MATHEMATICS

ON A HEXAGONIC STRUCTURE. I

BY

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0. A hexagonic structure is a geometry consisting of points and lines, in which each pair of points can be joined by a path containing at most 3 lines, but where no 2-, 3-, 4- or 5-gons exist. A geometry of this type is produced by the fixed lines of the triality between the points and spaces of both kinds on a 6-dimensional quadric in a 7-dimensional projective space. This triality can be adequately described by means of an octave algebra. The fixed lines come out as totally singular subalgebras.

However, there are more trialities, the geometrically most interesting of them being related with a semi-linear automorphism of the octave algebra. To these also belong hexagonic structures. To adapt the algebraic apparatus, a new multiplication has to be introduced. Then a similar treatment is possible.

It is the object of this paper to characterize the first example by means of its geometric properties. To this end one may use some peculiarities of its group, which is an extension of the exceptional group G_2. The available tools do not suffice to treat the case of a field of characteristic 2.

The split octave algebra. Trialities

1. The split octave algebra C over a commutative field K is uniquely characterized by the following conditions:

C is a composition algebra over K with unit e;

the norm Q is a non-degenerate and isotropic quadratic form;

$\dim C = 8$.

A short treatment of the construction and the subsequent properties has been given by VAN DER BLIJ and SPRINGER [1] §§ 1, 2 and [2].

The multiplication in C satisfies:

$$Q(xy) = Q(x)Q(y) \quad (x, y \in C). \tag{1.1}$$

The associated bilinear form for Q

$$(x, y) = Q(x+y) - Q(x) - Q(y)$$

is non-degenerate. If the characteristic $\chi(K) \neq 2$, Q is non-defective. Q is of index 4. By

$$x + \bar{x} = (x, e)\, e \tag{1.2}$$

an involution $x \to \bar{x}$ in C is defined. The following formulae hold:

$$(1.3) \qquad \begin{aligned} &x(\bar{x}y) = Q(x)y, \\ &x(\bar{y}z) + y(\bar{x}z) = (x, y)z, \\ &\overline{xy} = \bar{y}\,\bar{x}, \\ &(xy, z) = (x, z\bar{y}) = (y, \bar{x}z). \end{aligned}$$

From $x^2 - (x, e)x + Q(x)e = 0$ it follows that

$$(1.4) \qquad a^2 = 0 \Leftrightarrow Q(a) = (a, e) = 0.$$

The totally singular linear subspaces of C with respect to Q, on which $Q \equiv 0$, of dimensionality 1, 2, 3, 4 are called *points, lines, planes, spaces* respectively. The last mentioned are maximal and of two kinds, having the form aC or Ca for some $a \in C$ with $a \neq 0$, $Q(a) = 0$. The dimensionality of the intersection of spaces belonging to the same kind is even, to different kinds it is odd. For isotropic a, b we have

$$(1.5) \qquad \begin{aligned} &Ka = Kb && \Leftrightarrow aC = bC, \\ &Ka \neq Kb,\ (a, b) = 0 && \Leftrightarrow aC \cap bC = a(\bar{b}C), \\ &(a, b) \neq 0 && \Leftrightarrow aC \cap bC = 0; \\ &ab = 0 && \Leftrightarrow \dim aC \cap Cb = 3, \\ &ab \neq 0 && \Leftrightarrow aC \cap Cb = Kab. \end{aligned}$$

A plane determines uniquely a pair of spaces aC, Cb, such that it is contained in both. If a is isotropic

$$(1.6) \qquad x \in aC \Leftrightarrow \bar{a}x = 0.$$

In C we may choose a basis $x_0, y_0, x_1, x_2, x_3, y_1, y_2, y_3$ with the following properties:

$$x_0 + y_0 = e,\ (x_0, y_0) = 1,\ Q(x_0) = Q(y_0) = 0,$$

so $x_0{}^2 = x_0$, $y_0{}^2 = y_0$, $x_0y_0 = y_0x_0 = 0$;

$$x_1, x_2, x_3 \in x_0C \cap Cy_0,\ y_1, y_2, y_3 \in y_0C \cap Cx_0,$$

so $y_0x_i = x_ix_0 = x_0y_i = y_iy_0 = 0$,
$x_0x_i = x_iy_0 = x_i$, $y_0y_i = y_ix_0 = y_i$, and $x_i{}^2 = y_i{}^2 = 0 \quad (i = 1, 2, 3)$;

$$(x_i, y_j) = \delta_{ij}\ (i, j = 1, 2, 3),$$

so $x_iy_j = -\delta_{ij}x_0$, $y_ix_j = -\delta_{ij}y_0$, and after suitable normalization $x_ix_{i+1} = -x_{i+1}x_i = y_{i+2}$, $y_iy_{i+1} = -y_{i+1}y_i = x_{i+2}$ ($i = 1, 2, 3$, and $i+1$, $i+2$ reduced mod 3).

