High precision results for a two-point boundary value problem

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ABSTRACT

In this note we establish results of high accuracy for the two-point boundary value problem

$$y'' = n \sinh ny \tag{1a}$$

with the boundary conditions

$$y(0) = 0\,, \quad y(1) = 1, \tag{1b}$$

where n is real and positive. At the same time a derivation of the approximation for large n

$$y'(0) \doteq 8\, e^{-n}(1 - 2e^{-n/2} + 2e^{-n}), \tag{2}$$

which is deduced in [1] from numerical results, is obtained in the course of the development.

1. INTRODUCTION

The boundary value problem stated in (1) has been investigated by many authors (see [2] and the references given there). The difficulty of the numerical solution increases rapidly for, let us say, $n > 5$, and a number of methods have been proposed. The authors usually assess the accuracy by comparing their results, in particular the missing initial condition $y'(0)$, and $y'(1)$, with the results of other authors. It seems therefore worthwhile to make the "exact" values of $y'(0)$ for the continuous solutions of (1) available for comparison purposes. By "exact" values we mean values accurate to at least 14 significant digits. The method can furnish easily more digits if desired.

It has been known for some time that the solution to the problem can be expressed in terms of elliptic functions [3, 4]. Here we will however present the solution in terms of an expansion directly from the equation. This approach also enables us to prove the result in (2). Our lowest order approximation includes even one more term in the expansion of $y'(0)$ (see eq. (10) below).

2. THE IMPLICIT REPRESENTATION OF $y'(0)$

The differential equation can be reduced in the well-known manner to the first order equation

$$y'^2(x) = y'^2(0) + 4 \sinh^2(ny/2), \tag{3}$$

and therefore

$$y'^2(1) = y'^2(0) + 4 \sinh^2(n/2). \tag{4}$$

The integration of (3) and the boundary condition at $x = 1$ lead to

$$\int_0^1 [y'^2(0) + 4 \sinh^2(ny/2)]^{-1/2} dy = 1.$$

By using the simplifying notation $y'(0) = 2a$, $u = ny/2$ and by introducing

$$J(s) = \int_s^{\infty} (a^2 + \sinh^2 u)^{-1/2} du$$

we obtain

$$n = \int_0^{n/2} (a^2 + \sinh^2 u)^{-1/2} du = J(0) - J(n/2). \tag{5}$$

The idea is to expand the integrand and integrate term by term.

For n sufficiently large, where a^2 is small, this can be carried out directly for $J(n/2)$. In $J(0)$, on the other hand, a^2 is always dominant near the lower limit of integration, so that $J(0)$ must be transformed to make an expansion in a^2 possible. We therefore write

$$J(0) = J(u_0) + \int_0^{u_0} (a^2 + \sinh^2 u)^{-1/2} du$$

where u_0 is defined by

$$\sinh^2 u_0 = a. \tag{6}$$

By introducing the new variable v

$$\sinh u = a/\sinh v,$$

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the second integral transforms to $J(u_0)$ also, so that

$$J(0) = 2J(u_0) \quad (7)$$

and

$$n = 2J(u_0) - J(n/2). \quad (8)$$

This "folding" of $J(0)$ around u_0 to obtain $2J(u_0)$ is the crucial step, since we can now take full advantage of the fact that a^2 is small.

3. THE LEADING TERM

It is remarkable that the lowest order approximation of (5) already gives a very good approximation to $y'(0)$. In

$$n = 2\int_{u_0}^{\infty} (a^2 + \sinh^2 u)^{-1/2} du - \int_{n/2}^{\infty} (a^2 + \sinh^2 u)^{-1/2} du$$

we neglect the a^2 to obtain

$$n = 2\int_{u_0}^{\infty} du/\sinh u - \int_{n/2}^{\infty} du/\sinh u = 2 \log \coth (u_0/2) + \tanh (n/4).$$

