Computation of Newton sum rules for associated and co-recursive classical orthogonal polynomials

Pierpaolo Natalini\textsuperscript{a,*}, Paolo Emilio Ricci\textsuperscript{b}

\textsuperscript{a}Dipartimento di Matematica, Università degli Studi di Roma Tre, L.go San Leonardo Murialdo, 1-00146 Roma, Italy
\textsuperscript{b}Dipartimento di Matematica, Università degli Studi di Roma “La Sapienza”, P.le A. Moro, 2-00185 Roma, Italy

Received 11 October 1999

Abstract


MSC: 33B15; 65L99

Keywords: Associated polynomials; Co-recursive polynomials; Newton sum rules

1. Introduction

Starting from the three-term recurrence relation

\[
\alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x) = xP_n(x), \quad n \geq 0,
\]

\[
P_{-1} = 0, \quad P_0 = 1,
\]

\(\alpha_n, \beta_n, \gamma_n \) real, \(\alpha_n > 0 \ \forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\), associated orthogonal polynomials of order \(c\) (\(c\) integral number) are the set \(\{P_n(x,c)\}_{n \in \mathbb{N}_0}\) defined by the following integral perturbation of indexes...
of (1.1):
\[
\begin{align*}
\alpha_{n+c} P_{n+1}(x; c) + \beta_{n+c} P_n(x; c) + \alpha_{n+c-1} P_{n-1}(x; c) &= xP_n(x; c), \quad n \geq 0 \\
P_{-1}(x; c) &= 0, \quad P_0(x; c) = 1.
\end{align*}
\] (1.2)

Obviously \( P_n(x; 0) \equiv P_n(x) \quad \forall n \in \mathbb{N}_0. \)

The associated orthogonal polynomials have been studied by T.S. Chihara [7,8]. They appear in connection with stationary birth and death processes (see also M.E. Ismail et al. [11]) i.e. Markov processes with non negative integral state variables.

Another class of orthogonal polynomials, the so-called co-recursive polynomials, are defined by adding a real perturbation \( \delta \) to the first coefficient \( \delta_0 \) of recurrence relation (1.1), i.e., considering the set \( \{ Q_n(x; \delta) \}_{n \in \mathbb{N}_0} \) defined by the following recurrence relation:
\[
\begin{align*}
\alpha_n Q_{n+1}(x; \delta) + (\beta_n + \delta \delta_{n,0}) Q_n(x; \delta) + \alpha_{n-1} Q_{n-1}(x; \delta) &= xQ_n(x; \delta), \quad n \geq 0, \\
Q_{-1}(x; \delta) &= 0, \quad Q_0(x; \delta) = 1.
\end{align*}
\] (1.3)

Obviously \( Q_n(x; 0) \equiv P_n(x) \quad \forall n \in \mathbb{N}_0. \)

The co-recursive orthogonal polynomials have been studied by Chihara [6], Letessier [12–14]. They appear in connection with potential scattering (see [20]). Co-recursive associated polynomials \( Q_n(x; \delta, c) \) have also been introduced.

Recently, the distribution of zeros and first Newton sum rules of associated, co-recursive and co-recursive associated polynomials in terms of the entries \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) of the Jacobi matrix have been studied by Ifantis et al. [9], Ifantis and Siafarikas [10]. They give explicit expressions for the first Newton sum rules of the associated and co-recursive associated of the classical orthogonal polynomials.

The differential equations satisfied by such polynomials, have been introduced by Belmedhi and Ronveaux [1], Zarzo et al. [21] for the associated of classical polynomials and by Ronveaux and Marcellan [18], Ronveaux et al. [19] for the co-recursive case.

By using these equations and some preceding results by Dehesa et al. [4], Ricci [17] and Natalini [15], we are able to obtain numerical results for any order Newton sum rules of the above-mentioned polynomial sets.