For brevity we shall call a basis with these properties *normal.*

2. Let P be the orthogonal geometry of Q, i.e. the set of all totally singular subspaces of C ordered by inclusion. Let $Q = \{a \in C \mid Q(a) = 0, a \neq 0\}$

and $a_\#^0 = a_\# = Ka$, $a_\#^1 = C\bar{a}$, $a_\#^2 = \bar{a}C$ $(a \in Q)$. In this and the next two sections, i will designate an element of the additive group of order 3. Let $Q^i = \{a_\#^i \mid a \in Q\}$. The sets of spaces of both kinds Q^1, Q^2 and the set of points Q^0 of P are projectively isomorphic. Define an incidence-relation between elements of the Q^i by

$$a_\#^i \, I \, b_\#^{i+1} \Leftrightarrow ab = 0 \qquad (a, b \in Q),$$

then spaces of different kinds are incident iff their intersection is 3-dimensional. In the incidence-structure $T = \ulcorner Q^0, Q^1, Q^2; I \urcorner$ the three sets Q^i take symmetric places. T is a symmetric T-geometry (see TITS [1] § 3).

Let $\mathscr{G}_P$ and $\mathscr{G}_T$ be the groups of P and T respectively. The latter consists of the *T-collineations* (TITS [2] § 3.3), i.e. the permutations of $Q^0 \cup Q^1 \cup Q^2$ that preserve the relation I. A $\pi \in \mathscr{G}_T$ permutes the Q^i, so there is a homomorphism $\pi \to [\pi]$ of $\mathscr{G}_T$ in S_3. This map is onto, for the T-collineations

$$\varrho : a_\#^i \to a_\#^{i+1},$$

$$\psi : a_\#^0 \to \bar{a}_\#^0, \; a_\#^1 \to \bar{a}_\#^2, \; a_\#^2 \to \bar{a}_\#^1$$

satisfy $[\varrho] = (012)$, $[\psi] = (12)$. As $\varrho^3 = \psi^2 = \mathrm{id}$, $\varrho\psi = \psi\varrho^2$, they generate a subgroup $S \subset \mathscr{G}_T$ isomorphic to S_3.

For $\pi \in \mathscr{G}_P$, the restriction $\pi \mid Q^0 \cup Q^1 \cup Q^2$ is a T-collineation. Inversely, if $\pi \in \mathscr{G}_T$ and $[\pi] = (0)$ or (12), then π can be extended uniquely to a $\pi' \in \mathscr{G}_P$. So $\mathscr{G}_P \subset \mathscr{G}_T$. Now consider the kernel $\mathscr{G}_P{}^+ = \mathscr{G}_T{}^+$ of the homomorphism $\mathscr{G}_T \to S_3$. For two elements α, β of any group, write $\alpha^\beta = \beta^{-1}\alpha\beta$. The inner automorphism $\pi \to \pi^\varrho$ of $\mathscr{G}_T$ induces an outer automorphism of $\mathscr{G}_P{}^+$ of order 3 (see § 3).

Let $t \to t_\#$ be the natural homomorphism of the group of semi-similarities $\Gamma O_8(K, Q)$ in $\mathscr{G}_P$ (DIEUDONNÉ [1] I § 10). By CHOW's theorem (see e.g. DIEUDONNÉ [1] III § 3) this map is onto. The kernel, isomorphic to K^* (multiplicative group of K), consists of the homothetic transformations, so $t_\# = K^*t$. Write $\Gamma O_8{}^+ (K, Q)$ for the inverse image of $\mathscr{G}_P{}^+$. Take $t_0 \in \Gamma O_8{}^+ (K, Q)$ and t_1, t_2 such that $t_{0\#}{}^\varrho = t_{1\#}$, $t_{0\#}{}^{\varrho^2} = t_{2\#}$. If $\pi \mid Q^0$ is induced by t_0, then $\pi \mid Q^i$ by t_i (in the coordinates of Q^i).