But $\coth (u_0/2) = (1 + \cosh u_0)/\sinh u_0 \doteq 2/\sqrt{a}$ if we approximate $\cosh u_0 = (1 + a)^{1/2} \doteq 1$ by its leading term, so that

$$-n \doteq \log [a/4 \tanh (n/4)],$$

$$a \doteq 4e^{-n} \tanh (n/4).$$

Hence, the lowest order approximation of $y'(0)$ is

$$y_0'(0) = 8e^{-n} \tanh (n/4). \quad (9)$$

As we will show below, this result is accurate to (including) terms of order $e^{-5n/2}$, i.e., in an expansion in powers of $e^{-n/2}$

$$y'(0) = 8e^{-n}(1 - 2e^{-n/2} + 2e^{-n} - 2e^{-3n/2} + \ldots) \quad (10)$$

is actually correct: the next term in (8) furnishes a contribution of $0(ne^{-3n})$. This confirms the results (2), obtained numerically in [1].

4. THE EXPANSION IN a

There is no basic difficulty in expanding the integrands in (8) in terms of a^2, provided that, in $J(n/2)$, $a < \sinh(n/2)$. In $J(u_0)$ we always have $a^2 < \sinh^2 u_0 = a$. Therefore, an estimate shows that the expansion is certainly valid for $n > 1$. Although the expansion can be carried out quite generally, we will just show this by including one more term and then simply state the general result.
We write

$$J(s) = \int_s^{\infty} (1 + a^2/\sinh^2 u)^{-1/2} du/\sinh u$$

$$= \int_s^{\infty} du/\sinh u - (a^2/2)\int_s^{\infty} du/\sinh^3 u + (3a^4/8)\int_s^{\infty} du/\sinh^5 u - \ldots$$

or

$$J(s) = (1 + \frac{a^2}{4} + \ldots) \log \coth (s/2) - \frac{a^2 \cosh s}{4 \sinh^2 s} + \frac{3a^4 \cosh s}{32 \sinh^4 s} \ldots \quad (11)$$

For $J(n/2)$ we need to retain only

$$J(n/2) = -(1 + \frac{a^2}{4}) \log \tanh (n/4) - \frac{a^2 \cosh (n/2)}{4 \sinh^2 (n/2)},$$

whereas for $J(u_0)$ more terms are required

$$J(u_0) = (1 + 1/4a^2) \log [(1 + \sqrt{1+a})^2/a] - a\sqrt{1+a}/2 + 3a^2/16.$$

From (8) it follows that

$$n = (1 + \frac{a^2}{4}) \log \left\{\frac{(1 + \sqrt{1+a})^2}{a} \tanh (n/4)\right\} - \frac{a}{2} - \frac{a^2}{16} + \frac{a^2 \cosh (n/2)}{4 \sinh^2 (n/2)}.$$

If we set $a_1 = a_0(1 + \epsilon)$ with

$$a_0 = 4e^{-n} \tanh (n/4)$$

we obtain, after a simple expansion,

$$\epsilon = \frac{a_0^2}{4}\left[n - 1 + \frac{\cosh (n/2)}{\sinh^2 (n/2)}\right],$$

or as the next approximation (cf. (9))

$$y'(0) = y_0'(0)\left\{1 + \frac{y_0'^2(0)}{16}\left[n - 1 + \frac{\cosh (n/2)}{\sinh^2 (n/2)}\right]\right\} + 0(e^{-4n}) \quad (12)$$

The term $y_0'(0)$ is therefore indeed correct to $0(ne^{-3n})$.
The general expansion can now proceed along the same line.
All integrals are of the form

$$\int_s^{\infty} du/\sinh^{2k+1} u, \quad k = 0, 1, 2, \ldots$$

and are expressible as a finite sum in terms of $\log \coth(s/2)$ and $\cosh s/\sinh^{2j} s$, $j = 1, 2, \ldots$. For $s = n/2$ all these terms are therefore known functions, and for $s = u_0$ the terms are functions of a.
If we introduce