2. Explicit expression for the Newton sum rules of associated and co-recursive associated OPS

In the above-mentioned papers, Ifantis et al. [9], Ifantis and Siafarikas [10] proved the following results.

For any fixed integral \( N \in \mathbb{N} \), denoted by \( \{ e_n \}_{n=0,1,\ldots,N-1} \) an orthonormal basis of the euclidean space \( E^N \), introduce the operators (matrices):
\[
A = \begin{pmatrix}
\alpha_c & & \\
& \ddots & \\
& & \alpha_c + N-1
\end{pmatrix},
\] (2.1)
\[ B = \begin{pmatrix} \beta_c & \cdots & \\ \vdots & \ddots & \\ \beta_{c+N-1} & & \end{pmatrix} , \]  
(2.2)

\[ V = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} , \]  
(2.3)

\[ V^* = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \end{pmatrix} \]  
(2.4)

and consider the operator \( T_0 := AV^* + VA + B \). Then, for associated polynomials the following representation formula for the Newton sum rules of polynomials \( \{P_n(x,c)\}_{n \in \mathbb{N}} \) holds true:

**Proposition 2.1.** Let \( N \in \mathbb{N} \) be a positive integral number. Denote by \( \lambda_n(c) \) \((n = 0, 1, \ldots, N - 1)\) zeros of the associated polynomial \( P_N(x; c) \). Then for any \( k \in \mathbb{N} \)

\[ \sum_{n=0}^{N-1} j_n^k = \sum_{n=0}^{N-1} (T_0^k e_n, e_n). \]  
(2.5)

**Remark 2.1.** The second-hand side of Eq. (2.5) is independent on the choice of the orthonormal basis \( \{e_n\}_{n=0,1,\ldots,N-1} \) of \( E^N \) (see [9]).

**Remark 2.2.** The matrix associated to the operator \( T_0 \) is the Jacobi-type matrix:

\[ T_0 = \begin{pmatrix} \beta_c & \alpha_c & 0 \\ \alpha_c & \beta_{c+1} & \alpha_{c+1} \\ \vdots & \vdots & \ddots \\ \alpha_{c+N-3} & \beta_{c+N-2} & \alpha_{c+N-2} \\ 0 & \alpha_{c+N-2} & \beta_{c+N-1} \end{pmatrix} . \]  
(2.6)

In the above-mentioned papers [9,10] the authors give analytic expression for the first Newton sum rules in terms of the sequences \( \{\alpha_{n+c}\}_{n \in \mathbb{N}} \) and \( \{\beta_{n+c}\}_{n \in \mathbb{N}} \).

Consider now a positive real parameter \( \gamma \) and co-recursive associated polynomials \( \{Q_n(x; \beta, c)\} \). Then a new family \( \{Q_n(x; \beta, c, \delta)\}_{n \in \mathbb{N}} \) called scaled co-recursive polynomials can be introduced as follows:

\[ \alpha_{n+c} Q_{n+1} + (\beta_{n+c} + \beta \delta_{n,0}) Q_n + \alpha_{n-1+c} Q_{n-1} = x[1 - (\gamma - 1) \delta_{n,0}] Q_n, \quad n \geq 0, \]

\[ Q_{-1} = 0, \quad Q_0 = 1, \]

where \( Q_n \equiv Q_n(x; \beta, \gamma, c) \).
Introduce the further operators (matrices)

\[
C = \begin{pmatrix}
\gamma \\
1 \\
\vdots \\
1
\end{pmatrix} = \text{diag}(\gamma, 1, \ldots, 1); \\
(2.7)
\]

\[
T = C^{-1/2}(T_0 + \beta P_0(x; c))C^{-1/2}; \\
(2.8)
\]

and consider the generalized eigenvalues problem

\[ [T_0 + \beta P_0(x; c)]Z = \lambda CZ. \]

