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Spectral properties for perturbations of unitary operators ☆

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ABSTRACT

Consider a unitary operator U_0 acting on a complex separable Hilbert space $\mathcal{H}$. In this paper we study spectral properties for perturbations of U_0 of the type,

$$U_\beta = U_0 e^{iK\beta},$$

with K a compact self-adjoint operator acting on $\mathcal{H}$ and β a real parameter. We apply the commutator theory developed for unitary operators in Astaburuaga et al. (2006) [1] to prove the absence of singular continuous spectrum for U_β. Moreover, we study the eigenvalue problem for U_β when the unperturbed operator U_0 does not have any. A typical example of this situation corresponds to the case when U_0 is purely absolutely continuous. Conditions on the eigenvalues of K are given to produce eigenvalues for U_β for both cases finite and infinite rank of K, and we give an example where the results can be applied.

1. Introduction and notation. Abstract setting

Consider a unitary operator U_0 acting on a complex separable Hilbert space $\mathcal{H}$ with inner product $\langle , \rangle$ conjugate linear in the first component. Let us denote by $\{E_0(\cdot)\}$ the spectral family associated to U_0; in other words,

$$\langle \phi, U_0^n \phi \rangle = \int_{\mathbb{T}} e^{in\theta} \, d\langle \phi, E_0(\theta)\phi \rangle$$

for all $\phi \in \mathcal{H}$ and $n \in \mathbb{Z}$, with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

In what follows we denote by $\mathfrak{G}_\infty$ the set of compact operators defined on $\mathcal{H}$. For $K \in \mathfrak{G}_\infty$ and self-adjoint we define the perturbed (U_0 the unperturbed) unitary operator U_β as follows

$$U_\beta = U_0 e^{i\beta K} \tag{1.1}$$

where β is a real parameter.

The identity $U_\beta - U_0 = (e^{i\beta K} - I)U_0$ implies that $U_\beta - U_0 \in \mathfrak{G}_\infty$, so by Weyl's theorem the essential spectrum of U_0 and U_β coincide, see [9].

In [1] the authors developed a well posed commutator theory for unitary operators (for a general theory about commutators see [3,7]). Precisely, they proved that if U and A satisfy

(a) the first commutator $U^*AU - A$ is densely defined and it admits a bounded extension satisfying a Mourre's inequality, $U^*AU - A \geqslant \alpha I + C$ for some positive constant α, compact operator C and self-adjoint operator A,
(b) the second order commutator $[A, U^*AU]$ is densely defined and it admits a bounded extension,

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then the spectrum of U has no singular continuous component and only a finite number of eigenvalues of finite multiplicity in $S^1 = \{z\colon\ |z| = 1\}$, see Theorem 3.3 in [1]. Moreover, if (a) holds with $C = 0$ then the spectrum of U is purely absolutely continuous in S^1. We shall apply these results to prove the absence of singular continuous spectrum for U_β. We mention [2] for the instability problem of embedded eigenvalues in the self-adjoint case.

The purpose of Section 2 is to find conditions on the parameters $\{\lambda_n\}$ and the orthonormal set $\{u_n\}$, given by the spectral decomposition of K (Riez–Fisher theorem), which guarantees that U_β satisfies (a), (b). For simplicity we only consider the case $U_0^* A U_0 - A = I$.

In Section 3 we study the eigenvalue problem for U_β under the assumptions that U_0 does not have any eigenvalues. A typical example of this sort corresponds to the case when U_0 is purely absolutely continuous.

We recall that U_0 is *purely absolutely continuous* if its spectral measure $d\langle\phi, E_0(\theta)\phi\rangle$ has a Radon–Nikodym derivative, with respect to the Lebesgue measure $\frac{d\langle\phi, E_0(\theta)\phi\rangle}{d\theta} = F_\phi(\theta)$ belonging to $L^1(\mathbb{T})$, for any $\phi \in \mathcal{H}$, see [4,8,9] as references.

The eigenvalue problem for U_β consists of finding a vector $\psi \in \mathcal{H}$, $\psi \neq 0$ such that $U_\beta \psi = z_0 \psi$, with $z_0 = e^{iE}$, $E \in \mathbb{R}$. Actually the eigenvalue problem $U_0, e^{iK\beta}\psi = z_0\psi$ is equivalent to

$$U_0\big(I - e^{i\beta K}\big)\psi = (U_0 - z_0 I)\psi. \tag{1.2}$$

Notice that $(U_0 - z_0 I)\psi \neq 0$ if z_0 is not an eigenvalue for U_0.

On the other hand, K is compact and self-adjoint, and hence by the Riez–Fisher theorem there exist an orthonormal set $\{u_n\}_n \subset \mathcal{H}$ and a collection of real numbers $\{\lambda_n\}_n$ such that

$$K = \sum_{n=1}^{\infty} \lambda_n \langle u_n, \cdot\rangle u_n, \tag{1.3}$$

with $\lambda_n \to 0$ as n tends to infinity, see [8]. Using this representation for K, the eigenvalue problem (1.2) becomes

$$\sum_{n=1}^{\infty} \big(1 - e^{i\beta\lambda_n}\big)\langle u_n, \psi\rangle U_0 u_n = (U_0 - z_0 I)\psi, \tag{1.4}$$

where the above identity holds in the strong sense in $\mathcal{H}$.

