class for $P \in I^k(G)$ is $w_E(P) \in H^{2k}(\Omega^*(M))$ which defines a multiplicative homomorphism

$$w_E : I^*(G) \to H(\Omega^*(M))$$

and it is called the Chern–Weil homomorphism.

Let $\pi : E \to M$ be a principal $G$-bundle on a differentiable manifold $M$. Suppose $\omega$ is a connection in $E$ with the associated curvature form $F_{\omega} \in \Omega^2(E, \mathfrak{g})$. We have $F_{\omega}^k = F_{\omega} \wedge \cdots \wedge F_{\omega} \in \Omega^{2k}(E, \mathfrak{g})$ for $k \geq 1$ so $P \in I^k(G)$ gives rise to the integrand $P(F_{\omega}^k) \in \Omega^{2k}(E)$. For $P \in I^*(G)$, $w_E(P)$ called the characteristic class of $E$ corresponding to $P$, that is, $w_E(P) = [P(F_{\omega}^k)]$.

**Definition 2.2 (Simplicial form).** A simplicial n-form $\varphi$ on a simplicial manifold $M = \{M_k\}$ is a sequence of $n$-forms $\varphi(p)$ on $\Delta^n \times M_k$ such that $(\varepsilon^j \times \text{id})^* \varphi(p) = (\text{id} \times d_j)^* \varphi(p-1)$ on $\Delta^{p-1} \times M_k$, for all $j = 0, \ldots, p$ and $p = 0, \ldots$, where $\Delta^n$ is the standard $p$-simplex, $\varepsilon^j$ is the $j$-th face map and $d_j$ is the $j$-th face operator.

**Definition 2.3 (Chern–Simons form).** Let $\pi : E \to M$ be a principal $SU(2)$-bundle and $\omega$ an $su(2)$-valued 1-form on $E$, where $su(2)$ is the Lie algebra of $SU(2)$. The real valued 4-form $\tilde{p}$ on $E$ given by

$$\tilde{p} = \frac{1}{8\pi^2} \text{Tr}(F_\omega \wedge F_\omega),$$

denote the real-valued map $\text{Tr} : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ is given by $\text{Tr}(B, A) = \text{Tr} BA$, is the lift of a unique 4-form $p$ on $M$, that is, $\pi^*(p) = \tilde{p}$. The 3-form

$$T P(\omega) = \frac{1}{8\pi^2} \text{Tr} \left( \omega \wedge F_\omega - \frac{1}{3} \omega \wedge \omega \wedge \omega \right)$$

is called the Chern–Simons form.

One can easily see that the form $p$ in Definition 2.3 represents the second Chern class of the bundle $\xi(\omega)$, that is, $c_2(E) = \frac{1}{8\pi^2} \text{Tr}(F_\omega \wedge F_\omega)$ (see Phillips and Stone [14]).

In order to compute a related formula for the variational formula of the Chern–Simons classes for a given bundle $F \to |S|$. (see Akyar [2] for the definition of a bundle on a simplicial set) with a connection $\omega$, we need the difference of Chern–Simons forms considered as a difference of differential characters. We study a certain graded ring $H^*(M)$ for a smooth manifold $M$ which is the ring of differential characters on $M$. If $A \subset \mathbb{R}$ is a proper subgroup, a differential character (mod $A$) is a homomorphism $f$ from the group of smooth singular $k$-cycles to $\mathbb{R}/A$, whose coboundary is the mod $A$ reduction of some (necessarily closed) differential form $w \in \Omega^{k+1}(M)$. One can see that $f$ uniquely determines not only $w$ but a class $u \in H^{k+1}(M, \mathbb{A})$ whose real image is cohomologous to the de Rham class of $w$. One can recall the Chern–Simons form as a differential character using a lift of Weil homomorphism due to Cheeger and Simons [4] as follows:

Let $\xi = (E, M, \omega)$ be a principal $G$-bundle, where $G$ is a Lie group. Let $\Omega^*(M)$ denote the set of closed forms. The Chern–Weil homomorphism constructs a homomorphism $w : I^k(G) \to H^{2k}(BG, \mathbb{R})$, where $BG$ is the classifying space of $G$ and a natural transformations $W : I^k(G) \to \Omega^*_{cl} M$ such that the following diagram of natural transformations commutes