In the above-mentioned paper [9], it is shown that, for any fixed \( N \in \mathbb{N} \), the eigenvalues of this problem are zeros of the polynomial \( Q_n(x; \beta, \gamma, c) \) (\( \beta, \gamma \) real, \( \gamma > 0 \), \( c \) integral number). Then the following proposition holds true:

**Proposition 2.2.** Let \( N \in \mathbb{N} \) be a positive integral number, and denote by \( \omega_n(\beta, \gamma, c) \) (\( n = 0, 1, \ldots, N - 1 \)) the zeros of \( Q_n(x; \beta, \gamma, c) \). Then for any \( k \in \mathbb{N} \)

\[
\sum_{n=0}^{N-1} \omega_n^k(\beta, \gamma, c) = \sum_{n=0}^{N-1} (T^k e_n, e_n). \\
(2.9)
\]

Obviously the above representation formula (2.9) holds true in particular, for the co-recursive associated polynomials \( Q_n(x; \beta, c) \). In this case, we assume \( \gamma = 1 \), \( T = T_0 + \beta P_0(x; c) \) and \( \omega_n(\beta, 1, c) = \omega_n(\beta, c) \).

Even for the considered general case analytic expressions for the first Newton sum rules in terms of the sequence \( \{x_{n+c}\} \), \( \{\beta_{n+c}\} \) and parameters \( \beta, \gamma \) can be found in [9].

### 3. Representation of Newton sum rules in terms of coefficients of differential equations

Explicit expression of the Newton sum rules of classical and semiclassical OPS in terms of coefficients of the corresponding differential equation are given by Ricci [17], Natalini [15]. However, it is worth to note that some year before our papers was printed, results of Case where generalized by Buendia et al. [4], which considered polynomials satisfying differential equations with polynomial coefficients, but not necessarily of hypergeometric type.

In the same paper [4], the above-mentioned authors introduced a recurrent formula for computing polynomial coefficients in terms of the coefficients of the differential equation satisfied by same polynomials.

For shortness, we will limit ourselves to present here the representation formulas which appear in [17,15], but in our paper [16], we have extended the possibility to apply our formulas even in the more general situation considered in [4]. This is important, since the differential equation satisfied by co-recursive OPS of classical polynomials is not, in general, of hypergeometric type.
Formula appearing in [17,15], is based on the so-called Case sum rules (see [5]). Suppose the polynomial

\[ P_N(x) = \prod_{i=1}^{N} (x - x_i) = x^N - u_1 x^{N-1} + \cdots + (-1)^N u_N, \]

satisfy the generalized hypergeometric-type differential equation of order \( r \):

\[ \sum_{i=0}^{r} g_i(x) y^{(i)}(x) = 0, \tag{4.1} \]

where \( g_i(x) = \sum_{j=0}^{i} a_j^{(i)} x^j \), and consider the Newton sum rules

\[ y_h = \sum_{k=1}^{N} x_h^k \]

and the Case sum rules

\[ J^{(i)}_h = \sum_{\not= (l_1, \ldots, l_i)} \frac{x_h^{l_i}}{\prod_{k=1}^{i} (x_{l_i} - x_h)}, \]

where the symbol \( \not= \) means that the sum runs over all \( l_j (j = 1, \ldots, N) \), subject to the condition that all \( l \) are different. The following recurrence relation for the Newton sum rules holds for any \( s \geq 0 \):

\[ \sum_{i=2}^{r} \sum_{j=0}^{i} a_j^{(i)} J^{(i)}_{s+j} = -a_0^{(1)} y_s - a_1^{(1)} y_{s+1}. \]

In [17] a representation formula for the Case sum rules \( J^{(i)}_h \) in terms of the coefficients of the considered polynomial and therefore (by Newton formulas) in terms of the first \( y_i (t \leq s) \) is given.