Our main goal is to give conditions on U_0, the spectral representation of K, the real parameter β and $z_0 = e^{iE}$ that guarantee the existence of non-trivial solution of (1.4), for rank K finite or infinite. The perturbation of rank one of a purely absolutely continuous U_0 was studied in [6].

1.1. Notations

In this paper U_0 represents a unitary operator acting on $\mathcal{H}$, K is a self-adjoint compact operator defined on $\mathcal{H}$ with spectral decomposition (1.3) and $U_\beta = U_0 e^{i\beta K}$ is called the perturbed operator with β a real parameter. The self-adjoint operator A is called a *conjugate operator for* U_o with domain $\mathcal{D}(A)$.

We denote by l^2 the Hilbert space of complex sequences $f = (f(n))_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |f(n)|^2 < \infty$, with inner product $\langle f, g\rangle = \sum_{n=1}^{\infty} \overline{f(n)} g(n)$. In some example we will work with $l^2(\mathbb{Z})$, the Hilbert space of complex sequences $f = (f(n))_{n=-\infty}^{\infty}$ such that $\sum_{n=-\infty}^{\infty} |f(n)|^2 < \infty$. Also, we denote by $\mathbb{C}^M$ the corresponding finite dimensional version and by δ_{ij} the Kronecker delta.

2. Absence of singular continuous spectrum

We start this section by given a briefly introduction to commutator theory for unitary operators and how the existence of these commutators has consequences on the spectral properties of U_β.

For a unitary operator U and a self-adjoint operator A, the commutators C_1, C_2 are formally defined by

$$C_1 = U^* A U - A, \qquad C_2 = [A, C_1] := AC_1 - C_1 A.$$

C_1 is called the commutator of first order and C_2 the commutator of second order. Let us called (A), (B), (C) the following hypotheses:

(A) There exists a self-adjoint operator A on the Hilbert space $\mathcal{H}$ such that $U_0^* A U_0 - A = I$ on the domain $\mathcal{D}(A)$.
(B) The range of K is a subset of $\mathcal{D}(A)$.
(C) The range of K is a subset of $\mathcal{D}(A^2)$.

Clearly (B) follows from (C). If U_0 and A satisfy (A) then $U_0^{*n} A U_0^n - A = nI$ for all $n \in \mathbb{Z}$ and $|\langle\psi, U_0^n\psi\rangle|^2 \leqslant \frac{c}{n^2}\|A\psi\|^2$, for all $\psi \in \mathcal{D}(A)$. This inequality proves that the Radon–Nikodym derivative $F_\psi(\theta)$ belongs to $L^2(\mathbb{T})$ for ψ on a dense subspace of $\mathcal{H}$.

Moreover, condition (A) also shows that such A is not bounded below nor above, since for all positive integers n we have that

$$\langle U_0^n\psi, AU_0^n\psi\rangle \geqslant n\|\psi\|^2 + \langle\psi, A\psi\rangle, \qquad \langle U_0^{*n}\psi, AU_0^{*n}\psi\rangle \leqslant \langle\psi, A\psi\rangle - n\|\psi\|^2.$$

Actually, the sequence $\{\|Au_j\|\}_j$ is frequently unbounded.

Using (A) and (B), it is straightforward to check that the commutator of first order $C_{1,\beta}$ for the perturbed operator U_β is formally

$$C_{1,\beta} = U_\beta^* AU_\beta - A = \left(e^{-i\beta K}Ae^{i\beta K} - A\right) + I.$$

If we denote by $K' = e^{-i\beta K} - I$, we have that K' is compact, normal and

$$e^{-i\beta K}Ae^{i\beta K} - A = K'A + AK'^* + K'AK'^*.$$

In the proof of Theorem 4.1 in [1], it is shown that if AK is compact and (B) holds then the operators $K'A$, AK'^*, $K'AK'^*$ are compact. If in addition we impose (C) together with A^2K bounded then $C_{2,\beta}$ is also bounded.

Theorem 2.1. *Assume that conditions* (A), (B) *are satisfied for the unperturbed operator U_0. Suppose that*

$$\sum_{j=1}^{\infty}|\lambda_j|^2\|Au_j\|^2 < \infty. \tag{2.1}$$

Then AK and $(e^{-i\beta K}Ae^{i\beta K} - A)$ are compact operators on $\mathcal{H}$ with norm

$$\left\|\left(e^{-i\beta K}Ae^{i\beta K} - A\right)\right\| \leqslant \gamma|\beta|\left(\sum_{j=1}^{\infty}|\lambda_j|^2\|Au_j\|^2\right)^{1/2},$$

for some positive constant γ.

Proof. By the spectral representation, $K = \sum_{j=1}^{\infty}\lambda_j\langle u_j,\cdot\rangle u_j$. We may define $K_N = \sum_{j=1}^{N}\lambda_j\langle u_j,\cdot\rangle u_j$, $K'_N = \sum_{j=1}^{N}(e^{-i\beta\lambda_j} - 1)\langle u_j,\cdot\rangle u_j$. It is easy to see that

$$\|AK_N\| \leqslant \left(\sum_{j=1}^{N}|\lambda_j|^2\|Au_j\|^2\right)^{1/2} \quad \text{and} \quad \left\|AK_N'^*\right\| \leqslant |\beta|\left(\sum_{j=1}^{N}|\lambda_j|^2\|Au_j\|^2\right)^{1/2}.$$

Using (2.1) we obtain that AK_N and $AK_N'^*$ converge, as N tends to infinity, to the compact operators AK and AK'^* respectively (in operator norm) and $\|AK'^*\| \leqslant |\beta|(\sum_{j=1}^{\infty}|\lambda_j|^2\|Au_j\|^2)^{1/2}$.