Now $ab_\#^i \, I \, \bar{a}_\#^{i-1}, \; \bar{b}_\#^{i+1}$ implies $(t_i ab)_\#^i \, I \, (t_{i-1}\bar{a})_\#^{i-1}, \; (t_{i+1}\bar{b})_\#^{i+1}$. It follows easily that

$$(2.1) \qquad t_i \, ab = \lambda_i \, \hat{t}_{i-1}a \cdot \hat{t}_{i+1}b, \; \lambda_i \in K,$$

where $\hat{t}a = \overline{t\bar{a}}$, $\hat{t}_\# = (t_\#)^\psi$.

The semi-similarities t_i belong to the same automorphism of K, and their multipliers r_i satisfy $r_i = \lambda_i^2 r_{i-1} r_{i+1}$, $r_i \lambda_{i-1} \lambda_{i+1} = 1$. Each of the three $t_{0\#}, t_{1\#}, t_{2\#}$ determines the others uniquely by the condition (2.1) for one value of i. The three formulae (2.1) for different i are equivalent to one another and to

$$(2.2) \qquad \hat{t}_i \, ab = \lambda_i \, t_{i+1}a \cdot t_{i-1}b, \; \lambda_i \in K.$$

In the case of an octave algebra with anisotropic Q, the formulae (2.1) hold for the group of similarities $GO_8^+(K, Q)$ (see VAN DER BLIJ and SPRINGER [2] § 1).

3. Let L be the set of lines of P. A line $l \in L$ may be understood as union of the points it contains or as intersection of the spaces of the same kind it is contained in. Write $l' = \{a \in Q \mid C\bar{a} \supset l\}$, $l'' = \{a \in Q \mid \bar{a}C \supset l\}$, then l' and l'' are lines, and we have by (1.6) $l \cdot l' = l'' \cdot l = 0$.

(3.1) Lemma. *Let $l, m, n \in L$. Then*

$$l \cdot m = m \cdot n = 0 \Rightarrow n \cdot l = 0.$$

Proof: Take independent $a_1, a_2 \in m$ and $b \in n$. By (1.5) $l = C\bar{a}_1 \cap C\bar{a}_2 = (Ca_1)\bar{a}_2$. If $x \in C$, then $b((xa_1)\bar{a}_2) = (\bar{b}, xa_1)\bar{a}_2 - (\bar{a}_1\bar{x})(\bar{b}\bar{a}_2) = 0$ ∎ It follows that $(l')' = l''$, $((l')')' = l$.

In this section, if $l \in L$, write $l^i = \{a_{\#}{}^i \mid a \in l\}$, and $L^i = \{l^i \mid l \in L\}$. Let $L' = \{\ulcorner l^i, (l')^{i+1}, (l'')^{i+2} \urcorner \mid l \in L\}$ be the set of the triples of incident lines of L^0, L^1, L^2. Now a T-collineation π, preserving incidence, maps L' on L'. For brevity, write $\ulcorner l^i \urcorner$ for the triple of L' containing $l^i \in L^i$. If $\ulcorner l^0, m^1, n^2 \urcorner \in L'$, denote the element of L^i in this triple by $\ulcorner l^0, m^1, n^2 \urcorner^i$. If $a_{\#} = l_1 \cap l_2$, $(a \in Q,\ l_1, l_2 \in L)$, $\pi a_{\#}{}^0$ is the intersection of, or the space spanned by $(\pi \ulcorner l_1{}^0 \urcorner)^0$ and $(\pi \ulcorner l_2{}^0 \urcorner)^0$. So the representation of $\mathscr{G}_T$ in L' is faithful. The representation of $\mathscr{G}_T$ in $Q^0 \cup Q^1 \cup Q^2 \cup L'$ is called the triality-group of Q (KUIPER [1] § 5).