$$\beta = a(1 + \sqrt{1+a})^{-2}$$

and hence

$$a = 4\beta/(1 - \beta)^2,$$

furthermore $\sigma = 1/\sinh^2(n/2)$

$\gamma = \cosh(n/2)/2$

$a_0 = 1,\ a_k = -(1-1/2k)\,a_{k-1}$ for $k = 1, 2, \ldots$

and the functions

$$I(\alpha) = \sum_{k=0}^{\infty} (a_k\ \alpha^k)^2$$

$$S(n,\alpha) = \sum_{k=1}^{\infty} a_k^2\, \alpha^k \sum_{q=1}^{\infty} (\sqrt{1+\alpha}\ \alpha^{k-q} - \gamma\,\sigma^q \alpha^k)/q\ a_q$$

then, continuing the expansion as in (11), we obtain

$$n = I(\alpha)\,\log[\tanh(n/4)/\beta] + S(n,\alpha). \qquad (13)$$

It is quite feasible to obtain, by a systematic expansion of α in terms of n, more terms in α (as outlined above for the next higher term ϵ). However, it is more convenient to solve (13) by iteration. We start with the approximation α_0 or α_1 (see (9) and (12)) and compute by consecutive substitution

$$\beta = \tanh(n/4)\,\exp\left[\{S(n,\alpha) - n\}/I(\alpha)\right]. \qquad (14)$$

The lowest approximation above corresponds to $S(n,0) = 0,\ I(0) = 1,\ \beta = \alpha/4$, leading to (9).
The value of the slope at $x = 1$ is found by (4)

$$y'(1) = 2\sqrt{\alpha^2 + 1/\sigma}.$$

The iterative solution of (14) is quite suitable for machine computation even down to $n = 1$, and it has also been implemented on a Hewlett-Packard 67, where the relative error of $\pm 5.10^{-9}$ can be achieved.
No reference has been made to elliptic functions. But instead of evaluating $I(\alpha)$ by the series as written, we can use the known transformation for the complete elliptic integral $K = (\pi/2)\ I(\alpha)$ [5, eq. 773.2], applied repeatedly for faster convergence.

Although the numerical solution of the problem presents no difficulty for $n < 1$, for completeness' sake an expansion for small n can be easily carried out with the result

$$y'(0) = 1 - \frac{n^2}{6} + \frac{n^4}{90} \ \ldots$$

which joins $y'(0)$ for $n = 1$ with an error of 0.1 %.

5. NUMERICAL RESULTS FOR y'(0) AND y'(1)

In table I the results for $y'(0)$, accurate to all digits given, the relative errors $E_1 = [y'(0) - y_0'(0)]/y'(0)$ from (9) and E_2 from (12), and $y'(1)$ are listed. There is no need to include values for $n > 20$, since $y_0'(0)$ is correct to 16 digits. For $y'(1)$, the approximation $y'(1) = 2\sinh(n/2)$ furnishes 16 digits for $n \geqslant 14$. There are fewer than 20 iterations required to solve (14) for $n = 1$, six iterations for $n = 5$, and even fewer for larger n.

6. THE APPROXIMATION TO THE FULL SOLUTION

The basic idea to derive the good approximation for $y'(0)$ can also be used to obtain an approximation to the full solution $y(x)$.
From (3) it follows that