\[
\begin{array}{ccc}
I^*(G) & \xrightarrow{w} & H^*(BG, \mathbb{R}) & \xleftarrow{r} & H^*(BG, \mathbb{A}) \\
\downarrow{W} & & \downarrow{C_\mathbb{R}} & & \downarrow{C_\mathbb{A}} \\
\Omega^*_{cl} M & \xrightarrow{\text{DR}} & H^*(M, \mathbb{R}) & \xleftarrow{r} & H^*(M, \mathbb{A})
\end{array}
\]
where $C_A, C_R$ are provided by the theory of characteristic classes and $D_R$ is the de Rham homomorphism, where $A$ is a proper subring of $R$. If $P \in \mathcal{P}(G)$, $u \in \mathcal{H}^*(BG, \Lambda)$ and $F_{\omega}$ is the curvature form of $\xi$, where $\xi \in \varepsilon(G)$ and $\varepsilon(G)$ is the category of the objects which are triples $\xi = (E, M, \omega)$ with morphisms being connection-preserving bundle maps, then $W(P) = P(F_0, \ldots, F_{\omega})$ and $C_A(u) = u(\xi)$ is the characteristic class. Set

\begin{align*}
K^k(G, \Lambda) &= \{ (P, u) \in \mathcal{P}(G) \times \mathcal{H}^k(BG, \Lambda) \mid w(P) = r(u) \}, \\
R^k(M, \Lambda) &= \{ (w, u) \in \Omega_c^k(M) \times \mathcal{H}^k(M, \Lambda) \mid r(u) = [w] \}
\end{align*}

and $R^*(M, \Lambda) = \bigoplus_k R^k(M, \Lambda)$. Here $r : H^k(M, \Lambda) \to H^k(M, \mathbb{R})$ and $[w]$ is the de Rham class of $w$ and $\Omega^k_c$ denotes the closed $k$-forms with periods lying in $\Lambda$. The diagram (2.4) induces a map $W \times C_A : K^*(G, \Lambda) \to R^*(M, \Lambda)$. On the other hand, we have a ring which is given by

\[ \hat{H}^k(M, \mathbb{R}/\Lambda) = \{ f \in \text{Hom}(Z_k, \mathbb{R}/\Lambda) \mid f \circ \partial \in \Omega^k \}. \]

We set $\hat{H}^{-1}(M, \Lambda) = \Lambda$.

**Definition 2.5 (Differential character).** $\hat{H}^*(M, \mathbb{R}/\Lambda) = \bigoplus \hat{H}^k(M, \mathbb{R}/\Lambda)$ is a graded $\Lambda$-module whose objects are called differential characters.

There exists a unique natural transformation

\[ S : K^*(G, \Lambda) \to \hat{H}^*(M, \mathbb{R}/\Lambda) \]

such that the diagram

\[ \begin{array}{ccc}
K^*(G, \Lambda) & \xrightarrow{W \times C_A} & R^*(M, \Lambda) \\
S & \downarrow{\delta_1, \delta_2} & \downarrow{\delta_1, \delta_2} \\
\hat{H}^*(M, \mathbb{R}/\Lambda) & & \end{array} \]

commutes and $S_{P, u} \in \hat{H}^{2k-1}(M, \mathbb{R}/\Lambda)$ is called the Chern–Simons class.

**Theorem 2.6. (Cheeger and Simons [4])** Let $(P, u) \in K^2k(G, \Lambda)$. For each $\xi \in \varepsilon(G)$, there exists a unique $S_{P, u} \in \hat{H}^{2k-1}(M, \mathbb{R}/\Lambda)$ satisfying:

1. $\delta_1(S_{P, u}(\omega)) = P(F_{\omega})$.
2. $\delta_2(S_{P, u}(\omega)) = u(\xi)$.
3. If $\xi \in \varepsilon(G)$ and $\phi : \xi \to \xi$ is a morphism then $\phi^*(S_{P, u}(\hat{\omega})) = S_{P, u}(\omega)$.

For a pair $(P, u)$, we have the Chern–Simons class for $\xi = (E, M, \omega)$ with $F_{\omega}^{k+1} = 0$ (see Dupont and Kamber [7]). We can give a more geometric interpretation of the Chern–Simons class.

**Proposition 2.7.** Suppose that $M^{2k-1} = \partial W^{2k}$ is an oriented manifold and $\xi$ extends to $\hat{\xi}$ over $W$. Let $\hat{\omega}$ be any extension of $\omega$ in $\hat{E}$. Setting $\hat{\xi} = (\hat{E}, W, \hat{\omega})$ and we have the morphism $\xi \to \hat{\xi}$. Thus $S_{P, u}(\hat{\omega})|_{M^{2k-1}} = S_{P, u}(\omega)$. Moreover

\[ S_{P, u}(\omega), [M] = \int_W P(F_{\hat{\omega}}) \mod \mathbb{Z}. \]