L can be mapped on L' by $l \to \ulcorner l^i \urcorner$ in three ways, corresponding to the interpretation of Q as coordinatization of the three Q^i. So $\pi \in \mathscr{G}_T$ induces three transformations π_i of L, corresponding to $\ulcorner l^i \urcorner \to (\pi \ulcorner l^i \urcorner)^i$ respectively. As $\varrho \ulcorner l^i \urcorner = \ulcorner l^{i+1} \urcorner$, we have $\pi_{i+1} = (\pi^\varrho)_i$. The induced transformation is unique iff $\pi = \pi^\varrho$.

Every T-collineation is of the form $\varphi\pi$, $\varphi \in S$, $\pi \in \mathscr{G}_T{}^+$. Suppose $(\varphi\pi)^\varrho = \varphi\pi$, then $[\varphi^\varrho] = [\varphi]$, so $\varphi = \varrho^i$. As $\varrho^\varrho = \varrho$, it follows that $\pi^\varrho = \pi$. Let $\pi = t_{\#}$. Now (2.2) takes the form $\hat{t}\, ab = \lambda\, ta \cdot tb$. In consequence te must belong to the centre Ke of C, so $te = \varkappa e$, $\varkappa \in K$, and $\varkappa^2 = r$. A suitable multiple of t has $\varkappa = r = \lambda = 1$. Then $\hat{t}a = ta$. So the semi-similarity t can be chosen to be an automorphism of C as a ring (short: a *semi-automorphism* of C). All automorphisms of C as a ring are semi-linear, since the centre Ke must be invariant as a whole. They are semi-similarities, because $x^2 - (x, e)\, x + Q(x)\, e = 0$. Write $\mathscr{G}_C$ for the group of semi-automorphisms. We have $\mathscr{G}_C \subset \mathscr{G}_T{}^+$.

We now prove that the automorphism $\pi \to \pi^\varrho$ of $\mathscr{G}_T{}^+$ is outer. Suppose we have $\sigma \in \mathscr{G}_T{}^+$ such that $\pi^\varrho = \pi^\sigma$ for all $\pi \in \mathscr{G}_T{}^+$. Now $\sigma^\varrho = \sigma$, so $\sigma \in \mathscr{G}_C$. Consider π, induced by t_0: $x \to \bar{a}x\bar{a}$ with $Q(a) \neq 0$. It follows from MOUFANG's identity $a(xy)a = (ax)(ya)$ that we may take in (2.2) $t_1 : x \to ax$ and $t_2 : x \to xa$. We have $t_1{}^\sigma = \varkappa t_2$ $(\varkappa \in K)$, so $\sigma^{-1}a \cdot x = \varkappa xa$. This leads to a contradiction.

We have the following representation of S in L:

$$\varrho l = l'', \ \varrho^2 l = l',$$

$$\psi_0 l = \bar{l}, \ \psi_1 l = \overline{l'} = \bar{l}'', \ \psi_2 l = \bar{l}' = \overline{l''}.$$

In the case of a field K of characteristic $\chi(K) \neq 2$ the representation in L can be obtained from the Lie-algebra $L(Q)$ of Q (see VAN DER BLIJ and SPRINGER [2] § 3).

4. A *triality* is a T-collineation of order 3 permuting points and spaces of both kinds cyclically. Thus, if $\tau \in \mathscr{G}_T{}^+$ is induced by $t_0 \in \Gamma O_8{}^+(K, Q)$, $\varrho\tau$ will be a triality iff $(\varrho\tau)^3 = \tau^{\varrho^2}\tau^{\varrho}\tau = \mathrm{id}$, so iff $t_{i+2}t_{i+1}t_i = \varkappa \,\mathrm{id}$, $\varkappa \in K$. All trialities can be classified (see TITS [2] § 5.2 and SPRINGER [1]). The automorphism of K belonging to t_i has order 3, and for each such automorphism there is essentially only one triality, except in the case of the identical automorphism of K. Every automorphism of K can be extended to a semi-automorphism of C. We shall consider only trialities with $\tau \in \mathscr{G}_C$.