$$x = \int_0^y [y'^2(0) + 4\sinh^2(n\eta/2)]^{-1/2}\, d\eta$$

or

$$nx = \int_0^{ny/2} (\alpha^2 + \sinh^2 u)^{-1/2}\, du = J(0) - J(ny/2)$$

Table 1

n	y'(0)	E_1	E_2	y'(I)
1	$8.45202\ 68530\ 9951\cdot10^{-1}$	$1.5\cdot10^{-1}$	$3.2\cdot10^{-2}$	1.43183 78623 6849
2	$5.18621\ 21926\ 9340\cdot10^{-1}$	$3.5\cdot10^{-2}$	$3.3\cdot10^{-3}$	2.40693 98312 4707
3	$2.55604\ 21556\ 2933\cdot10^{-1}$	$1.0\cdot10^{-2}$	$3.0\cdot10^{-4}$	4.26622 28618 0282
4	$1.11880\ 16477\ 0749\cdot10^{-1}$	$2.6\cdot10^{-3}$	$1.9\cdot10^{-5}$	7.25458 35747 6858
5	$4.57504\ 61406\ 3187\cdot10^{-2}$	$5.5\cdot10^{-4}$	$8.3\cdot10^{-7}$	$1.21004\ 95450\ 7778\cdot10$
6	$1.79509\ 49489\ 5458\cdot10^{-2}$	$1.0\cdot10^{-4}$	$2.9\cdot10^{-8}$	$2.00357\ 57896\ 3586\cdot10$
7	$6.86750\ 96950\ 5692\cdot10^{-3}$	$1.8\cdot10^{-5}$	$8.6\cdot10^{-10}$	$3.30852\ 55288\ 0148\cdot10$
8	$2.58716\ 94189\ 6258\cdot10^{-3}$	$2.9\cdot10^{-6}$	$2.3\cdot10^{-11}$	$5.45798\ 34455\ 5734\cdot10$
9	$9.65584\ 54107\ 6174\cdot10^{-4}$	$4.7\cdot10^{-7}$	$5.8\cdot10^{-13}$	$9.00060\ 22309\ 1603\cdot10$
10	$3.58337\ 78463\ 0814\cdot10^{-4}$	$7.2\cdot10^{-8}$	$1.4\cdot10^{-14}$	$1.48406\ 42115\ 6010\cdot10^{2}$
12	$4.89106\ 21759\ 1979\cdot10^{-5}$	$1.6\cdot10^{-9}$	$7.1\cdot10^{-18}$	$4.03426\ 31474\ 0561\cdot10^{2}$
15	$2.44451\ 30237\ 4325\cdot10^{-6}$	$5.2\cdot10^{-12}$	–	$1.80804\ 18613\ 7169\cdot10^{3}$
20	$1.64877\ 31827\ 8040\cdot10^{-8}$	$3.2\cdot10^{-16}$	–	$2.20264\ 65749\ 4068\cdot10^{4}$

and hence (cf. eq. (8))

$$nx = 2J(u_0) - J(ny/2).$$

If y is sufficiently large, i.e., $a < \sinh(ny/2)$, then we drop all a's, except in $\coth(u_0/2) = 2/\sqrt{a}$. As in section 3, it follows that

$$nx = \log(4/a) + \log\tanh(ny/4).$$

By using the approximation for a from (9) or directly from the condition

$$y(1) = \frac{4}{n}\tanh^{-1}(ae^n/4) = 1$$

we obtain the result

$$y_0(x) = \frac{4}{n}\tanh^{-1}[e^{-n}\tanh(n/4)\, e^{nx}]. \qquad (15)$$

This approximation is not valid for small y, but for large n the error even at x = 0, is still only

$$y_0(0) \doteq y_0'(0)/2n$$

It is noteworthy that such a drastic simplification leads to rather accurate results for n = 10 and x > 0.5 (see table II).

Table II

	n = 10			
x	y(x)		x	y(x)
0.00	0.0	1)	0.60	7.22893 1229E-3
	0.0	2)		7.2281 E-3
	0.179 E-4	3)		7.22897 5 E-3
	-0.118 E-11	4)		7.22893 1213E-3
0.05	1.86728 1378E-5		0.70	1.96640 6314E-2
	1.86728 1374E-5			1.9648 E-2
	2.95 E-5			1.96640 79 E-2
	1.86728 1306E-5			1.96640 6310E-2
0.10	4.21118 9937E-5		0 80	5.37303 2947E-2
	4.21118 9922E-5			•5.341 E-2
	4.870 E-5			5.37303 346 E-2
	4.21118 9889E-5			5.37303 2935E-2
0.20	1.29964 1161E-4		0.90	1.52114 0768E-1
	1.29964 1124E-4			1.452 E-1
	1.324 E-4			1.52114 0775E-1
	1.29964 1157E-4			1.52114 0764E-1
0.30	3.58978 4022E-4		0.95	2.76267 7347E-1
	3.58978 31 E-4			2.39 E-1
	3.5987 E-4			2.76267 7341E-1
	3.58978 4014E-4			2.76267 7338E-1
0.40	9.77902 7739E-4		0.98	4.48233 0406E-1
	9.77900 8 E-4			3.23 E-1
	9.7823 E-4			4.48233 0387E-1
	9.77902 7718E-4			4.48233 0387E-1
0.50	2.65902 0496E-3		1.00	1.0
	2.65898 1 E-3			0.395
	2.65914 E-3			1.0
	2.65902 0490E-3			1.0