**Proof.** We have

\[ S_{P, u}(\omega), [M] = \{ \psi^* \hat{s}, [M] \} = \{ \psi^* \hat{s}, [\partial W] \} = \{ \delta \psi^* \hat{s}, [W] \} = \{ \mathfrak{I}([P(F_{\hat{\omega}})]), [W] \} = \int_W P(F_{\hat{\omega}}). \]

Here $P(F_{\hat{\omega}})$ is the Chern–Simons form, $\mathfrak{I}$ is the integration map, $\psi : M \to \hat{E}/G = BG$ is the classifying map and $\psi^* \hat{s} \in C^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ is called the Chern–Simons character. Moreover $\psi^* \hat{s} \mod \mathbb{Z}$ is called the cohomology class. \( \square \)
Corollary 2.8. For two given connections \(\omega_1\) and \(\omega_2\) in a bundle \(E \to M\), we have \(\tilde{\omega} = (1-t)\omega_0 + t\omega_1 + t\omega_2\) in \(M \times [0,1] = W^k\) and the difference of the characters is the reduction of a form. In other words, the difference of the Chern–Simon classes on the manifold \(M\) is given as a variational formula

\[
\{S_{p,u}(\omega_2), [M]\} - \{S_{p,u}(\omega_1), [M]\} = \int_W P(F_{\tilde{\omega}}) = \int_M TP(\omega_1, \omega_2),
\]

where \(TP(\omega_1, \omega_2) = \int_0^1 i_{d\omega/dt} P(F_{\omega}) dt\), where \(i_{d\omega/dt}\) is the usual interior product in the \(t\)-variable.

Example 2.9. Suppose \(E = M \times G\) and \(M = M^{2k-1}\) is a closed, oriented manifold, \(P \in \mathcal{L}(G)\). Take \(W = M \times [0,1]\) and for any connection \(\omega\) in \(E\) and \(\omega_{MC}\) we write

\[
\tilde{\omega} = (1-t)\omega_{MC} + t\omega = \omega_{MC} + tA
\]

which is a connection in \(\tilde{E} = W \times G\), where \(A = \omega - \omega_{MC}\). We get

\[
\{S_{p,u}(\omega), [M]\} - \{S_{p,u}(\omega_{MC}), [M]\} = \int_{M \times [0,1]} P(F_{\omega}),
\]

since \(S_{p,u}(\omega_{MC}) = 0\) we have

\[
\{S_{p,u}(\omega), [M]\} = \int_M TP(A),
\]

where \(TP(A) \overset{\text{def}}{=} \int_0^1 i_{d\omega/dt} P(F_{\omega}) dt\) and \(TP(A)\) is an algebraic expression in \(A\).

3. Applications to gauge field theory

In this section, we evaluate the Chern–Simons form on a prismatic set for a given bundle \(F \to |S|\) with a connection \(\omega\) using the prismatic subdivision of the simplicial set \(S\). For clarification we recall some notations and explanations from Akyar and Dupont [3]:

Definition 3.1. Given \(p \geq 0\), a \((p+1)\)-multi-simplicial set \(S\) is a sequence \(\{S_{q_0,...,q_p}\}\) which is a simplicial set in each variable \(q_j, j = 0, \ldots, p\) and such that the face and degeneracy operators

\[
d_j^k : S_{q_0,...,q_p} \to S_{q_0,...,q_{i-1},...,q_p},
\]

\[
s_j^k : S_{q_0,...,q_p} \to S_{q_0,...,q_{i+1},...,q_p}
\]

for \(k = 0, \ldots, p\), satisfying the simplicial identities and so that they commute with \(d_m^n, s_m^n\) for \(k \neq l\), where \(l = 0, \ldots, p\).

Definition 3.2 (Prismatic set). A prismatic set \(PS\) is a sequence \(\{P, P S\} = \{P_{p} S_{q_0,...,q_p}\}\) of \((p+1)\)-multi-simplicial sets together with face operators \(d_k : P_{p} S_{q_0,...,q_p} \to P_{p-1} S_{q_0,...,q_{-1},...,q_p}\) commuting with \(d^k_m\) and \(s^k_m\) (saying \(d^k_m = s^k_m = \text{id}\) on the right) such that \(\{P, PS\}\) is a \(\Delta\)-set. Explicitly,

\[
P_{p} S_{q_0,...,q_p} := S_{q_0+,...,q_p+},
\]

where \(q_0 + \cdots + q_p = q\), is the \((p+1)\)-prismatic set with face operators

\[
d^k_j : P_{p} S_{q_0,...,q_i,...,q_p} = S_{q+p} \to P_{p} S_{q_0,...,q_{i-1},...,q_p} = S_{q+p-1}
\]

which are defined by

\[
d^k_j := d_{q_0+,...,q_{i-1}+j},
\]

