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ℝ-factorizability and ω-uniform continuity in topological groups ☆

Li-Hong Xie [a,*], Shou Lin [b,c]

[a] School of Mathematics, Sichuan University, Chengdu 610065, PR China
[b] Department of Mathematics, Zhangzhou Normal University, Zhangzhou 363000, PR China
[c] Institute of Mathematics, Ningde Normal University, Ningde 352100, PR China

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ABSTRACT

In this paper the concept of property ω-U is introduced in topological groups. The main results are that (1) every Lindelöf topological group and every totally bounded topological group have property ω-U; (2) a topological group is ℝ-factorizable if and only if it is an ω-narrow group with property ω-U; (3) $\mathscr{M}$-factorizable groups are preserved by open continuous homomorphisms, which gives a positive answer to a problem posed by A.V. Arhangel'skiĭ and M. Tkachenko.

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1. Introduction

A topological group G is a group G with a topology such that the product mapping of $G \times G$ onto G associating xy with arbitrary $x, y \in G$ is jointly continuous and the inverse mapping of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. All topological groups considered here are assumed to be Hausdorff.

Let G be a topological group. Recall that a real-valued function f on G is *left uniformly continuous*, if $f : (G, \mathscr{V}_G^l) \to (\mathbb{R}, \mathscr{U})$ is a uniformly continuous function, where $\mathscr{V}_G^l$ is the left uniform structure on G and $\mathscr{U}$ is the uniform structure on $\mathbb{R}$. This means that for every $\varepsilon > 0$, there exists $O \in \mathscr{V}_G^l$ such that $|f(x) - f(y)| < \varepsilon$ whenever $(x, y) \in O$. Similarly, f is called *right uniformly continuous*, if $f : (G, \mathscr{V}_G^r) \to (\mathbb{R}, \mathscr{U})$ is a uniformly continuous function, where $\mathscr{V}_G^r$ is the right uniform structure on G. A real-valued function f on G is *uniformly continuous*, if f is both left and right uniformly continuous. J.M. Kister [6] called that a topological group G has *property U* provided that each continuous real-valued function f on G is uniformly continuous. It is well known that every compact topological group has property U and clearly every discrete

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* Corresponding author.
E-mail addresses: xielihong2011@yahoo.cn (L.-H. Xie), shoulin60@163.com (S. Lin).

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group has property U. Kister [6, Corollary 2] had shown that a locally compact group with property U is either discrete or compact.

A topological group G is said to be *totally bounded* [12] if, for each neighborhood V of the identity in G, a finite number of translates of V covers G. W.W. Comfort and K.A. Ross [2, Theorems 1.5 and 2.7] have shown that every pseudocompact topological group has property U, but a totally bounded topological group need not have property U. Therefore, the above analysis naturally leads us to consider what properties the continuous real-valued functions defined on totally bounded groups have.

In this paper, we introduce the concept of property ω-U in Definition 4.1, which is weaker than property U, in topological groups. It is shown that every totally bounded group and every Lindelöf group have property ω-U.

Some decomposition theorems of topological groups are obtained by property ω-U. A topological group G is $\mathbb{R}$-*factorizable* [8,9] if, for every continuous real-valued function f on G, there exist a continuous homomorphism $p : G \to K$ onto a second-countable topological group K and a continuous function $h : K \to \mathbb{R}$ such that $f = h \circ p$. Some characterizations of $\mathbb{R}$-factorizable and related m-factorizable and $\mathscr{M}$-factorizable groups are gave in terms of property ω-U (Theorems 4.9, 5.8 and 5.11). It is showed that an open continuous homomorphic image of an $\mathscr{M}$-factorizable group is $\mathscr{M}$-factorizable, which affirmatively answers a problem posed by A.V. Arhangel'skiĭ and M. Tkachenko in [1, Open Problem 8.4.4].

2. ω-Uniform continuity in uniform spaces

Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces [3,12]. A mapping $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is called *uniform continuous* if for every $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ such that $(f(x), f(x')) \in V$ whenever $(x, x') \in U$.

Let $(X, \mathscr{U})$ be a uniform space. Put $U[x] = \{y \in X \mid (x, y) \in U\}$ for each $U \in \mathscr{U}$. Recall that a continuous real-valued function $f : X \to \mathbb{R}$ is *uniformly continuous* if, for every $\varepsilon > 0$, there exists $U \in \mathscr{U}$ such that $f(U[x]) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$ for all $x \in X$. It is well known that every continuous real-valued function on a compact uniform space is uniformly continuous. We introduce the concept of ω-uniform continuity as a generalization of the uniform continuity in uniform spaces.

Definition 2.1. Let $(X, \mathscr{U})$ be a uniform space. A function $f : X \to \mathbb{R}$ is called *ω-uniformly continuous* if, for every $\varepsilon > 0$, there is a countable family $\mathscr{V} \subseteq \mathscr{U}$ satisfying that for each $x \in X$ there exists $V_x \in \mathscr{V}$ such that

$$f\big(V_x[x]\big) \subseteq \big(f(x) - \varepsilon, f(x) + \varepsilon\big).$$

Remark 2.2. It is easy to see that uniformly continuous $\Rightarrow$ ω-uniformly continuous $\Rightarrow$ continuous. However, the converses are not true, see Remark 2.7 and Theorem 4.9.

Consider a uniform space $(X, \mathscr{U})$ and a pseudometric ρ on the set X. The pseudometric ρ is called *uniform with respect to $\mathscr{U}$* if for every $\varepsilon > 0$ there exists $V \in \mathscr{U}$ such that $\rho(x, y) < \varepsilon$ whenever $(x, y) \in V$.