On the other hand, $K'A$ can be extended to a compact operator with the same norm as AK'^*, thus the norm of $K'AK'^*$ is bounded by $\|K\|\|AK'^*\|$, concluding the proof. □

Corollary 2.2 *(Mourre's inequality). With the hypotheses of Theorem* 2.1 *there exists a compact operator $\tilde{K}$ such that $C_{1,\beta}$ satisfies*

(i) $C_{1,\beta} = I + \tilde{K}$.
(ii) *There are constants $\alpha, \beta_0 > 0$ such that $C_{1,\beta} \geqslant \alpha I$ for all $|\beta| < \beta_0$.*

Next, we study the existence of the commutator of second order, $C_{2,\beta} = [A, C_{1,\beta}] = [A, e^{-i\beta K}Ae^{i\beta K}]$.

Theorem 2.3. *Assume that conditions* (A), (B), (C) *are satisfied for the unperturbed operator U_0. Suppose that*

$$\sum_{j=1}^{\infty}|\lambda_j|^2\left\|A^2u_j\right\|^2 < \infty. \tag{2.2}$$

Then $C_{2,\beta}$ is a bounded operator on $\mathcal{H}$.

Proof. In the same way that we proved that AK is compact, using (2.2), we obtain that A^2K is compact. Then $C_{2,\beta}$ is bounded since

$$C_{2,\beta} = AK'A + A^2K'^* + AK'AK'^* - \left(K'A^2 + AK'^*A + K'AK'^*A\right). \qquad \square$$

We summarize the above results in the following theorem.

Theorem 2.4. *Assume that K, A and U_0 satisfy the hypotheses of Theorem 2.3. Then $U_\beta = U_0 e^{i\beta K}$ does not have singular continuous spectrum, and it has at most a finite number of eigenvalues of finite multiplicity. In addition, if $|\beta|$ is sufficiently small, then U_β has purely absolutely continuous spectrum.*

This theorem says, for the case that U_0 is purely absolutely continuous with spectrum S^1, that the possible eigenvalues of U_β are embedded in the absolutely continuous spectrum of U_β.

As an example consider the Shift operator U_0 on $l^2(\mathbb{Z})$ defined in a complete orthonormal basis of $l^2(\mathbb{Z})$ $\{\dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots\}$ by

$$U_0 e_j = e_{j+1}, \quad j \in \mathbb{Z}.$$

Its adjoint becomes $U_0^* e_j = e_{j-1}$, $n \in \mathbb{Z}$. Let us define the conjugate operator A for U_0 as follows: $Ae_j = je_j$, with domain $\mathcal{D}(A) = \{u \in l^2(\mathbb{Z})\colon \sum_k k^2 |\langle e_k, u\rangle|^2 < \infty\}$. It is easy to see that $U_0^{*n} A U_0^n - A = nI$ on $\mathcal{D}(A)$, for all $n \in \mathbb{Z}$.

The corresponding hypotheses of Theorem 2.4 are fulfilled if

$$\sum_{j,k} k^4 |\lambda_j|^2 \left|\langle e_k, u_j\rangle\right|^2 < \infty.$$

Equivalently, we may consider $L^2[0, 2\pi]$ with the usual inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} \bar{f}(x) g(x)\,dx$ and the unitary operator $(U_0 f)(x) = e^{ix} f(x)$ acting there. In this context, $A = -i\frac{d}{dx}$ with domain

$$\mathcal{D}(A) = \left\{u\colon \text{absolutely continuous},\ u_x \in L^2[0, 2\pi],\ u(0) = u(2\pi)\right\}.$$

It is well known that $U_0^* A U_0 - A = I$ and $\sum_j |\lambda_j|^2 \|(u_j)_{xx}\|^2 < \infty$ guarantees that hypotheses of Theorem 2.4 are fulfilled.

Another example is the Floquet operator corresponding to the kicks on the Shift operator. That is, U_0 is the unitary operator $U_0 : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^m)$ defined as $U_0 = e^{-iy\cdot\nabla}$ with $y \in \mathbb{R}^m$, $\|y\| = 1$. The perturbed operator is given by $U_\beta = U_0 e^{i\beta K}$, where K is a self-adjoint compact operator on $L^2(\mathbb{R}^m)$. Define the conjugate operator A by $(Af)(x) = (x \cdot y) f(x)$, where $x \cdot y$ stands for the usual dot product on $\mathbb{R}^m$. It is known that the first commutator for U_0 satisfies $U_0^* A U_0 - A = I$.

U_0 is purely absolutely continuous with spectrum S^1, see [7]. If we assume (A) and in addition that AK is compact, then the first commutator $C_{1,\beta}$ is bounded and it satisfies Mourre's inequality. On the other hand, (B) and boundness of $A^2 K$ assure that the second commutator $C_{2,\beta}$ is bounded. Finally, it follows that under the assumptions of Theorem 2.3 that all conditions mentioned above hold.