\(j = 0, \ldots, q_i, i = 0, \ldots, p\), and degeneracy operators

\[
s^k_j : P_{p} S_{q_0,...,q_i,...,q_p} = S_{q+p} \to P_{p} S_{q_0,...,q_{i+1},...,q_p} = S_{q+p+1}
\]

which are defined by

\[
s^k_j := s_{q_0+,...,q_{i-1}+j},
\]
The face maps \( d_i : P_p S_{q_0 \ldots q_p} \rightarrow P_{p-1} S_{q_0 \ldots q_p} \) are the operators corresponding to the inclusions \( \Delta^q \rightarrow \Delta^q + i \rightarrow \Delta^q + p \rightarrow 1 \). Moreover if similarly there are given degeneracy operators \( s_k : P_p S_{q_0 \ldots q_p} \rightarrow P_{p+1} S_{q_0 \ldots q_k q_{k+1} \ldots q_p} \) making \( P_p S \) a simplicial set, a prismatic set \( P S \) is called a strong prismatic set.

**Example 3.3 (Triangulated fibre bundles).** Given a smooth fibre bundle \( \pi : Y \rightarrow Z \) with \( \dim Y = m + n \), \( \dim Z = m \) and compact fibres possibly with boundary. By a theorem of Johnson [13], there are smooth triangulations \( K \) and \( L \) of \( Y \) and \( Z \), respectively and a simplicial map \( \pi' : K \rightarrow L \) in the following commutative diagram:

\[
\begin{array}{ccc}
|K| & \xrightarrow{\approx} & Y \\
|\pi'| & \xrightarrow{\pi} & |L| \xrightarrow{\approx} Z
\end{array}
\]

and the horizontal maps are homeomorphisms which are smooth on each simplex. Let \( K \) be an ordered simplicial complex as in Dwyer and Henn [12, Section 3] and let \( |K| = \bigsqcup_{k \in K_k} \Delta^k \times \tau / \sim \), \( k = 0, \ldots, \dim K \), be the geometric realization.

A simplex \( \tau \) in \( K \) has vertices \( \tau = (b_0^0, b_0^1, \ldots, b_0^p) \) with \( \sigma = (a_0, \ldots, a_p) \) in \( L \) such that \( \pi'(b_j^i) = a_i \). So geometrically, for an open simplex \( \sigma \) in \( L \), we have

\[
\pi^{-1}(\bar{\sigma}) \approx |\bar{\sigma}| \times \bigsqcup_{\tau \in \pi^{-1}(\sigma)} \Delta^{|q_0 - q_p|} \times \tau
\]

where \( \Delta^{|q_0 - q_p|} = \Delta^q \times \cdots \times \Delta^q \).

In order to see how the prismatic subdivision appears, we recall the following example from Akyar and Dupont [3].

**Example 3.4 (Prismatic triangulation of a simplicial set).** Let \( f : S \rightarrow \{*\} \) be a simplicial map of simplicial sets, where \( \{*\} \) is the simplicial set with one element in each degree and

\[
P_p(f)_{q_0 \ldots q_p} = \{ (\bar{\sigma}, \sigma) \in \{*\} \times S_{q_0 \ldots q_p + p} \mid f(\sigma) = \mu_{q_0 \ldots q_p}(\bar{\sigma}) \}
\]

where \( \mu_{q_0 \ldots q_p} = \hat{s}_{q+p} \circ S_{q_0 + \ldots + q_p + 1} \circ \cdots \circ S_{q_0 + \ldots + q_{p-1} + 1} \circ \cdots \circ \hat{s}_{q_0} \circ S_{q_0}, \) where “=” means that the term is left out. We call \( P_p(f) = P_p S \) the \( p \)-th prismatic subdivision of \( S \).

We have for each \( p \) the geometric realization

\[
|P_p S| = \bigsqcup_{q_0 \ldots q_p} \Delta^{|q_0 - q_p|} \times S_{q_0 + \ldots + q_p + p} / \sim
\]

with equivalence relation “\( \sim \)” generated by the face and degeneracy maps

\[
e^i_j : \Delta^{|q_0 - q_i| - 1 - q_p} \rightarrow \Delta^{|q_0 - q_i| - q_p}
\]

and

\[
\eta^i_j : \Delta^{|q_0 - q_i| - q_p} \rightarrow \Delta^{|q_0 - q_i| - 1 - q_p},
\]

respectively. The sequences of \( \Delta \)-spaces \( \{|P_p S|\} \) give the fat realization:

\[
\|\{P_p S\}\| = \bigsqcup_{p \geq 0} \Delta^p \times |P_p S| / \sim,
\]

with equivalence relation “\( \sim \)” generated by the face operators \( |d_i| : |P_p S| \rightarrow |P_{p-1} S| \) which are given by \( |d_i| = \pi_i \times d_i \) with the natural projections \( \pi_i : \Delta^{|q_0 - q_i|} \rightarrow \Delta^{|q_0 - q_i| - 1 - q_p} \), where \( i = 0, \ldots, p \). For each \( t \in \Delta^p \), the map \( |P_p S| \rightarrow |S| \) is a homeomorphism. In particular, \( \|\{P_p S\}\| \rightarrow |S| \) is a homotopy equivalence.

**Example 3.5 (Nerve of coverings).** (Dupont and Ljungmann [8]) Given a covering \( \mathcal{U} = \{U_i\} \) of \( Z \), there is a covering \( \mathcal{V} = \{W_i = \pi^{-1}(U_i)\} \) of \( Y \), and for each \( i \), \( V^i \) is an open cover of \( W_i \) giving a covering \( \mathcal{V} = \bigcup V^i \). Then \( P_p \mathcal{N}(\mathcal{V}/\mathcal{U})_{q_0 \ldots q_p} = \left( \bigcup_{V^i_{j_0} \cap \cdots \cap V^i_{j_p} \neq \emptyset} |V^i_{j_0} \cap \cdots \cap V^i_{j_p}| \right) \), where \( V^i_{j} \in \mathcal{V}^i \), is given with face and degeneracy maps as inclusions.

**Note.** Here \( \mathcal{U} = \{U_i = s(t_{a_i})\} \) where \( a_i \in t^p \) is a 0-simplex in \( L \) and \( \mathcal{V}^i = \{V^i_{j} = s(t_{b_i}^j)\} \), where \( b_i^j \in \pi^{-1}(a_i) \cup R^0 \).

Let us recall the transition functions from Akyar [2] for a given bundle over the realization of a simplicial set \( S \) and \( \sigma \in S_p \). Given a bundle \( F \rightarrow |S| \) and a set of trivializations, we get for each face \( \tau \) of say \( \dim \tau = q < p = \dim \sigma \) in \( \sigma \), a transition function \( \psi_{\sigma, \tau} : \Delta^q \rightarrow G \) as follows: The bundle map \( \theta \) given by the diagram
\[\Delta^q \times (d_{l_p-i_0}\sigma) \times G \xrightarrow{\Theta} \Delta^p \times (\sigma) \times G\]

\[\Delta^q \times d_{l_p-i_0}\sigma \xrightarrow{\varepsilon_{l_0-i_p}} \Delta^p \times \sigma\]

where \(d_i\sigma = \tau, \Theta = \varphi_0 \circ \varepsilon_{l_0-i_p} \circ \varphi^{-1}_{d_{l_p-i_0}\sigma}, \varepsilon_{l_0-i_p} = \varepsilon_{l_0} \circ \cdots \circ \varepsilon_{i_p}, \varepsilon_{l_0} = \varepsilon_{l_0} \circ \cdots \circ \varepsilon_{l_p}\) and \(d_{l_p-i_0} = d_{l_p} \circ \cdots \circ d_{i_0}\), determines \(\nu_{\sigma, t}\) by the formula

\[\Theta(t, g) = (\varepsilon_{l_0} \circ \cdots \circ \varepsilon_{l_p}(t), \nu_{\sigma, t}(t)g), \quad t \in \Delta^q, \quad g \in G.\]

Let \(\tilde{P}_p S_{q_0+\cdots+q_p} := S_{q_0+\cdots+q_p+2p+1}\) be another prismatic set for a simplicial set \(S\) given with face and degeneracy operators inherited from the ones of \(S_{q_0+2p+1}\) (see Akyar and Dupont [3]). There is a map \(h : ||\tilde{P}_p S.|| \rightarrow ||P, S.||\) defined by \(h(t, s^0, \ldots, s^p, x) = (t, s^0, \ldots, s^p, d_{q_0+1} \circ d_{q_0+q_1+3} \circ \cdots o d_{q_0+2p+1})\), \(x \in S_{q_0+2p+1}\), where \(q = q_0 + \cdots + q_p\) and \(||\tilde{P}_p S.|| = \bigsqcup_{p \geq 0} \Delta^p \times \Delta^q_0 \times \tilde{P}_p S_{q_0+\cdots+q_p}/\sim\) is defined with the equivalence relation given similarly as described for \(||P, S.||\).