Lemma 2.3. *([3, Corollary 8.1.11]) For every uniformity $\mathscr{U}$ on a set X and every $V \in \mathscr{U}$ there exists a pseudometric ρ on X which is uniform with respect to $\mathscr{U}$ and satisfies the condition $\{(x, y) \mid \rho(x, y) < 1\} \subseteq V$.*

Remark 2.4. Let $(X, \mathscr{U})$ be a uniform space. For every $V \in \mathscr{U}$, take a pseudometric ρ_V satisfying the conditions in Lemma 2.3. By letting $xE_V y$ whenever $\rho_V(x, y) = 0$ an equivalent relation E_V on the set X is defined. Let X_V be the quotient set of E_V. By letting $\overline{\rho}_V([x], [y]) = \rho_V(x, y)$ for all $[x], [y] \in X_V$ a metric $\overline{\rho}_V$ on the set X_V is defined. Let $\mathscr{U}_V$ be the uniformity on the set X_V induced by the metric $\overline{\rho}_V$. It follows from Lemma 2.3, that letting $f_V(x) = [x]$, we define a uniformly continuous mapping $f_V : (X, \mathscr{U}) \to (X_V, \mathscr{U}_V)$.

A uniform space $(X, \mathscr{U})$ is metrizable if there exists a metric ρ on the set X such that the uniformity induced by ρ coincides with the original uniformity $\mathscr{U}$. It is well known that a uniformity $\mathscr{U}$ on a set X is induced by a metric ρ if and only if the uniformity $\mathscr{U}$ has a countable base [3].

Theorem 2.5. *Let $(X, \mathscr{U})$ be a uniform space and $f : X \to \mathbb{R}$ be a function. The following are equivalent.*

(1) *f is ω-uniformly continuous*;
(2) *there exist a uniformly continuous function $g : (X, \mathscr{U}) \to (Y, \mathscr{V})$ onto a metrizable uniform space $(Y, \mathscr{V})$ and a continuous function $p : Y \to \mathbb{R}$ such that $f = p \circ g$.*

Proof. (1) $\Rightarrow$ (2). Let $f : X \to \mathbb{R}$ be ω-uniformly continuous. By Definition 2.1, for each $n \in \mathbb{N}$ there exists a countable family $\zeta_n \subseteq \mathscr{U}$ satisfying that for every $x \in X$ there exists $V_x \in \zeta_n$ such that $f(V_x[x]) \subseteq (f(x) - \frac{1}{n}, f(x) + \frac{1}{n})$. Put $\zeta = \bigcup_{n \in \mathbb{N}} \zeta_n$. Then $|\zeta| \leqslant \omega$. According to Lemma 2.3, for each $V \in \zeta$ there exists a pseudometric ρ_V on the set X which is uniform with respect to $\mathscr{U}$ and satisfies the condition $\{(x,y) \mid \rho_V(x,y) < 1\} \subseteq V$. Therefore, there exists a uniformly continuous function $g_V : (X,\mathscr{U}) \to (X_V, \mathscr{U}_V)$, where g_V and $(X_V, \mathscr{U}_V)$ are defined according to Remark 2.4. Define

$$g = \Delta_{V\in\zeta} g_V : (X,\mathscr{U}) \to \left(\prod_{V\in\zeta} X_V, \prod_{V\in\zeta} \mathscr{U}_V\right),$$

where $\Delta_{V\in\zeta} g_V$ is the diagonal product of the family $\{g_V \mid V \in \zeta\}$. Since g_V is uniformly continuous for each $V \in \zeta$ and the Cartesian product $(\prod_{V\in\zeta} X_V, \prod_{V\in\zeta} \mathscr{U}_V)$ is a metrizable uniform space, $g = \Delta_{V\in\zeta} g_V$ is uniformly continuous.

Claim. *$f(x_1) = f(x_2)$ for all $x_1, x_2 \in X$ satisfying $g(x_1) = g(x_2)$.*

Indeed, assume to the contrary, and choose $x_1, x_2 \in X$ and $n \in \mathbb{N}$ such that

$$g(x_1) = g(x_2) \quad \text{and} \quad f(x_1) \notin \left(f(x_2) - \frac{1}{n}, f(x_2) + \frac{1}{n}\right).$$

By the property of ζ_n, for x_2 there exists $V \in \zeta_n$ such that

$$f\bigl(V[x_2]\bigr) \subseteq \left(f(x_2) - \frac{1}{n}, f(x_2) + \frac{1}{n}\right).$$

From $g(x_1) = g(x_2)$ it follows that $g_V(x_1) = g_V(x_2)$, hence $\rho_V(x_1, x_2) = 0$, thus

$$(x_2, x_1) \in \bigl\{(x,y) \in X \times X \bigm| \rho_V(x,y) < 1\bigr\} \subseteq V$$

by Lemma 2.3 and Remark 2.4. Therefore, $x_1 \in V[x_2]$, which implies that

$$f(x_1) \in f\bigl(V[x_2]\bigr) \subseteq \left(f(x_2) - \frac{1}{n}, f(x_2) + \frac{1}{n}\right).$$

This contradiction completes the proof of the claim.

From the claim it follows that there is a function $p : g(X) \to \mathbb{R}$ such that $f = p \circ g$. It remains to prove that the function p is continuous.