3. Eigenvalue problem

In this section we begin to study the eigenvalue problem (1.2). First we mention a result, proven in [6], that says that if $z = e^{iE}$ is not an eigenvalue of U_0 then

$$\lim_{r \to 1^\pm} \left(U_0 - e^{iE}\right)\left(U_0 - re^{iE}\right)^{-1} = I \tag{3.1}$$

strongly.

Also we will use the following result, see [9] for details. If U_0 is absolutely continuous, $d\langle\psi, E_0(\theta)\phi\rangle = F_{\psi,\phi}(\theta) d\theta$ with $F_{\psi,\phi}(\theta) = \frac{d\langle\psi, E_0(\theta)\phi\rangle}{d\theta}$ belonging to $L^1(\mathbb{T})$ for any $\psi, \phi \in \mathcal{H}$.

Now we state a lemma that will be used later.

Lemma 3.1. *Let U_0 be unitary, $u, v \in \mathcal{H}$ with $\|u\| = 1$ and $E \in [0, 2\pi)$. Assume that $z = e^{iE}$ is not an eigenvalue of U_0 and*

$$v = \lim_{r \to 1^-} \left(U_0 - re^{iE}\right)^{-1} u \quad \textit{in the strong sense}.$$

Then

(i) $(U_0 - e^{iE}) v = u$.
(ii) *For all $\psi \in \mathcal{H}$,*

$$d\left\langle\psi, E_0(\theta) u\right\rangle = \left(e^{i\theta} - e^{iE}\right) d\left\langle\psi, E_0(\theta) v\right\rangle.$$

(iii) *If in addition U_0 is purely absolutely continuous*

$$\int_0^{2\pi} \cot\left((E - \theta)/2\right) d\left\langle\psi, E_0(\theta) u\right\rangle \quad \textit{exists}.$$

Proof. By (3.1) part (i) follows at once. The second statement is a direct consequence of item (i) and the spectral theorem. To prove (iii) we apply (ii) and notice that U_0 is purely absolutely continuous, so the corresponding Radon–Nikodym derivative $G_{\psi,v}(\theta) = \frac{d\langle\psi, E_0(\theta)v\rangle}{d\theta}$ is an $L^1(\mathbb{T})$ function and $(e^{i\theta} - e^{iE})\cot((E-\theta)/2)$ is uniformly bounded. □

Let us remind that $\{u_j\}_j$ and $\{\lambda_j\}_j$ are the corresponding vectors and real numbers coming from the spectral representation (1.3) of the compact self-adjoint operator K. Let us enumerate the following hypotheses.

(H1) There is $E \in [0, 2\pi)$ such that e^{iE} is not an eigenvalue of U_0 and $v_j := \lim_{r\to 1}(U_0 - re^{iE})^{-1}u_j$ exists in the strong sense for each j.
(H2) $\sum_{j=1}^{\infty} |\lambda_j|^2 \|v_j\|^2 < \infty$, if rank of K is not finite.

Assume that (H1), (H2) hold for U_0 and K. Thus the eigenvalue problem (1.4) can be written as

$$\sum_{j=1}^{M}\bigl(1 - e^{i\beta\lambda_j}\bigr)\langle u_j, \psi\rangle U_0 v_j = \psi, \tag{3.2}$$

where M is the rank of K.

By taking product with $\langle u_p, \cdot\rangle$ in the above identity, we may represent Eq. (3.2) in $\mathbb{C}^M$ ($M = \infty$ is allowed; in that case the Hilbert space is just l^2). In this framework Eq. (3.2) becomes

$$\sum_{j=1}^{M}\bigl(1 - e^{i\beta\lambda_j}\bigr)\langle u_p, U_0 v_j\rangle f(j) = f(p), \tag{3.3}$$

where $v_j = \lim_{r\to 1}(U_0 - re^{iE})^{-1}u_j$, and $f \in \mathbb{C}^M$ with $f(j) = \langle u_j, \psi\rangle$.

By Lemma 3.1, we know that $U_0 v_j = u_j + e^{iE} v_j$ and since $\{u_j\}$ is an orthonormal set, (3.3) becomes

$$\sum_{j=1}^{M}\bigl(1 - e^{i\beta\lambda_j}\bigr)\bigl\langle u_p, e^{iE} v_j\bigr\rangle f(j) = e^{i\beta\lambda_p} f(p). \tag{3.4}$$

Clearly, if M is finite, $(T_M f)(p) = \sum_{j=1}^{M}(1 - e^{i\beta\lambda_j})\langle u_p, e^{iE} v_j\rangle f(j)$ is well defined on $\mathbb{C}^M$. If $M = \infty$ we assume (H2). Anyway, conditions (H1), (H2) imply that $T := T_\infty$ is compact in l^2 and $\|T_\infty\|^2 \leqslant |\beta|^2 \sum_{j,p=1}^{\infty} |\lambda_j|^2 |\langle u_p, v_j\rangle|^2 < \infty$.

Under these assumptions the corresponding characteristic equation (1.4), in the l^2 framework, is

$$(Tf)(p) = e^{i\beta\lambda_p} f(p), \quad \text{for all } p \geqslant 1. \tag{3.5}$$

We will see that Eq. (3.5) sometimes admits only the trivial solution in l^2, more precisely, for β sufficiently small $f = 0$ is the only solution.