The transition functions are used to define the classifying mapping \(\tilde{m} : ||\tilde{P}_p S.|| \rightarrow BG\) (see also Akyar and Dupont [3]), the map \(\tilde{m} : \tilde{F} \rightarrow EG\) and a connection \(\tilde{\omega}\) in \(\tilde{F} \rightarrow ||\tilde{P}_p S.||\), where \(\tilde{F} \cong \tilde{m}^* (E G)\).

We assume that we already have a connection \(\hat{\omega}'\) in the bundle \(\tilde{F} \rightarrow ||\tilde{P}_p S.||\) and the other one \(\hat{\omega}\) can be defined as

\[\hat{\omega} = \tilde{m}^* (\omega), \quad \hat{\omega} = \tilde{m}^* \left(\sum_{i=0}^p t_i^* g^{-1}_i(t) \cdot d g_i(t)\right),\]

where \(\omega\) is the universal connection in the universal bundle \(EG \rightarrow BG\). The bundle over \(||\tilde{P}_p S.||\) enables us to find the variational formula of the Chern–Simons classes for the bundle over \(S.\).

Now, we compare these two connections \(\hat{\omega}, \hat{\omega}'\) in \(\tilde{F} \rightarrow ||\tilde{P}_p S.||\) so that we give the variational formula of the Chern–Simons class for \(F \rightarrow |S.|\) with \(\hat{\omega}\) and \(\hat{\omega}'\) as a difference form.

**Definition 3.6 (The Chern–Simons functional).** The Chern–Simons form on each simplex \(x\) of \(||\tilde{P}_p S.||\) is

\[\langle S_{p, u}(\hat{\omega}), ||\tilde{P}_p S.|| \rangle_x = \int_{\Delta^p \times \Delta^q_0} P(F_{\hat{\omega}}). \quad (3.7)\]

We can conclude the main result as the following theorem.

**Theorem 3.8.** For two connections \(\hat{\omega}\) and \(\hat{\omega}'\) in a bundle \(\tilde{F} \rightarrow ||\tilde{P}_p S.||\), the variational formula of the Chern–Simons classes on each simplex \(x\) of \(||\tilde{P}_p S.||\) is given as a difference form

\[\langle S_{p, u}(\hat{\omega}'), ||\tilde{P}_p S.|| \rangle_x - \langle S_{p, u}(\hat{\omega}), ||\tilde{P}_p S.|| \rangle_x = \int_{I \times \Delta^p \times \Delta^q_0} P(F_{\hat{\omega}}) \quad (3.9)\]

where \(\tilde{\omega} = (1-t)\hat{\omega} + t\hat{\omega}'\) and \(TP(\hat{\omega}, \hat{\omega}') = \int_0^1 \frac{dh}{dt} P(F_{\hat{\omega}}) \ dt.\)

**Example 3.10.** Let us consider two different compatible transition functions corresponding to parallel transport functions by varying with respect to the parameter \(t\) in \(\Delta^p\), saying that \(t' = t + \lambda t\). Varying \(t\) defines two different classifying maps which also determine two connections \(\hat{\omega}_1\) and \(\hat{\omega}_2\) in the bundle \(\tilde{F} \rightarrow ||\tilde{P}_p S.||\). The variational formula is given by

\[\langle S_{p, u}(\hat{\omega}_1), ||\tilde{P}_p S.|| \rangle - \langle S_{p, u}(\hat{\omega}_2), ||\tilde{P}_p S.|| \rangle = \int_{I \times \Delta^p \times \Delta^q_0} P(F_{\hat{\omega}_2})\]

where \(\tilde{\omega}_2 = (1-t)\hat{\omega}_1 + t\hat{\omega}_2\), as the variation of the connection for the Chern–Simons classes.

**Note.** Construction of a prismatic set which corresponds to the nerve of the covering by stars of vertices and a classifying map on it gives us a principal G-bundle with a connection and explicit formulas for characteristic classes via the usual Chern–Weil and Chern–Simons theory.
On the other hand, the Chern–Simons functional $S_{p,u}(\omega)$ can be used as the Lagrangian of a quantum field theory by fixing one of the connections, let us call it $\omega_0$ and this leads us to path integrals $Z(M) = \int e^{2\pi i S_{p,u}(\omega)} d\omega$ over the space of all connections on the 3-manifold (see Freed [9], Huebschmann [10], Witten [16], Dijkgraaf and Witten [11], Rabin [15]).

Theorem 3.8 can be incorporated into the paper by Dupont and Ljungmann [8]. If one wants to have a better formula (see Theorem 1.1 in [8]) one can use Deligne cohomology which leads a combinatorial version of the Chern–Simons function. In other words, Theorem 3.8 is a way to avoid the use of Deligne cohomology.