Let $y \in g(X)$ and $\varepsilon > 0$. Take an $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Choose a point $x \in X$ with $g(x) = y$. For x there exists $V \in \zeta_n$ such that

$$f\bigl(V[x]\bigr) \subseteq \left(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\right) = \left(p(y) - \frac{1}{n}, p(y) + \frac{1}{n}\right) \subseteq \bigl(p(y) - \varepsilon, p(y) + \varepsilon\bigr).$$

Put

$$B = \bigl\{z \in X_V \bigm| \overline{\rho}_V\bigl(\pi_V(y), z\bigr) < 1\bigr\},$$

where $\overline{\rho}_V$ is defined according to Remark 2.4 and $\pi_V : \prod_{V'\in\zeta} X_{V'} \to X_V$ is the projection. And set

$$W = g(X) \cap \left(\prod_{V'\in\zeta\setminus\{V\}} X_{V'} \times B\right).$$

Clearly, W is a neighborhood of y. Now we shall prove that $p(W) \subseteq (p(y) - \varepsilon, p(y) + \varepsilon)$, which implies that p is continuous.

Indeed, from Lemma 2.3 and Remark 2.4 it follows that

$$g^{-1}(W) = g_V^{-1}(B) = \bigl\{z \in X \bigm| \rho_V(x,z) < 1\bigr\} \subseteq V[x].$$

Thus,

$$p(W) = f\bigl(g^{-1}(W)\bigr) \subseteq f\bigl(V[x]\bigr) \subseteq \bigl(f(x) - \varepsilon, f(x) + \varepsilon\bigr) = \bigl(p(y) - \varepsilon, p(y) + \varepsilon\bigr).$$

(2) $\Rightarrow$ (1). There exist a uniformly continuous function $g : (X,\mathscr{U}) \to (Y,\mathscr{V})$ onto a metrizable uniform space $(Y,\mathscr{V})$ and a continuous function $p : Y \to \mathbb{R}$ such that $f = p \circ g$. Since $(Y,\mathscr{V})$ is metrizable, there exists a countable base μ of the uniformity $\mathscr{V}$. Put $\gamma = \{\psi^{-1}(V) \mid V \in \mu\}$, where $\psi = (g,g) : X \times X \to Y \times Y$. Then $|\gamma| \leqslant \omega$ and $\gamma \subseteq \mathscr{U}$ by the uniform continuity of g. Take any $\varepsilon > 0$. Since p is continuous, there exists $V \in \mu$ such that $p(V[g(x)]) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$. From $\psi^{-1}(V) \in \gamma$ and

$$f\bigl(\psi^{-1}(V)[x]\bigr) \subseteq p\bigl(V\bigl[g(x)\bigr]\bigr) \subseteq \bigl(f(x) - \varepsilon, f(x) + \varepsilon\bigr)$$

it follows that f is ω-uniformly continuous. $\square$

From Theorem 2.5 it easily follows the following result.

Corollary 2.6. *Every continuous real-valued function f on a metrizable uniform space is ω-uniformly continuous.*

Remark 2.7. "ω-Uniformly continuous" cannot be replaced by "uniformly continuous" in Corollary 2.6. For instance, $\mathbb{R}$ with usual uniformity is metrizable, but not all continuous real-valued functions on $\mathbb{R}$ are uniformly continuous. It implies that ω-uniformly continuous $\nRightarrow$ uniformly continuous.

3. ω-Uniform continuity in topological groups

Let G be a topological group. Denote by $\mathscr{N}_s(G, e)$ the family of all open symmetric neighborhoods at the identity e of G in this paper. We introduce the concept of ω-uniform continuity as a generalization of the uniform continuity on topological groups.

Definition 3.1. A real-valued function f on a topological group G is *left* (resp. *right*) *ω-uniformly continuous* if, for every $\varepsilon > 0$, there exists a countable family $\mathcal{U} \subseteq \mathscr{N}_s(G, e)$ such that for every $x \in G$, there exists $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$ (resp. whenever $yx^{-1} \in U$).

Definition 3.2. A real-valued function f on a topological group G is *ω-uniformly continuous* if f is both left and right ω-uniformly continuous.

Remark 3.3. (1) If we consider a topological group G as a uniform space $(G, \mathscr{V}_G^l)$ or $(G, \mathscr{V}_G^r)$, where $\mathscr{V}_G^l$ and $\mathscr{V}_G^r$ are left and right uniformities, respectively, then Definition 3.1 is equivalent to Definition 2.1.

(2) ω-Uniformly continuous $\nRightarrow$ uniformly continuous by Remark 2.7.

According to the definitions, one can easily obtain the following results.

Theorem 3.4. *Let f be a real-valued function defined on a topological group. Then*

(1) *if f is left* (*resp. right*) *uniformly continuous, then f is left* (*resp. right*) *ω-uniformly continuous*;
(2) *if f is uniformly continuous, then f is ω-uniformly continuous.*

Recall that a topological space X is a *P-space* if every G_δ-set in X is open. Similarly, a *P-group* is a topological group whose underlying space is a P-space.

Theorem 3.5. *Let f be a real-valued function defined on a P-group G. Then*

(1) *f is left* (*resp. right*) *uniformly continuous if and only if f is left* (*resp. right*) *ω-uniformly continuous*;
(2) *f is uniformly continuous if and only if f is ω-uniformly continuous.*

The following theorem gives a characterization of left or right ω-uniformly continuous functions on a topological group.

Theorem 3.6. *Let f be a real-valued function defined on a topological group G. The following are equivalent.*

(1) *f is left* (*resp. right*) *ω-uniformly continuous*;
(2) *there exists a countable family $\mathcal{U}_f \subseteq \mathscr{N}_s(G, e)$ satisfying that for every point $x \in G$ and $\varepsilon > 0$, there exists $U \in \mathcal{U}_f$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U$* (*resp. $yx^{-1} \in U$*).