Proposition 3.2. *Assume that conditions* (H1) *and* (H2) *are fulfilled and choose β such that $\|T_M\| < 1$ ($M = \infty$ is included). Then in $\mathbb{C}^M$ (or l^2 for $M = \infty$) the only solution of* (3.5) *is the trivial one.*

Proof. Assume that $Tf = Df$, with D the unitary operator defined as $(Df)(p) = e^{i\beta\lambda_p} f(p)$. Thus, $\|Tf\| = \|Df\| = \|f\|$, so if $f \neq 0$ contradicts the fact that $\|T\| < 1$. □

We shall see in the next section that if K has finite rank then the eigenvalue problem can be reduced to an eigenvalue problem for matrices in $\mathbb{C}^m$, where m is the rank of K.

Theorem 3.3. *Consider U_0 an absolutely continuous operator and $E \in [0, 2\pi)$. Let $\{u_j\}_j$ be an orthonormal set on $\mathcal{H}$ satisfying hypothesis* (H1)*. Then*

$$\langle u_p, U_0 v_j\rangle = \frac{1}{2}\delta_{pj} + \frac{i}{2}\int_0^{2\pi} \cot\bigl((E-\theta)/2\bigr)\, d\bigl\langle u_p, E_0(\theta)u_j\bigr\rangle \tag{3.6}$$

and $\langle u_p, U_0 v_j\rangle = -\overline{\langle u_j, U_0 v_p\rangle}$.

Proof. By (H1) and the spectral theorem one has that

$$\begin{aligned}\langle u_p, U_0 v_j\rangle &= \Big\langle u_p, U_0\Big(\lim_{r\to 1}\big(U_0 - re^{iE}\big)^{-1}u_j\Big)\Big\rangle\\ &= \lim_{r\to 1}\big\langle u_p, U_0\big(U_0 - re^{iE}\big)^{-1}u_j\big\rangle\\ &= \lim_{r\to 1}\int_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-re^{iE}}\,d\big\langle u_p, E_0(\theta)u_j\big\rangle.\end{aligned}$$

But it is well known that $\frac{e^{i\theta}}{e^{i\theta}-re^{iE}} \to \frac{1}{2} + \frac{i}{2}\cot((E-\theta)/2)$, for $\theta \neq E$. So, it remains to prove that the limit can be carried inside the integral.

Let us write $F_{p,j}(\theta) = \frac{d\langle u_p, E_0(\theta)u_j\rangle}{d\theta}$ and $G_{p,j}(\theta) = \frac{d\langle u_p, E_0(\theta)v_j\rangle}{d\theta}$. Note that $F_{p,j}, G_{p,j} \in L^1(\mathbb{T})$ and by Lemma 3.1, $F_{p,j}(\theta) = (e^{i\theta} - e^{iE})G_{p,j}(\theta)$. Thus,

$$\left|\frac{e^{i\theta}}{e^{i\theta}-re^{iE}}\right|\big|F_{p,j}(\theta)\big| = \left|\frac{e^{i\theta}-e^{iE}}{e^{i\theta}-re^{iE}}\right|\big|G_{p,j}(\theta)\big| \leqslant 2\big|G_{p,j}(\theta)\big|.$$

Thus, using the Lebesgue's dominated convergence theorem we get (3.6).

The conjugate property follows easily from the facts that $\cot((E-\theta)/2)$ is real and $\overline{d\langle u_j, E_0(\theta)u_p\rangle} = d\langle u_p, E_0(\theta)u_j\rangle$. Actually, weaker conditions can be imposed to get the identity (3.6), see [5]. □

3.1. K of finite rank

In this section we shall study the eigenvalue problem (1.2) for the perturbed operator $e^{i\beta K}$, K a finite rank operator. We will see, as expected, that the eigenvalue problem (1.2) is essentially an eigenvalue problem for finite matrices.

Assuming that K has rank m, as we mentioned, the eigenvalue problem to solve is

$$\sum_{j=1}^{m}\big(1 - e^{i\beta\lambda_j}\big)\langle u_p, U_0 v_j\rangle f(j) = f(p), \quad \text{for all } p = 1,\dots,m.$$

Let us represents the above identities by

$$A\hat{x} = \hat{x}, \tag{3.7}$$

where $\hat{x} \in \mathbb{C}^m$ is the column vector $\hat{x} = [\langle u_1, \psi\rangle \dots \langle u_m, \psi\rangle]^t$, and $A = (a_{pj})$ is an $m \times m$ matrix with coefficients in $\mathbb{C}$ defined by

$$a_{pj} = \big(1 - e^{i\beta\lambda_j}\big)\langle u_p, U_0 v_j\rangle.$$

The matrix A can be decomposed as $A = BD$, so the eigenvalue problem (3.7) becomes

$$(BD - I)\hat{x} = \hat{0}, \tag{3.8}$$

where D is an $m \times m$ diagonal matrix with entries $d_j = 1 - e^{i\lambda_j\beta}$ and B is the $m \times m$ matrix given by

$$B = (b_{pj}), \quad \text{with } b_{pj} = \langle u_p, U_0 v_j\rangle. \tag{3.9}$$

Note that D depends on the parameters β and λ_j, the corresponding eigenvalues of K. In addition, if the matrix D is invertible then $\mathrm{Null}(A - I) \neq \{0\}$ if and only if $\mathrm{Null}(B - D^{-1}) \neq \{0\}$.

In the case that D is not invertible, some columns of A are zero, so the rank of K is less than m or $\lambda_j\beta$ is a multiple of 2π for some j. From now on we assume that D is invertible.