Note. When $\tilde{F} \to \|P,S,|\|$ is topologically trivial, one usually takes $\tilde{\omega}$ to be the trivial connection and $S_{p,u}(\tilde{\omega})$ to be zero; for a flat connection $\omega$, the forms $TP(\omega, \tilde{\omega})$ are then closed and $S_{p,u}(\tilde{\omega})$ calculates the customary Chern–Simons invariants of the flat connection $\tilde{\omega}$.

4. Another approach to integration with Deligne cohomology

The present work can be incorporated into the paper by Dupont and Ljungmann [8]. Now we can interpret $\int_{\Delta^p \times \Delta^{q_0-q_p} P(F)}$ given by (3.7) in terms of integration of simplicial forms and Deligne cohomology (in order to have a combinatorial version for (3.9)). If we fix $\tilde{\omega}$ and only consider the first term in (3.9) we get a better formula for the Chern–Simons form on each simplex of $\|P,S,|\|$. Let us recall some facts from [8].

**Theorem 4.1.** (Dupont and Ljungmann [8]) Given a fiber bundle $\pi : Y \to Z$ with compact, oriented $n$-dimensional fibers and suitable coverings $V$ and $U$ of $Y$ and $Z$, respectively. Then there is a map

$$
\int_{\{Y/Z\}} : \Omega^{*-n}((|N\nu|)) \to \Omega^*(|NU|),
$$

where $N\nu$, $NU$ denote the nerves of the coverings $V$, $U$, respectively, that is, given an open cover $U = \{U_i\}$ of $Z$, the nerve $NU = \{NU(p)\}$ of the covering is given by $NU(p) = \bigcup_{U_{i_0} \cap \ldots \cap U_{i_p}}$, here $U_{i_0} \cap \ldots \cap U_{i_p}$. $NU$ is a simplicial manifold with the face $d_j : U_{i_0 \ldots i_p} \to U_{i_0 \ldots i_j \ldots i_p}$ and degeneracy maps $s_j : U_{i_0 \ldots i_p} \to U_{i_0 \ldots \hat{i}_j \ldots i_p}$ given as inclusions.

**Definition 4.2.** A simplicial $n$-form $w = \{w^{(p)}\}$ on $NU$ can be given in a similar way as in Definition 2.2 considering $M$ as $NU$.

**Definition 4.3** (Prismatic forms). A prismatic $n$-form is a collection $w = \{w_{q_0 \ldots q_p}\}$ of forms $w_{q_0 \ldots q_p} \in \Omega^n(\Delta^p \times \Delta^{q_0-q_p} \times P^1 N(V/U))$ satisfying the equivalence relations (see Definition 3.3 in Dupont and Ljungmann [8]). A form is called normal if it also satisfies the equivalence relations for the degeneracy maps.

The complex of normal prismatic form is $\Omega^*(|PNV/U|)$ and we have

$$
\Omega^*(|PNV/U|) = \bigoplus_{p+q+r=n} \Omega^{p,q,r}(|PNV/U|),
$$

where $q = q_0 + \ldots + q_p$ and $\Omega^{p,q,r}(|PNV/U|)$ is the set of forms of degree $p$ in the barycentric coordinates of the first simplex, of degree $q_0$ in the second and so on and finally of degree $r$ in some local coordinates on the nerve of the covering. Thus $\Omega^*(|PNV/U|)$ becomes a triple-complex.

Given a triangulation $L$ of a smooth $m$-manifold $Z$ there is an open cover $U$ given by the stars $\text{st}(\sigma)$, where $a \in L^0$ and $\forall \sigma \in L, \text{st}(\sigma) = \bigcup_{a \in L^0, \sigma \subseteq \tau} |\tau|$ is the closed star which inherits a natural triangulation $L_\sigma$ from $L$. This gives $|L_\sigma| \cong \text{st}(\sigma)$.

**Definition 4.4.** i) The triangulated nerve $NL$ is the simplicial complex with $p$-simplices given by $N_p L = \bigcup_{\sigma \in L^p} |L_\sigma|$ for $\sigma = (a_0, \ldots, a_p)$ the face and degeneracy operators $d_j, s_j$ are inclusions.

ii) A $p$-simplicial form $w$ on $|NL|$ is a collection of forms $w = \{w^{(p)}\}$ living on $\bigcup_{\sigma \in L^p} \Delta^p \times \Delta^1 \times L_\sigma^{(i)}$, where $L_\sigma^{(i)}$ is the discrete set of $i$-simplices in $L_\sigma$.