1) y(x) in [6], 2) $y_\ell(x)$ eq. (16), 3) $y_0(x)$ eq. (15), 4) y(x) eq. (17).

Table III

	n = 5			
x	y(x)		x	y(x)
0.00	0.0	1)	0.70	(0.1514)
	(-1.99 E-6)	2)		0.1531614
0.05	0.0023114		0.75	(0.1944)
	(0.0023100)			0.1983240
0.10	0.00476807		0.80	(0.2497)
	(0.0047670)			0.2582165
0.20	(0.0107532)		0.90	(0.412)
	0.0107529			0.4550600
0.30	(0.0194831)		0.94	(0.503)
	0.0194850			0.5910385
0.40	(0.03318)		0.98	(0.614)
	0.0320037			0.8114850
0.50	(0.05536)		1.00	(0.679)
	0.05543735			1.0000000
0.60	(0.09166)			
	0.09204435			

1) $y_\ell(x)$ eq. (16), 2) $y_1(x)$ eq. (17), the less accurate approximation is in parentheses.

x	y(x) in [7]
0.00	0.0
0.50	0.005543
0.75	0.1983
0.94	0.5910
0.98	0.8114
1.00	1.0

For small y we can approximate y(x) by the solution

$$y_\ell(x) = \frac{y'(0)}{n}\sinh nx \qquad (16)$$

of the linearized equation (1)

$$y_\ell'' = n^2 y_\ell.$$

Whereas $y_0(x)$ furnishes an upper bound for the solution, $y_\ell(x)$ gives a lower bound. In order to decide which of the two solutions should be adopted, we simply find the x_m in table II for which the difference $y_0(x) - y_\ell(x)$ is a minimum and use $y_\ell(x)$ below it, $y_0(x)$ above it. Actually, the next approximation to y(x) is quite easy to obtain, as outlined in the next section, and gives considerably better results.

It is not surprising that $y_0(x)$ is independent of the unknown initial condition y'(0). If we integrate (3) for y'(0) = 0, and impose the end condition y(0) = 1 we obtain

$$\int_y^1 d\eta/\sinh(n\eta/2) = 2(1-x)$$

and the result (15) follows immediately.

7. HIGHER ORDER EXPANSION

Since the lowest order solution $y_0(x)$ is so simple, we compute the next term

$y(x) = y_0(x) + a^2 y_1(x)$.

If we use (3) then the a^2 terms lead to

$$2y_0' \, y_1' = 4 + 4ny_1 \sinh(ny_0/2) \cosh(ny_0/2)$$

or

$$y_1' - n \cosh(ny_0/2)\, y_1 = \mathrm{cosech}\,(ny_0/2).$$

By using identities for the hyperbolic functions, this equation is integrated by the standard formula [5, 891.1]. All integrations can be carried out in closed form and the result is

$$y_1(x) = \frac{z}{4n(1-z^2)} \left\{ z^2 - T^2 - \frac{1}{z^2} + \frac{1}{T^2} + 4n(1-x) \right\} \tag{17}$$

where $T = \tanh(n/4)$ and $z = e^{-n} \tanh(n/4)\, e^{nx}$. This permits a simple evaluation of the approximation $y_1(x)$.

For $n = 10$, this result furnishes in general eight significant figures. Even for $n = 5$ we agree with the (apparently truncated) four figures in [7], but the values $y(x)$ based on (17) can be expected to be accurate to five places or more, depending on x.

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