The next proposition summarizes some properties of the matrix $A = BD$. We assume that U_0 is purely absolutely continuous, so Theorem 3.3 is valid.

Proposition 3.4. *Assume that the $m \times m$ diagonal matrix D with entries $d_j = 1 - e^{i\beta\lambda_j}$ is invertible and define $C = B - D^{-1}$ with B defined in* (3.9)*. Then $C = iR$ with R hermitian. Moreover, the eigenvalues of B lie on the line $\{\frac{1}{2} + it\colon t \in \mathbb{R}\}$.*

Proof. The first part follows using Theorem 3.3 and the identity $d_j^{-1} = 1/2 + i/2\cot(\lambda_j\beta/2)$.

Consider α an eigenvalue for B with $Bw = \alpha w$, $w = [w_1 \dots w_m]^t$. Using that $B = D^{-1} + iR$,

$$\big\langle w, D^{-1}w\big\rangle + i\langle w, Rw\rangle = \langle w, Bw\rangle = \alpha\|w\|^2.$$

But, $\langle w, D^{-1}w\rangle = \frac{1}{2}\|w\|^2 + \frac{i}{2}\sum_{j=1}^{m}|w_j|^2\cot(\lambda_j\beta/2)$ and $\langle w, Rw\rangle$ is real. By taking the real part in the above identities we obtain that $\Re\alpha = 1/2$. □

Thus, we have proved that $A\hat{x} = \hat{x}$ has a non-trivial solution if and only if 0 is an eigenvalue for the hermitian matrix R.

Now to go further one needs to look at the entries of R. Let us denote as $G(x)$ the kernel $G(x) = \frac{1}{2}\cot(x/2)$ for $0 < x < 2\pi$. By hypothesis (H1) and since U_0 is purely absolutely continuous we know that

$$c_j(E) = \int_0^{2\pi} G(E-\theta)\, d\langle u_j, E_0(\theta)u_j\rangle \tag{3.10}$$

is well defined.

The principal diagonal of R has real entries given by

$$r_{jj} = \int_0^{2\pi} G(E-\theta)\, d\langle u_j, E_0(\theta)u_j\rangle - G(\lambda_j\beta/2) = c_j(E) - G(\lambda_j\beta/2). \tag{3.11}$$

On the other hand, the off diagonal entries of R do not depend on the real parameter β.

Example. Let K be a real finite linear combination of rank one orthogonal projectors, that is, $K = \sum_{j=1}^{m} \lambda\langle u_j, \cdot\rangle u_j$. Using (3.8), the equation $BD\hat{x} = \hat{x}$ becomes

$$B\hat{x} = \frac{1}{1-e^{i\lambda\beta}}\hat{x} = \left(\frac{1}{2} + \frac{i}{2}\cot(\lambda\beta/2)\right)\hat{x}.$$

By Proposition 3.4, if $B\hat{w} = \alpha_0\hat{w}$ then $\alpha_0 = \frac{1}{2} + it_0$, for some real t_0. The image of cotangent is the whole real line, thus for every eigenvalue α of B we can choose a unique β, depending on α, with $0 < \beta < 2\pi/\lambda$ such that $\alpha = \frac{1}{2} + it = \frac{1}{2} + \frac{i}{2}\cot(\lambda\beta/2)$.

In this way we have proved that for such K and E there exists β restricted to the open interval $]0, 2\pi/\lambda[$ such that $U_\beta\psi = e^{iE}\psi$ has a non-trivial solution.

Consider the eigenvalue problem $U_\beta\psi = z\psi$ with $z = e^{iE}$ and $K = \sum_{j=1}^{m} \lambda_j\langle u_j, \cdot\rangle u_j$. We associate to this problem the following Hermitian $m \times m$ matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ \overline{r_{12}} & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \\ \overline{r_{1n}} & \overline{r_{2n}} & \dots & r_{mm} \end{bmatrix} \tag{3.12}$$

where $r_{pj} = -ib_{pj}$ are given by (3.9) for $p \neq j$ and by (3.11) for $p = j$. Note that for $p \neq j$, r_{pj} depend on E but not on β or λ_j. So far, we have proved the following result.

Proposition 3.5. *Assume that the entries of the matrix R are well defined for a given real number E. Then $z = e^{iE}$ is an eigenvalue of U_β if and only if* $\det(R) = 0$.

Our next result shows that the rank two case is solvable in a rather general context.

Theorem 3.6. *Let K be a rank two operator defined by $K = \lambda_1\langle u_1, \cdot\rangle u_1 + \lambda_2\langle u_2, \cdot\rangle u_2$ with u_1, u_2 orthogonal and satisfying* (H1) *for a real number E, $0 \leqslant E \leqslant 2\pi$. Assume that λ_1, λ_2 have the same sign. Then there exists β_0 such that has e^{iE} is an eigenvalue for the perturbed operator $U_{\beta_0} = U_0 e^{i\beta_0 K}$.*

Proof. Let us write $D := D(\beta)$. It is enough to consider $\lambda_1 \geqslant \lambda_2 > 0$. It is more convenient to work with the matrix equation $(B - D^{-1}(\beta))\hat{x} = \hat{0}$, which has a non-trivial solution if and only if there exists β such that $\det(B - D^{-1}(\beta)) = 0$.