Theorem 3.7. *Let G be a topological group. Every* (*resp. bounded*) *continuous real-valued function on G is left ω-uniformly continuous if and only if every* (*resp. bounded*) *continuous real-valued function on G is right ω-uniformly continuous.*

4. ω-Uniform continuity and ℝ-factorizable topological groups

In this section, we apply the concept of ω-uniform continuity into studying the class of $\mathbb{R}$-factorizable topological groups. Kister's property U is defined in Section 1. Comfort and Ross [2] called that a topological group G has *property BU* if each bounded continuous real-valued function on G is uniformly continuous.

Definition 4.1. A topological group G has *property ω-U* (resp. *property $B\omega$-U*) if each (resp. bounded) continuous real-valued function on G is ω-uniformly continuous.

Remark 4.2. (1) The uniform structure on G should be taken to be either the left or right uniform structure. It often happens that these structures do not coincide. Nevertheless, according to Theorem 3.7, the definitions of properties ω-U and $B\omega$-U are unambiguous.

(2) According to the definitions of properties ω-U and $B\omega$-U and Theorem 3.5, every topological group with property U (resp. BU) has property ω-U (resp. $B\omega$-U).

It is well known that a topological group has property BU if and only if it has property U [2, Theorem 2.8].

Theorem 4.3. *A topological group has property $B\omega$-U if and only if it has property ω-U.*

Proof. It is obvious that property ω-U implies property $B\omega$-U. Suppose that a topological group G has property $B\omega$-U and let f be a continuous real-valued function on G. Thus, the bounded continuous function $(-n) \vee f \wedge n$ must be ω-uniformly continuous for all $n \in \mathbb{N}$. Using this fact and Theorem 3.6, one can easily obtain that f is ω-uniformly continuous, thus G has property ω-U. $\square$

Theorem 4.4. *Every Lindelöf topological group has property ω-U.*

Proof. Let G be a Lindelöf topological group. According to Theorem 3.7 and Definition 4.1, it suffices to show that every continuous real-valued function f on G is left ω-uniformly continuous. Since G is Lindelöf, for each $n \in \mathbb{N}$ one can easily find a family $\mathscr{U}_{f,n} = \{V_j \mid j \in \omega\} \subseteq \mathscr{N}_s(G, e)$ and a subset $A_{f,n} = \{h_j \mid j \in \omega\} \subseteq G$ satisfying that:

(i) $G = \bigcup_{j\in\omega} h_j V_j$;
(ii) $f(h_j V_j^2) \subseteq (f(h_j) - \frac{1}{n}, f(h_j) + \frac{1}{n})$ for each $j \in \omega$.

Put $\mathscr{U}_f = \bigcup_{n\in\mathbb{N}} \mathscr{U}_{f,n}$. We shall show that $\mathscr{U}_f$ satisfies the condition (2) in Theorem 3.6, which implies that f is left ω-uniformly continuous. It is obvious that $|\mathscr{U}_f| \leqslant \omega$ and $\mathscr{U}_f \subseteq \mathscr{N}_s(G, e)$. Let $h \in G$ and $\varepsilon > 0$. There is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$. According to (i) there exists $j_0 \in \omega$ such that $h \in h_{j_0} V_{j_0}$, where $h_{j_0} \in A_{f,n_0}$ and $V_{j_0} \in \mathscr{U}_{f,n_0} \subseteq \mathscr{U}_f$. From (ii) it follows that

$$f(hV_{j_0}) \subseteq f\big(h_{j_0} V_{j_0}^2\big) \subseteq \left(f(h_{j_0}) - \frac{1}{n_0}, f(h_{j_0}) + \frac{1}{n_0}\right) \subseteq \left(f(h_{j_0}) - \frac{\varepsilon}{2}, f(h_{j_0}) + \frac{\varepsilon}{2}\right),$$

that is,

$$\big|f(h) - f(y)\big| \leqslant \big|f(h) - f(h_{j_0})\big| + \big|f(h_{j_0}) - f(y)\big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

whenever $h^{-1}y \in V_{j_0}$. $\square$

Corollary 4.5. *Every subgroup of a topological group with a countable network has property ω-U, in particular, so does every subgroup of a second-countable topological group.*

Remark 4.6. "Property ω-U" in Theorem 4.4 and Corollary 4.5 cannot be replaced by "property U". For instance, the group $(\mathbb{R}, +)$ with the usual topology is second-countable, but it is well known that not all continuous real-valued functions on $(\mathbb{R}, +)$ are uniformly continuous.

A topological group G is said to be *ω-narrow* (i.e., *$\aleph_0$-bounded* [4]) if for each neighborhood V of the identity in G, there exists a countable subset $M \subseteq G$ such that $G = MV$.

Lemma 4.7. *([1, Corollary 3.4.19]) Let H be an ω-narrow topological group. Then for every open neighborhood U of the identity in H, there exists a continuous homomorphism π of H onto a second-countable topological group G such that $\pi^{-1}(V) \subseteq U$, for some open neighborhood V of the identity in G.*

Lemma 4.8. *Let H be an ω-narrow topological group and $f : H \to \mathbb{R}$ be either left or right ω-uniformly continuous. Then there exist a continuous homomorphism $\pi : H \to K$ onto a second-countable topological group K and a continuous function $p : K \to \mathbb{R}$ such that $f = p \circ \pi$.*

Proof. Suppose that f is left ω-uniformly continuous on H. According to Theorem 3.6, there exists a countable family $\mathscr{U}_f \subseteq \mathscr{N}_s(H, e)$ satisfying that for every point $x \in H$ and $\varepsilon > 0$, there exists $V \in \mathcal{U}_f$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in V$.