We recall an integration map $\int_{|Y/Z|} : \Omega^{*-n}(|NK|) \to \Omega^*(|NL|)$ from Dupont and Ljungmann [8]. For a simplex $\sigma = (a_0, \ldots, a_p) \in L$ we have a chain map

$$
AW : P_m C_k(K/L_\sigma) \to \bigoplus_{k_1+k_2=k} P_p C_{k_1}(K/\sigma) \otimes P_m C_{k_2}(K/L_\sigma)
$$

by

$$
AW(\tau) = \sum_{0 \leq q_1 \leq q_j} \tau^{q_0-q_p} \otimes \tau^{q_0-q_p}.
$$
where
\[ \tau_{q_0\cdots q_p} = [b_{01}^{i_0},\ldots,b_{q_0}^{i_0}\cdots|b_{01}^{i_p},\ldots,b_{q_p}^{i_p}] \]
and
\[ \tau_{q_0\cdot q_p} = [b_{01}^{i_0},\ldots,b_{q_0}^{i_0}\cdots|b_{01}^{i_j},\ldots,b_{q_j}^{i_j}\cdots|b_{01}^{i_m},\ldots,b_{q_m}^{i_m}], \]
for \( 0 \leq q_j \leq q_i \), where \( i, j = 0,\ldots, p \). Here \( P_p C_{q_0\cdot q_p} (K/L) \) is the free abelian group generated by \( P_p S(K/L)q_0\cdot q_p \) and \( P_p S(K/L)q_0\cdot q_p \subseteq S_{p+q_0+\cdots+q_p} \times S_p \) is the subset of pairs of simplices \( (\tau, \eta) \) so that \( q_i + 1 \) of the vertices in \( \tau \) lie over the \( i \)-th vertex in \( \eta \). Let \( (\tau, \eta) \in P_m C_K(K/L) \) then since \( \eta \) is a top-dimensional simplex in \( L_\sigma \), we have \( \sigma \subseteq \eta \). Let \( i_0,\ldots,i_p \in \{0,\ldots,m\} \) denote the indices of the corresponding vertices of \( \sigma \) in \( \eta \). Write \( \tau \propto \sigma = (b_{01}^{i_0},\ldots,b_{q_0}^{i_0}\cdots|b_{01}^{i_m},\ldots,b_{q_m}^{i_m}) \), where the \( i \)-th block \( |b_{01}^{i_1},\ldots,b_{q_1}^{i_1}| \) lies over the \( i \)-th vertex in \( \eta \). Then the integration map is given by
\[
\int_{\Delta^p \times \eta} w_{\tau_{q_0\cdot q_p}} = \sum_{\tau \in P_m S_K(\eta)} \sum_{0 \leq q_j \leq q_i} \epsilon(\tau) \int_{\Delta^p \times \eta} w_{\tau_{q_0\cdot q_p}}^{(p+q)},
\]
where \( w_{\tau_{q_0\cdot q_p}}^{(p+q)} \in \Omega^{*+n}(\Delta^p \times K_{q_0\cdot q_p}) \) and \( \epsilon(\tau) \) is the sign of \( \tau \) in \( [Y_\eta] \) and \( q = \sum_{i=0}^{p} q_i \). Here \( w \) is restricted to \( \Delta^p \times \eta \) and it is integrated along the fiber over \( \Delta^p \times \eta \) with respect to the map \( \Delta^p \times \eta \rightarrow \Delta^p \) given by
\[
(t_0,\ldots,t_{p+q}) \mapsto \left( \sum_{i=0}^{q_0} t_i, \sum_{i=q_0+1}^{q_0+q_1+1} t_i, \ldots, \sum_{i=q_0+\cdots+q_{p-1}+p}^{q_0+\cdots+q_p+1} t_i \right),
\]
and the map \( \tau_{q_0\cdot q_p} \rightarrow \eta \) is the restriction of \( \pi \).

For \( w \in \Omega^{|N|+k}([N^{|N|}]) \)
\[
\left( \int_{[Y/\pi]} w \right)_{|\Delta^p \times U_{q_0\cdot q_p}} = \int_{\Delta^p \times \eta} \hat{\Phi}^{*} f^{*} w
\]
where the right hand side denotes usual integration along the fibers, \( f : |P N \vee U| \rightarrow |N V| \) and \( \hat{\Phi} : |N W| \rightarrow |P N \vee U| \). This integration enables us to give a better formulation for (3.9) when \( \tilde{\omega} \) is fixed.

Acknowledgements

I would like to thank Johan L. Dupont for his interests and comments during the preparation of this paper and Aarhus University for hospitality.

References