By Proposition 3.4 we have that $\det(B - D^{-1}(\beta)) = 0$ if and only if $\det(R) = 0$, R hermitian. That is, $r_{11}r_{22} - |r_{12}|^2 = 0$. Note that r_{12} does not depend on β. Also, $\det(R) = 0$ if and only if

$$\left(c_1(E) - \frac{1}{2}\cot(\lambda_1\beta/2)\right) \cdot \left(c_2(E) - \frac{1}{2}\cot(\lambda_2\beta/2)\right) = |r_{12}|^2, \tag{3.13}$$

where $c_1(E), c_2(E)$ do not depend on β. So, we need to prove that for a given positive number $|r_{12}|^2$ there exists a solution β for Eq. (3.13).

The period of $g_j(\beta) = \frac{1}{2}\cot(\lambda_j\beta/2)$ is $T_j = \frac{2\pi}{\lambda_j}$ for each $j = 1, 2$, and the condition $\lambda_1 \geqslant \lambda_2 > 0$ implies that $T_1 \leqslant T_2$.

Let $h_j(x)$ denote the periodic functions $h_j(x) = c_j(E) - g_j(x)$ for $j = 1, 2$. Since λ_1, λ_2 have the same signs both functions go to $-\infty$ as $\beta \to 0^+$. On the other hand, $g_1(]0, T_1[) = \mathbb{R}$, which assures that the range of $h_1(x)$ as x runs over the interval $]0, T_1[$ is the whole real line. Since h_1 is continuous in the branch $]0, T_1[$ there exists $x_1 \in]0, T_1[$ with $h_1(x_1) = 0$.

Thus, $H(x) = h_1(x)h_2(x)$ approaches $+\infty$ as $t \to 0^+$ and $H(x_1) = 0$. Also, since $T_2 \geqslant T_1$, H is continuous in $]0, T_1[$. Therefore, the product function $H(x) = (c_1(E) - g_1(\beta)) \cdot (c_2(E) - g_2(\beta))$ attains any nonnegative number when x runs over $]0, T_1]$. □

Next, we settle a sort of converse result.

Theorem 3.7. *Let U_0 be a unitary operator without eigenvalues, and m be a positive integer. Consider E a real number and $\{u_1, \dots, u_m\}$ an orthonormal set satisfying condition* (H1)*. Then for any real t there exists a compact self-adjoint operator K of rank m, $K = \sum_{j=1}^m \lambda_j \langle u_j, \cdot \rangle u_j$, such that e^{iE} is an eigenvalue of $U_t = U_0 e^{itK}$.*

Proof. Let us define the function $\Delta : \mathbb{R}^m \to \mathbb{R}$ by $\Delta(x_1, x_2, \dots, x_m) = \det(M)$, where

$$M = \begin{bmatrix} c_1(E) - x_1 & r_{12} & \dots & r_{1n} \\ \overline{r_{12}} & c_2(E) - x_2 & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \\ \overline{r_{1n}} & \overline{r_{2n}} & \dots & c_m(E) - x_m \end{bmatrix}.$$

Let us write $\mathcal{N} = \{(x_1, x_2, \dots, x_m) \in \mathbb{R}^m\colon\ \Delta(x_1, x_2, \dots, x_m) = 0\}$. Taking $x_1 = x_2 = \dots = x_m = x$ one gets $M = Q - xI$, with Q hermitian. So, $\Delta(x_1, \dots, x_m) = p_Q(x)$ is the characteristic polynomial associated to Q. Since Q is hermitian the roots of $p_Q(x)$ must be real, so $\mathcal{N}$ is not empty.

Consider $(x_1, \dots, x_m)$ a solution of $\Delta(x_1, \dots, x_m) = 0$. Let us recall that $\cot(x/2)$ is a bijection between $]0, 2\pi[$ and $\mathbb{R} =]-\infty, \infty[$. Then for a given real number t it is possible to choose a vector $(\lambda_1, \dots, \lambda_m)$ with $0 < \lambda_j \frac{t}{2} < \pi$, for $j = 1, \dots, m$, and such that $x_j = \frac{1}{2}\cot(\lambda_j t/2)$. With these choices $R = M$ and $\det(R) = 0$, which completes the proof. □

3.2. Eigenvalues for K of infinite rank

We now study the eigenvalue problem $U_\beta \psi = z\psi$, where $U_\beta = U_0 e^{i\beta K}$ with K of infinite rank.

Suppose that conditions (H1), (H2) hold for K and $z = e^{iE}$. Define the linear operator S_K by $S_K \psi = \sum_j \lambda_j \langle u_j, \psi \rangle v_j$.

By conditions (H1), (H2) S_K is a bounded linear operator in $\mathcal{H}$ with norm $\|S_K\| \leqslant \sum_j |\lambda_j|^2 \|v_j\|^2$.

Following the same directions as we developed for the finite rank case, the eigenvalue problem for $U_\beta \psi = z\psi$ may be represented in the Hilbert space l^2 by

$$\sum_{j=1}^{\infty} \bigl(1 - e^{i\beta\lambda_j}\bigr) \langle u_j, \psi \rangle \langle u_p, U_0 v_j \rangle = \langle u_p, \psi \rangle, \tag{3.14}$$

with $\lambda_j \to 0$ and $v_j = S_K u_j$. Recall that $U_0 v_j = u_j + z v_j$ for all j.