**Proposition 3.1.** For any \( N \in \mathbb{N} \ (N \geq 2) \), \( h \in \mathbb{N}_0 \), \( i \in \mathbb{N} \), and such that \( 2 \leq i \leq N \), the following representation formula holds true:

\[ J^{(i)}_h = (i-1)! \sum_{k=0}^{N-i} (-1)^k \binom{N-k}{i} u_k \Phi_{N+i-h-k-1}(u_1, u_2, \ldots, u_N), \]

where \( \Phi_r(u_1, u_2, \ldots, u_N) \) denote the generalized Lucas polynomials of second kind in \( N \) variables defined in [2,3].

Another representation method of Newton sum rules is shown by Natalini and Ricci [16]. This method is based on the generalized Lucas polynomials of first kind (see e.g. [2]), and on the above-mentioned formula introduced by Buendia et al. [4].
Table 1

<table>
<thead>
<tr>
<th>( N = 9 )</th>
<th>( N = 12 )</th>
<th>( N = 15 )</th>
<th>( N = 18 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c = 1 )</td>
<td>( c = 2 )</td>
<td>( c = 3 )</td>
<td></td>
</tr>
</tbody>
</table>

(A) Associated Laguerre polynomials \( L_N^{(a)}(x; c) \) (\( a = 0 \))

\[
\begin{array}{cccc}
\mu_1 & 11 & 14 & 17 \\
\mu_2 & 210.7 & 351.83 & 528.86 \\
\mu_3 & 4909.6 & 10829 & 20240.2 \\
\mu_4 & 125703.2 & 368090.16 & 857686.3 \\
\mu_5 & 3394658.8 & 13246195.6 & 38582514.3 \\
\mu_6 & 94724728.1 & 494101308.83 & 1802225707.26 \\
\mu_7 & 2699863878.5 & 18877156981.6 & 86373808509.53 \\
\mu_8 & 78056028739.6 & 733279293589.5 & 4215395056098 \\
\end{array}
\]

(B) Associated Hermite polynomials \( H_N(x; c) \)

\[
\mu_{2i+1} = 0 \ \forall c, N, i
\]

\[
\begin{array}{cccc}
\mu_1 & 15 & 18 & 21 \\
\mu_2 & 361 & 538.5 & 751.8 \\
\mu_3 & 10335 & 19395 & 32663.4 \\
\mu_4 & 323673 & 769135.5 & 1368640.6 \\
\mu_5 & 10692975 & 32302503 & 79969113 \\
\mu_6 & 365527497 & 1407730279.5 & 4237580539.8 \\
\mu_7 & 12784159935 & 62906325381 & 23058032697 \\
\mu_8 & 4542251745017 & 2861172143509.5 & 12786018430325 \\
\end{array}
\]
Table 1. Continued

<table>
<thead>
<tr>
<th>$N$</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_2$</td>
<td>6.6</td>
<td>8.25</td>
<td>9.8</td>
<td>11.3</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>74</td>
<td>115.875</td>
<td>166.5</td>
<td>226</td>
</tr>
<tr>
<td>$\mu_6$</td>
<td>955</td>
<td>1922.0625</td>
<td>3377.85</td>
<td>5424.5</td>
</tr>
<tr>
<td>$\mu_8$</td>
<td>13218.5</td>
<td>34562.71875</td>
<td>74806.725</td>
<td>142796.5</td>
</tr>
</tbody>
</table>

(C) Associated Chebyshev polynomials of second kind $P_N^{(a,b)}(x;c)$ ($a = b = \frac{1}{2}$)

$\mu_{2i+1} = 0 \forall c, N, i$

c = 1, 2, 3

$\mu_2$ | 0.4  | 0.4583 | 0.46  | 0.475 |
| $\mu_4$ | 0.305| 0.322916| 0.3  | 0.34027 |
| $\mu_6$ | 0.2361 | 0.2552083| 0.26  | 0.274305 |
| $\mu_8$ | 0.1927085 | 0.212890625| 0.225 | 0.233072916 |

$(A')$ Co-recursive Laguerre polynomials $L_N^{(a)}(x;\beta)$ ($a = 0$)