Since H is ω-narrow, according to Lemma 4.7, for each $V \in \mathscr{U}_f$ there exists a continuous homomorphism π_V of H onto a second-countable topological group G_V such that $\pi_V^{-1}(U) \subseteq V$, for some open neighborhood U of the identity in G_V. Let $\pi = \Delta_{V \in \mathscr{U}_f} \pi_V$ be the diagonal product of the family $\{\pi_V \mid V \in \mathscr{U}_f\}$.

It is obvious that $\pi(H)$ is a second-countable topological group, since $\prod_{V \in \mathscr{U}_f} G_V$ is second-countable.

Claim. *$f(h_1) = f(h_2)$ for all $h_1, h_2 \in H$ satisfying $\pi(h_1) = \pi(h_2)$.*

Indeed, assume to the contrary, and choose $h_1, h_2 \in H$ and $\varepsilon > 0$ such that

$$\pi(h_1) = \pi(h_2) \quad \text{and} \quad f(h_2) \notin \big(f(h_1) - \varepsilon, f(h_1) + \varepsilon\big).$$

By the property of $\mathscr{U}_f$, for h_1 and ε there exists $V_{h_1,\varepsilon} \in \mathscr{U}_f$ such that $|f(h_1) - f(y)| < \varepsilon$ whenever $h_1^{-1} y \in V_{h_1,\varepsilon}$, which is equivalent to $f(h_1 V_{h_1,\varepsilon}) \subseteq (f(h_1) - \varepsilon, f(h_1) + \varepsilon)$. Therefore, there exists an open neighborhood U of the identity in $G_{V_{h_1,\varepsilon}}$ such that $\pi^{-1}_{V_{h_1,\varepsilon}}(U) \subseteq V_{h_1,\varepsilon}$ by the property of $\pi_{V_{h_1,\varepsilon}}$. Take an open neighborhood W of the identity in $G_{V_{h_1,\varepsilon}}$ such that $W^2 \subseteq U$. Put $g = \pi_{V_{h_1,\varepsilon}}(h_1)$, then $g = \pi_{V_{h_1,\varepsilon}}(h_2)$ by $\pi(h_1) = \pi(h_2)$, and

$$\begin{aligned} h_2 \in \pi^{-1}_{V_{h_1,\varepsilon}}(gW) &= \pi^{-1}_{V_{h_1,\varepsilon}}(g)\pi^{-1}_{V_{h_1,\varepsilon}}(W) \\ &= h_1 \pi^{-1}_{V^{h_1,\varepsilon}}(e)\pi^{-1}_{V_{h_1,\varepsilon}}(W) \subseteq h_1 \pi^{-1}_{V_{h_1,\varepsilon}}(W)\pi^{-1}_{V_{h_1,\varepsilon}}(W) \\ &= h_1 \pi^{-1}_{V_{h_1,\varepsilon}}\big(W^2\big) \subseteq h_1 \pi^{-1}_{V_{h_1,\varepsilon}}(U) \subseteq h_1 V_{h_1,\varepsilon}, \end{aligned}$$

which implies that

$$f(h_2) \in f(h_1 V_{h_1,\varepsilon}) \subseteq \big(f(h_1) - \varepsilon, f(h_1) + \varepsilon\big).$$

This contradiction completes the proof of the claim.

From the claim it follows that there is a function $p : \pi(H) \to \mathbb{R}$ such that $f = p \circ \pi$. It remains to prove that p is continuous.

Take any $\varepsilon > 0$, $g \in \pi(H)$ and choose a point $h \in H$ such that $g = \pi(h)$. According to $f = p \circ \pi$ and the property of $\mathscr{U}_f$ there exists $V_{h,\varepsilon} \in \mathscr{U}_f$ such that

$$f(hV_{h,\varepsilon}) \subseteq \big(f(h) - \varepsilon, f(h) + \varepsilon\big) = \big(p(g) - \varepsilon, p(g) + \varepsilon\big).$$

By the property of $\pi_{V_{h,\varepsilon}}$ above, there is an open neighborhood U containing the identity in $G_{V_{h,\varepsilon}}$ such that $\pi^{-1}_{V_{h,\varepsilon}}(U) \subseteq V_{h,\varepsilon}$. Choose an open neighborhood W of the identity in $G_{V_{h,\varepsilon}}$ such that $W^2 \subseteq U$. Put

$$O = \pi(H) \cap \left(W \times \prod_{V \in \mathscr{U}_f \setminus \{V_{h,\varepsilon}\}} G_V\right).$$

We claim that $p(gO) \subseteq (p(g) - \varepsilon, p(g) + \varepsilon)$, which implies that p is continuous.

In fact, since $g_{V_{h,\varepsilon}} = \pi_{V_{h,\varepsilon}}(h)$,

$$\begin{aligned} p(gO) &\subseteq f\big(\pi^{-1}(gO)\big) \\ &= f\left(\pi^{-1}\left(\pi(H) \cap \left(g_{V_{h,\varepsilon}} W \times \prod_{V \in \mathscr{U}_f \setminus \{V_{h,\varepsilon}\}} G_V\right)\right)\right) \\ &= f\big(\pi^{-1}_{V_{h,\varepsilon}}(g_{V_{h,\varepsilon}} W)\big) \subseteq f\big(h\pi^{-1}_{V_{h,\varepsilon}}(U)\big) \subseteq f(hV_{h,\varepsilon}) \\ &\subseteq \big(f(h) - \varepsilon, f(h) + \varepsilon\big) = \big(p(g) - \varepsilon, p(g) + \varepsilon\big). \end{aligned}$$

This completes the proof when f is left ω-uniformly continuous.