Let us define the operator T_z as follows

$$(T_z f)(p) = \sum_{j=1}^{\infty} \bigl(1 - e^{i\beta\lambda_j}\bigr) \langle u_p, z v_j \rangle f(j).$$

Under conditions (H1), (H2), T_z is compact on l^2 and the eigenvalue problem $U_\beta \varphi = z\varphi$ may be written as the following problem in l^2,

$$(T_z f)(p) = e^{i\beta\lambda_p} f(p). \tag{3.15}$$

Notice that v_j depend on z.

Theorem 3.8. *Assume that conditions* (H1), (H2) *are fulfilled. Then $f \in l^2$ is a solution of* (3.15) *if and only if $\psi = \sum_j (1 - e^{i\beta\lambda_j}) f(j) U_0 v_j$ is a solution of the eigenvalue problem $U_\beta \psi = z\psi$ with $f(j) = \langle u_j, \psi \rangle$ for all j.*

Proof. It only remains to prove that if $f \in l^2$ then $\psi = \sum_j (1 - e^{i\beta\lambda_j}) f(j) U_0 v_j$ is a vector belonging to $\mathcal{H}$, but this is a consequence of condition (H2) since

$$\sum_j |\lambda_j| \bigl|f(j)\bigr| \|U_0 v_j\| \leqslant \|f\|_{l^2} \Bigl(\sum_j |\lambda_j|^2 \|v_j\|_{l^2}^2 \Bigr)^{1/2}. \qquad \square$$

3.3. The shift operator

Now we shall apply the general framework developed above to the shift operator.

Let us take $f_n \in l^2$ with $f_n(k) = \delta_{nk}$ (Kronecker delta) and consider $g \in l^2$ a finite linear combination of f_n's. After reordering we may assume that $g = \sum_{n=1}^{N} a_n f_n$. First, we find conditions on K, β, E in such a way that g is a solution of (3.15).

Clearly, $(Tf_n)(p) = (1 - e^{i\beta\lambda_n})\langle u_p, zv_n\rangle$ and

$$\left(1 - e^{i\beta\lambda_n}\right)\langle u_p, zv_n\rangle = \begin{cases} 0 & \text{if } p \neq n, \\ e^{i\beta\lambda_n} & \text{if } p = n. \end{cases}$$

Then g is a solution of (3.15) if

$$\sum_{n=1}^{N} a_n\left(1 - e^{i\beta\lambda_n}\right)\langle u_p, zv_n\rangle = \begin{cases} a_p e^{i\beta\lambda_p} & \text{if } 1 \leqslant p \leqslant N, \\ 0 & \text{if } p > N. \end{cases} \tag{3.16}$$

If $\langle u_p, v_n\rangle = 0$ for all $p > N$ and $1 \leqslant n \leqslant N$, we obtain that

$$\sum_{n=1}^{N} a_n\left(1 - e^{i\beta\lambda_n}\right)\langle u_p, zv_n\rangle = a_p e^{i\beta\lambda_p},$$

which is just the equation given in (3.4) for the finite dimensional case.

Let U_0 be the shift operator on $l^2(\mathbb{Z})$. The operator U_0 is unitary, has purely absolutely continuous spectrum, and $\sigma(U_0) = \mathbb{T}$. Our goal is to define a compact operator K having infinite rank such that the perturbed operator $U_\beta = U_0 e^{i\beta K}$ has an eigenvalue $z = e^{iE}$.

Let us construct an orthonormal set $\{u_j\}_j$ and their corresponding $\{v_j\}_j$, $v_j = (U_0 - e^{iE})^{-1} u_j$, such that conditions (H1), (H2) are fulfilled.

Define the vector $u_j \in l^2$ as follows

$$u_j = ae_{3j+2} - zae_{3j+1}, \tag{3.17}$$

with $a \in \mathbb{C}$. Clearly $\mathfrak{B} = \{u_j\colon\ j \in \mathbb{Z}\}$ is an orthogonal set in $l^2(\mathbb{Z})$ and since $|z| = 1$, $\|u_j\|^2 = 2|a|^2$. Choosing a such that $|a|^2 = \frac{1}{2}$, the set $\mathfrak{B}$ is orthonormal.

It is easy to see that $v_j = ae_{3j+1}$ satisfies $(U_0 - z)v_j = u_j$ for all $j \in \mathbb{Z}$, so (H1) holds. A direct computing shows that $\langle u_j, v_p\rangle = 0$ for all $j \neq p$ and $\langle u_j, zv_j\rangle = -\frac{1}{2}$ and then its imaginary part is zero, so $\cot(\lambda\beta/2) = 0$.

Applying Theorem 3.8, z will be an eigenvalue if $|a|^2 = \frac{1}{2}$, and β, λ must satisfy $\beta\lambda = \pi$.

Proposition 3.9. *Consider the orthonormal set $\mathfrak{B} = \{u_j\colon\ j \in \mathbb{Z}\}$ where the vectors u_j are given by* (3.17) *with $|a|^2 = \frac{1}{2}$. Define the compact self-adjoint operator $K = \sum_{j\in\mathbb{Z}} \lambda_j \langle u_j, \cdot\rangle u_j$ and $U_\beta = U_0 e^{i\beta K}$, with U_0 the shift operator acting on $l^2(\mathbb{Z})$. Assume that β and λ_n satisfy $\beta\lambda_n = \pi$. Then $\psi = (1 - e^{i\beta\lambda_n})U_0 v_n$ satisfies $U_\beta\psi = e^{iE}\psi$.*

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