$\beta = 1$

$\mu_2$ | 9.1 | 12.083| 15.06 | 18.05 |
| $\mu_4$ | 153.3 | 276.25| 435.2 | 630.16 |
| $\mu_6$ | 3178.1 | 7800.83| 15555.6 | 27525.2 |
| $\mu_8$ | 72445.3 | 243291.25| 615737.6 | 1307558.16 |

$\beta = 2$

$\mu_2$ | 9.2 | 12.16 | 15.13 | 18.1 |
| $\mu_4$ | 153.8 | 276.6 | 435.3 | 630.3 |
| $\mu_6$ | 3180.5 | 7802.6 | 15557.13 | 27253.7 |
| $\mu_8$ | 72456.1 | 243299.3 | 615744.06 | 1307563.5 |

$\beta = 3$

$\mu_2$ | 9.3 | 12.25 | 15.2 | 18.16 |
| $\mu_4$ | 154.6 | 277.25 | 436 | 630.83 |
| $\mu_6$ | 3185 | 7806 | 15559.8 | 27256 |
| $\mu_8$ | 72480 | 243317.25 | 615758.4 | 1307575.5 |

$\mu_7$ | 1739599.2 | 8026324.6 | 25847277.13 | 66465323.1 |
$\mu_8$ | 43099143.2 | 274233586.6 | 1126254594.73 | 3531375759.1 |

$\mu_7$ | 10893333698 | 9589387308 | 50326074192.2 | 192150872152 |
$\mu_8$ | 27897937475.6 | 340708300904 | 2288953308520.2 | 10655028353365 |

$\mu_7$ | 10893337536 | 9589390193.25 | 503260764984.8 | 192150874075.5 |
$\mu_8$ | 27897960720 | 34070818337.25 | 2288953322467.2 | 10655028364987 |
Table 1. Continued

<table>
<thead>
<tr>
<th>$N = 9$</th>
<th>$N = 12$</th>
<th>$N = 15$</th>
<th>$N = 18$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(B'')$ Co-recursive Hermite polynomials $H_N(x; \beta)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\beta = 1$

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\mu_4$</th>
<th>$\mu_5$</th>
<th>$\mu_6$</th>
<th>$\mu_7$</th>
<th>$\mu_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.083</td>
<td>0.06</td>
<td>0.05</td>
<td>0.2</td>
<td>0.2083</td>
<td>0.16</td>
<td>0.138</td>
</tr>
<tr>
<td>4.1</td>
<td>5.583</td>
<td>7.06</td>
<td>8.5</td>
<td>30.3</td>
<td>58</td>
<td>94.7</td>
<td>140.416</td>
</tr>
<tr>
<td>0.27</td>
<td>0.2083</td>
<td>0.06</td>
<td>0.138</td>
<td>0.805</td>
<td>0.60416</td>
<td>0.483</td>
<td>0.4027</td>
</tr>
<tr>
<td>268.027</td>
<td>730.89583</td>
<td>1.54936</td>
<td>2.824638</td>
<td>2.7361</td>
<td>2.05208</td>
<td>1.6416</td>
<td>1.36805</td>
</tr>
<tr>
<td>2.55205</td>
<td>10.02022916</td>
<td>27.7352583</td>
<td>62.40984027</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\beta = 2$

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\mu_4$</th>
<th>$\mu_5$</th>
<th>$\mu_6$</th>
<th>$\mu_7$</th>
<th>$\mu_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.16</td>
<td>0.13</td>
<td>0.13</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
</tr>
<tr>
<td>4.4</td>
<td>5.83</td>
<td>7.26</td>
<td>8.72</td>
<td>32.6</td>
<td>59.75</td>
<td>96.1</td>
<td>141.583</td>
</tr>
<tr>
<td>1.2</td>
<td>0.916</td>
<td>0.73</td>
<td>0.61</td>
<td>6.61</td>
<td>4.9583</td>
<td>3.96</td>
<td>3.305</td>
</tr>
<tr>
<td>281.7</td>
<td>741.2083</td>
<td>1.5577616</td>
<td>2.8315138</td>
<td>3.305</td>
<td>3.305</td>
<td>3.305</td>
<td>3.305</td>
</tr>
<tr>
<td>2.62938</td>
<td>10.07822916</td>
<td>27.7816583</td>
<td>62.44850694</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\beta = 3$