Similarly, one can easily prove the result when f is right ω-uniformly continuous. □

The following is a main result in this section.

Theorem 4.9. *A topological group H is $\mathbb{R}$-factorizable if and only if it is an ω-narrow group with property ω-U.*

Proof. The sufficiency is obtained by Lemma 4.8. Conversely, suppose that H is an $\mathbb{R}$-factorizable topological group. Then H is ω-narrow [8, Lemma 2.2], so that it remains to show that H has property ω-U. Take any continuous real-valued f on H. Since H is $\mathbb{R}$-factorizable, there exist a continuous homomorphism $\pi : H \to K$ onto a second-countable topological group K and a continuous function $p : K \to \mathbb{R}$ such that $f = p \circ \pi$. Let $\mathscr{B}$ be a countable local base of the identity in K. Put $\mathscr{U}_f = \{\pi^{-1}(U) \mid U \in \mathscr{B}\}$. One can easily verify that $\mathscr{U}_f$ is a countable family of open neighborhoods of the identity in H and satisfies that for every point $x \in H$ and $\varepsilon > 0$, there exists $U_{x,\varepsilon} \in \mathcal{U}_f$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in U_{x,\varepsilon}$, which implies that the function f is left ω-uniformly continuous by Theorem 3.6. From Theorem 3.7 it follows that H has property ω-U. $\square$

Since there is a topological group G which is ω-narrow but not $\mathbb{R}$-factorizable [1, Example 8.2.1], from Theorem 4.9 it follows that there exists a continuous function on G, which is not ω-uniformly continuous.

Corollary 4.10. *([1, 8.1.b]) If H is an ω-narrow topological group with property U, then H is $\mathbb{R}$-factorizable.*

It is well known that every Lindelöf topological group is ω-narrow [1, Proposition 3.4.6]. According to Theorems 4.3 and 4.9, the following result is obvious.

Corollary 4.11. *([10, Theorem 5.5]) Every Lindelöf topological group is $\mathbb{R}$-factorizable.*

Since every totally bounded topological group is $\mathbb{R}$-factorizable [8, Corollary 1.14], the following result is obtained by Theorem 4.9.

Corollary 4.12. *Every totally bounded topological group has property ω-U.*

Remark 4.13. "Property ω-U" in Corollary 4.12 cannot be replaced by "property U", since every totally bounded topological group with property U is pseudocompact [2, Theorem 2.7].

Recall that a space X is said to be *pseudo-ω_1-compact* if every locally finite (equivalently, discrete) family of open sets in X is countable.

Corollary 4.14. *Let G be a topological group with property U. Then*

(1) *G is pseudo-ω_1-compact if and only if it is $\mathbb{R}$-factorizable*;
(2) *the continuous homomorphic image of G is $\mathbb{R}$-factorizable if G is ω-narrow.*

Proof. (1) It was proved that G is pseudo-ω_1-compact if and only if it is $\mathbb{R}$-factorizable when G is a P-group [11, Theorem 4.16]. Thus, we can assume that G is not a P-group and has property U.

Sufficiency. In [2, Theorem 2.2], it was proved that if a topological group has property U, then it is either totally bounded or a P-group. Thus G is totally bounded. According to the fact that a totally bounded topological group with property U is pseudocompact [2, Theorem 2.7], G is pseudo-ω_1-compact.

Necessity. Suppose that G is pseudo-ω_1-compact and has property U. According to [1, Proposition 3.4.31] and Remark 4.2, G is ω-narrow and has property ω-U. Thus G is $\mathbb{R}$-factorizable by Theorem 4.9.

(2) Suppose that G is an ω-narrow topological group with property U. It follows that G is $\mathbb{R}$-factorizable by Theorem 4.9. Since it is well known that a continuous homomorphic image of every $\mathbb{R}$-factorizable P-group is $\mathbb{R}$-factorizable [11, Corollary 5.9], it is enough to prove that the continuous homomorphic image of G is $\mathbb{R}$-factorizable when G is not a P-group.

In fact, in the sufficiency of the proof of (1), we have shown that G is pseudocompact when G is not a P-group with property U. Since a continuous homomorphic image of a pseudocompact (resp. an ω-narrow) topological group is pseudocompact (resp. ω-narrow [1, Proposition 3.4.2]) and every pseudocompact topological group has property U [2, Theorem 1.5], from Remark 4.2 and Theorem 4.9 it follows that the continuous homomorphic image of G is $\mathbb{R}$-factorizable. $\square$

Corollary 4.15. *([9, Theorem 3.8]) Every locally finite family of open subsets of a locally connected $\mathbb{R}$-factorizable topological group G is countable.*

Proof. Suppose that there exists an uncountable locally finite family of open subsets of G. Then there exists an uncountable discrete family $\{O_\alpha \mid \alpha < \omega_1\}$ of non-void open subsets of G [7, Lemma 1]. Since G is Hausdorff, it is completely regular. For every $\alpha < \omega_1$ pick a point $x_\alpha \in O_\alpha$ and define a continuous function $f_\alpha : G \to [0, 1]$ such that $f_\alpha(x_\alpha) = 1$