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\mu_4$</th>
<th>$\mu_5$</th>
<th>$\mu_6$</th>
<th>$\mu_7$</th>
<th>$\mu_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.25</td>
<td>0.2</td>
<td>0.16</td>
<td>0.25</td>
<td>0.76</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>3.5</td>
<td>2.625</td>
<td>2.1</td>
<td>1.75</td>
<td>2.625</td>
<td>2.1</td>
<td>1.75</td>
<td>1.75</td>
</tr>
<tr>
<td>41</td>
<td>66</td>
<td>101.1</td>
<td>145.75</td>
<td>41</td>
<td>66</td>
<td>101.1</td>
<td>145.75</td>
</tr>
<tr>
<td>35.75</td>
<td>26.8125</td>
<td>21.45</td>
<td>17.875</td>
<td>35.75</td>
<td>26.8125</td>
<td>21.45</td>
<td>17.875</td>
</tr>
<tr>
<td>80.25</td>
<td>815.0625</td>
<td>1616.7</td>
<td>2880.75</td>
<td>80.25</td>
<td>815.0625</td>
<td>1616.7</td>
<td>2880.75</td>
</tr>
<tr>
<td>362.865</td>
<td>272.15625</td>
<td>217.725</td>
<td>181.4375</td>
<td>362.865</td>
<td>272.15625</td>
<td>217.725</td>
<td>181.4375</td>
</tr>
<tr>
<td>3700.5</td>
<td>10881.5625</td>
<td>28424.325</td>
<td>62984.0625</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$(C'')$ Co-recursive Chebyshev polynomials of second kind $P^{(a,b)}_N(x;c)$ $(a = b = \frac{1}{2})$

$\beta = 1$

<table>
<thead>
<tr>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$\mu_4$</th>
<th>$\mu_5$</th>
<th>$\mu_6$</th>
<th>$\mu_7$</th>
<th>$\mu_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.083</td>
<td>0.06</td>
<td>0.05</td>
<td>0.2</td>
<td>0.2083</td>
<td>0.16</td>
<td>0.138</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5416</td>
<td>0.53</td>
<td>0.527</td>
<td>0.194</td>
<td>0.14583</td>
<td>0.116</td>
<td>0.0972</td>
</tr>
<tr>
<td>0.527</td>
<td>0.48958</td>
<td>0.46</td>
<td>0.45138</td>
<td>0.3194</td>
<td>0.23958</td>
<td>0.1916</td>
<td>0.15972</td>
</tr>
<tr>
<td>0.61805</td>
<td>0.5416</td>
<td>0.49583</td>
<td>0.46527</td>
<td>0.51215</td>
<td>0.38411</td>
<td>0.30729</td>
<td>0.25607</td>
</tr>
<tr>
<td>0.8177</td>
<td>0.68164</td>
<td>0.6</td>
<td>0.54557</td>
<td>0.8177</td>
<td>0.68164</td>
<td>0.6</td>
<td>0.54557</td>
</tr>
</tbody>
</table>

{(C')} Co-recursive Chebyshev polynomials of second kind $P^{(a,b)}_N(x;c)$ $(a = b = \frac{1}{2})$
Table 1. Continued

<table>
<thead>
<tr>
<th>$N$</th>
<th>$N = 9$</th>
<th>$N = 12$</th>
<th>$N = 15$</th>
<th>$N = 18$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.5</td>
<td>0.16</td>
<td>0.13</td>
<td>0.11</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.8</td>
<td>0.7916</td>
<td>0.73</td>
<td>0.6943</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>1.05</td>
<td>0.7916</td>
<td>0.63</td>
<td>0.527</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>2.527</td>
<td>1.98958</td>
<td>1.6</td>
<td>1.45138</td>
</tr>
<tr>
<td>$\mu_5$</td>
<td>4.805</td>
<td>3.60416</td>
<td>2.883</td>
<td>2.4027</td>
</tr>
<tr>
<td>$\mu_6$</td>
<td>10.4305</td>
<td>7.90104</td>
<td>6.383</td>
<td>5.37152</td>
</tr>
<tr>
<td>$\mu_7$</td>
<td>21.73263</td>
<td>16.29947</td>
<td>13.03958</td>
<td>10.86631</td>
</tr>
<tr>
<td>$\mu_8$</td>
<td>46.35937</td>
<td>34.83789</td>
<td>27.925</td>
<td>23.3164</td>
</tr>
</tbody>
</table>

| $\beta = 3$ |         |          |          |          |
| $\mu_1$ | 0.3     | 0.25     | 0.2      | 0.16     |
| $\mu_2$ | 1.4     | 1.2083   | 1.06     | 0.972    |
| $\mu_3$ | 3.25    | 2.4375   | 1.95     | 1.625    |
| $\mu_4$ | 10.305  | 7.82291  | 6.3      | 5.34027  |
| $\mu_5$ | 30.9583 | 23.21875 | 18.575   | 15.47916 |
| $\mu_6$ | 95.6736 | 71.834   | 57.52916 | 47.99305 |
| $\mu_7$ | 294.36979| 220.77734| 176.62187| 147.18489|
| $\mu_8$ | 907.81770| 680.93164| 544.8    | 454.04557|

*We note that results are independent on $c$.

Namely the following proposition holds true:

**Proposition 3.2.** Let be, again

$$P_N(x) = x^N - u_{N,1}x^{N-1} + u_{N,2}x^{N-2} - \cdots + (-1)^N u_{N,N}$$

solution of the differential equation

$$\sum_{k=0}^{m} g_k(x) y^{(k)} = 0,$$

where $g_k(x) = \sum_{j=0}^{l_k} a_j^{(k)} x^j$. Then the coefficients of $P_N$ can be computed recursively, in terms of the coefficients of the differential equation, by means of the following formula:

$$u_{h,j} = \frac{\sum_{p=1}^{j} (-1)^p u_{h,j-p} \sum_{k=0}^{m} (h - j + p)!/(h - j + p - k)! a_{k+p}^{(k)} \sum_{k=0}^{m} (h - j)!/(h - j - k)! a_{k+Q}^{(k)}}{\sum_{k=0}^{m} (h - j)!/(h - j - k)! a_{k+Q}^{(k)}},$$

where $u_{h,0} := 1$, and $Q := \max\{l_k - k; k = 0, 1, 2, \ldots, m\}$. 
Then, recalling definition of the generalized Lucas polynomials of first kind (see e.g. [3]), the representation formula:

\[ y_h = \sum_{k=1}^{N} \lambda_k^h = \Psi_{h+N-2}(u_{N;1}, u_{N;2}, \ldots, u_{N;N}) \]

holds true, which permits numerical computation of the considered Newton sum rules.

4. Numerical results

Recently, Belmedhi and Ronveaux [1], Zarzo et al. [21] and Ronveaux and Marcellan [18], Ronveaux et al. [19] gave explicit differential equations of the fourth-order satisfied by the associated or co-recursive of the classical orthogonal polynomials in terms of the coefficients of the original differential equations and parameters \( c \) or \( \beta \). By using the above-mentioned methods, starting from representation formulas of the Newton sum rules in terms of the coefficients of differential equation (4.1), we have computed numerically the first moments of the above considered associated or co-recursive of the classical Laguerre, Hermite and Jacobi polynomials. Results are shown in Table 1.

References