and $f_\alpha(G \setminus O_\alpha) = \{0\}$. Then $f = \sum_{\alpha<\omega_1} f_\alpha$ is continuous. Since G is $\mathbb{R}$-factorizable, f is ω-uniformly continuous by Theorem 4.9. Since G is locally connected, there exists a countable family $\mathcal{V}$ of non-void connected open neighborhoods at the identity of G satisfying that for every point $x \in G$ there exists $V \in \mathcal{V}$ such that $|f(x) - f(y)| < 1$ whenever $y \in xV$. Then for each $\beta < \omega_1$, there exists $V_\beta \in \mathcal{V}$ such that $x_\beta V_\beta \subseteq \bigcup_{\alpha<\omega_1} O_\alpha$. Since $x_\beta V_\beta$ is connected and $\{O_\alpha \mid \alpha < \omega_1\}$ is discrete, $x_\beta V_\beta \subseteq O_\beta$. Since $\mathcal{V}$ is countable, there are $V_0 \in \mathcal{V}$ and an uncountable subset $A \subseteq \omega_1$ such that $x_\alpha V_0 \subseteq O_\alpha$ for each $\alpha \in A$. Then $x_\alpha V_0 \cap x_\beta V_0 = \emptyset$ whenever $\alpha, \beta \in A$, $\alpha \neq \beta$. Let W be an open symmetric neighborhood of the identity of G such that $W^2 \subseteq V_0$. The group G is ω-narrow by Theorem 4.9. Therefore there exists a countable subset $K \subseteq G$ such that $G = WK$. Since A is uncountable, one can find a point $x \in K$ and distinct $\alpha, \beta \in A$ such that $\{x_\alpha, x_\beta\} \subseteq Wx$. Then $x_\beta^{-1} x_\alpha \in W^2 \subseteq V_0$, that is, $x_\alpha \in x_\beta V_0$, a contradiction with $x_\alpha V_0 \cap x_\beta V_0 = \emptyset$. □

Theorem 4.16. *Let G be a topological group with property ω-U (resp. property $B\omega$-U). If N is a closed normal subgroup of G, then the quotient group G/N has property ω-U (resp. property $B\omega$-U).*

Proof. Let $p : G \to G/N$ be a quotient homomorphism. Then p is an open continuous homomorphism [1, Theorem 1.5.1]. Take any (resp. bounded) continuous real-valued function f on G/N. Then $f \circ p$ is a (resp. a bounded) continuous real-valued function on G. Since G has property ω-U (resp. $B\omega$-U), $f \circ p$ is ω-uniformly continuous by Definition 4.1. According to Theorem 3.6, there exists a countable family $\mathcal{U}_{f\circ p} \subseteq \mathscr{N}_s(G, e)$ satisfying that for every $x \in G$ and $\varepsilon > 0$, there exists $U_{x,\varepsilon} \in \mathcal{U}_{f\circ p}$ such that $|f \circ p(x) - f \circ p(y)| < \varepsilon$ whenever $x^{-1}y \in U_{x,\varepsilon}$. Put $\mathscr{U}_f = \{p(U) \mid U \in \mathscr{U}_{f\circ p}\}$. Since p is an open homomorphism, one can easily verify that $\mathscr{U}_f$ satisfies the condition (2) in Theorem 3.6, which implies that f is (left) ω-uniformly continuous. So, G/N has property ω-U (resp. $B\omega$-U) by Theorem 3.7. □

Since the continuous homomorphic image of an ω-narrow group is ω-narrow [1, Proposition 3.4.2], according to Theorems 4.16 and 4.9 one can easily obtain the following result.

Corollary 4.17. *([9, Theorem 3.10]) An open continuous homomorphic image of an $\mathbb{R}$-factorizable topological group is $\mathbb{R}$-factorizable.*

5. ω-Uniform continuity and m-factorizable groups

A topological group G is called *m-factorizable* [1] (resp. *$\mathscr{M}$-factorizable* [1]) if for every continuous function $f : G \to M$ to a metrizable space M, there exist a continuous homomorphism $p : G \to K$ onto a second-countable (resp. first-countable) topological group K and a continuous function $g : K \to M$ such that $f = g \circ p$.

The following question is posed by A.V. Arhangel'skiĭ and M. Tkachenko in 2008. It is affirmatively answered in this section.

Question 5.1. ([1, Open Problem 8.4.4]) Is any quotient group of an $\mathscr{M}$-factorizable topological group $\mathscr{M}$-factorizable?

Definition 5.2. Let G be a topological group and (M, ρ) be a metric space. A function $f : G \to M$ is *left* (resp. *right*) *ω-uniformly continuous* if, for every $\varepsilon > 0$, there exists a countable family $\mathcal{U} \subseteq \mathscr{N}_s(G, e)$ satisfying that for every point $x \in G$, there exists $U \in \mathcal{U}$ such that $\rho(f(x), f(y)) < \varepsilon$ whenever $x^{-1}y \in U$ (resp. whenever $yx^{-1} \in U$).

Definition 5.3. Let G be a topological group and (M, ρ) be a metric space. A function $f : G \to M$ is *ω-uniformly continuous* if f is both left and right ω-uniformly continuous.

The invariance number $\mathrm{inv}(G)$ [1] of a semitopological group G is countable (notation: $\mathrm{inv}(G) \leqslant \omega$) if for each open neighborhood U of the neutral element e in G there exists a countable family γ of open neighborhoods of e such that for each $x \in G$, there exists $V \in \gamma$ satisfying $xVx^{-1} \subseteq U$. A topological group G such that $\mathrm{inv}(G) \leqslant \omega$ are also called *ω-balanced* [1].

Lemma 5.4. *([1, Theorem 3.4.18]) Let H be an ω-balanced topological group. Then, for every open neighborhood U of the identity in H, there exists a continuous homomorphism π from H onto a metrizable topological group G such that $\pi^{-1}(V) \subseteq U$, for some open neighborhood V of the identity in G.*

It is well known that a topological group G is metrizable if and only if G is first-countable [1, Theorem 3.3.12]. In the proof of Lemma 4.8, it does not use the order property of $\mathbb{R}$, but uses the metrizable property of $\mathbb{R}$, so, making a simple change, one can easily obtain the following result by Lemmas 4.7 and 5.4.

Lemma 5.5. *Let G be an ω-narrow (resp. an ω-balanced) topological group and $f : G \to M$ to a metric space (M, ρ) be either left or right ω-uniformly continuous. Then there exist a continuous homomorphism $p : G \to K$ onto a second-countable (resp. a first-countable) topological group K and a continuous function $h : K \to M$ such that $f = h \circ p$.*

The following result is obvious.

Lemma 5.6. *Let G be a topological group and (M,ρ) be a metric space. Then every continuous function from G into M is left ω-uniformly continuous if and only if every continuous function from G into M is right ω-uniformly continuous.*

According to Lemma 5.6, the following definition is unambiguous.

Definition 5.7. A topological group G has *property strong ω-U* if each continuous function $f: G \to M$ to a metric space (M,ρ) is ω-uniformly continuous.

Every m-factorizable topological group is ω-narrow [5]. According to Lemmas 5.5 and 5.6, one can easily obtain the following result by making a simple modification of the proof of Theorem 4.9.

Theorem 5.8. *A topological group G is m-factorizable if and only if it is ω-narrow and has property strong ω-U.*

Lemma 5.9. *([1, Theorem 3.4.22]) A topological group G is ω-balanced if and only if it is topologically isomorphic to a subgroup of a topological product of metrizable topological groups.*

Lemma 5.10. *Every $\mathscr{M}$-factorizable topological group is ω-balanced.*

Proof. Let G be an $\mathscr{M}$-factorizable topological group. To prove that G is ω-balanced it is enough to show that G is topologically isomorphic to a subgroup of a topological product of metrizable topological groups by Lemma 5.9.

Since G is a Hausdorff topological group, G is completely regular. Let $\gamma = \{f_\alpha \mid \alpha \in \Lambda\}$ be the family of all continuous real-valued functions on G. Then γ can separate points from closed subsets of G. Since G is $\mathscr{M}$-factorizable, there exist a continuous homomorphism $p_\alpha : G \to K_\alpha$ onto a first-countable topological group K and a continuous function $g_\alpha : K_\alpha \to \mathbb{R}$ such that $f_\alpha = g_\alpha \circ p_\alpha$ for each $\alpha \in \Lambda$. Since every first-countable topological group is metrizable, each K_α is metrizable. Put $\delta = \{p_\alpha \mid \alpha \in \Lambda\}$. We show that δ can separate points from closed subsets of G. Take any point x and closed subset F of G such that $x \notin F$. Then there exists $f_\alpha \in \gamma$ such that $f_\alpha(x) \notin \overline{f_\alpha(F)}$. We shall prove that $p_\alpha(x) \notin \overline{p_\alpha(F)}$, which implies that δ can separate points from closed subsets of G. Indeed, assume to the contrary, then $p_\alpha(x) \in \overline{p_\alpha(F)}$, thus

$$f_\alpha(x) = g_\alpha\big(p_\alpha(x)\big) \in g_\alpha\big(\overline{p_\alpha(F)}\big) \subseteq \overline{g_\alpha\big(p_\alpha(F)\big)} = \overline{f_\alpha(F)}$$

according to $f_\alpha = g_\alpha \circ p_\alpha$ and the continuity of g_α. This is a contradiction. Therefore $\Delta_{\alpha\in\Lambda} p_\alpha : G \to \prod_{\alpha\in\Lambda} K_\alpha$ is a topologically isomorphic embedding, where $\Delta_{\alpha\in\Lambda} p_\alpha$ is a diagonal product of the family δ. □

Making a simple modification of the proof of Theorem 4.9, one can easily obtain the following result according to Lemmas 5.5, 5.6 and 5.10.

Theorem 5.11. *A topological group G is $\mathscr{M}$-factorizable if and only if it is ω-balanced and has property strong ω-U.*

The following result is obvious.

Lemma 5.12. *An ω-balanced topological group is preserved by an open continuous homomorphism.*

The following theorem gives a positive answer to Question 5.1.

Theorem 5.13. *An $\mathscr{M}$-factorizable topological group is preserved by a quotient homomorphism.*

Proof. Let G be an $\mathscr{M}$-factorizable group and $p: G \to K$ be a quotient homomorphism, where K is a topological group. It is well known that f is open [1, Theorem 1.5.1]. Therefore K is ω-balanced by Lemmas 5.10 and 5.12. Let (M,ρ) be a metric space. According to Lemma 5.6 and Theorem 5.11, it is enough to show that every continuous function $f: K \to M$ is left ω-uniformly continuous. Since G is $\mathscr{M}$-factorizable, $f \circ p$ is ω-uniformly continuous by Theorem 5.11. Take any $\varepsilon > 0$. According to Definition 5.2 there exists a countable family $\mu \subseteq \mathscr{N}_s(G,e)$ satisfying that for every point $x \in G$, there exists $U \in \mu$ such that $\rho(f(p(x)), f(p(y))) < \varepsilon$ whenever $x^{-1}y \in U$. Put $\gamma = \{p(U) \mid U \in \mu\}$. Then γ is a countable family of open symmetric neighborhoods of the identity in K. For every point $x \in K$ take a point $z \in G$ such that $x = p(z)$. Then there exists $U \in \mu$ such that $\rho(f(p(z)), f(p(y))) < \varepsilon$ whenever $z^{-1}y \in U$, that is, there exists $p(U) \in \gamma$ such that $\rho(f(x), f(x')) < \varepsilon$ whenever $z^{-1}x' \in p(U)$, which implies that f is left ω-uniformly continuous. This completes the proof. □

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