Let τ be a semi-automorphism of order 3 of C. The automorphism of K belonging to τ will be denoted by τ as well. We introduce a new product in C, which will prove to be adapted to the description of the geometry of $\varrho\tau$ (see SPRINGER [1]):

$$a \star b = \tau^2\bar{a} \cdot \tau\bar{b} \quad (a, b \in C). \tag{4.1}$$

The following formulae are easily derived from those of § 1:

$$Q(a \star b) = \tau^2 Q(a)\ \tau Q(b); \tag{4.2}$$

$$(\alpha a) \star (\beta b) = \tau^2\alpha\ \tau\beta\ (a \star b) \quad (\alpha, \beta \in K); \tag{4.3}$$

$$\begin{aligned} &(x \star x) \star x = \tau Q(x)\ x, \\ &x \star (x \star x) = \tau^2 Q(x)\ x; \\ &(x \star y) \star z + (z \star y) \star x = \tau(x, z)\ y, \\ &x \star (y \star z) + z \star (y \star x) = \tau^2(x, z)\ y; \\ &(x \star y, z) = \tau(z \star x, y) = \tau^2(x, y \star z). \end{aligned} \tag{4.4}$$

Every space is of the form $a \star C$ or $C \star a$ with $a \neq 0$, $Q(a) = 0$. Suppose $a, b \neq 0$, $Q(a) = Q(b) = 0$, then we have

$$\begin{aligned} &Ka = Kb && \Leftrightarrow a \star C = b \star C, \\ &Ka \neq Kb,\ (a, b) = 0 && \Leftrightarrow \dim a \star C \cap b \star C = 2, \\ &(a, b) \neq 0 && \Leftrightarrow a \star C \cap b \star C = 0; \\ &a \star b = 0 && \Leftrightarrow \dim a \star C \cap C \star b = 3, \\ &a \star b \neq 0 && \Leftrightarrow a \star C \cap C \star b = Ka \star b; \end{aligned} \tag{4.5}$$

$$\begin{aligned} &x \in a \star C \Leftrightarrow x \star a = 0, \\ &x \in C \star b \Leftrightarrow b \star x = 0. \end{aligned} \tag{4.6}$$

Suppose $t_0, t_1, t_2 \in \Gamma O_8^+(K, Q)$ such that

$$t_{1\#} = t_{0\#}^{\varrho\tau}, \quad t_{2\#} = t_{0\#}^{(\varrho\tau)^2},$$

then from (2.1) follows

$$t_i(a \star b) = \lambda_i(t_{i-1}a \star t_{i+1}b). \tag{4.7}$$

A semi-linear transformation t of C will be called a $\star$-automorphism if $t(a \star b) = ta \star tb$. We denote the group of $\star$-automorphisms of C by $\mathscr{G}_C{}^*$.

5. A line l will be fixed for the triality $\varrho\tau$ if $l = \varrho\tau l = (\tau l)''$, so if $l \cdot \tau l = 0$, $l \star l = 0$. This means $a \star b = 0$ for all $a, b \in l$. A point $a_\#$ is called *autoconjunct* (short: a.c.) if $a \star a = 0$. This implies $a_\# \subset C \star a = \varrho\tau a_\#$ and $a_\# \subset a \star C = (\varrho\tau)^2 a_\#$. By (4.2) $a \star a = 0$ implies $Q(a) = 0$. A fixed line for $\varrho\tau$ will be called an autoconjunct line (instead of "droite fixe" TITS [2] § 4.1). All its points are a.c. To an a.c. point $a_\#$ corresponds a plane $a \star C \cap C \star a$, containing $a_\#$. This will be called the *centric* plane with *centre* $a_\#$ and its points will be called *centripetal* points (instead of "plan spécial" and "points spéciaux" TITS l.c.). Let $b_\#$ be centripetal, $a_\#$ a.c. and $b \in a \star C \cap C \star a$. This implies $a \in b \star C \cap C \star b$ (by (4.6)), so if $b \star b \neq 0$ we have $a \in Kb \star b$. It follows that $(b \star b) \star (b \star b) = 0$. This relation characterizes all centripetal points. We shall say that two a.c. points $a_\#$ and $b_\#$ are *conjunct* if they lie on an a.c. line or coincide.

For a given semi-automorphism τ of order 3 write $\Gamma = \{a \in C \mid a \star a = 0\}$. So $a \in \Gamma$ means $a_\#$ a.c. for $\varrho\tau$. Define a relation

$$a \frown b \Leftrightarrow a \star b = 0 \quad (a, b \in \Gamma).$$

It follows from (4.6) that $a \frown b$ implies $a, b \in b \star C \cap C \star a$, so a, b are dependent or $b \star a = 0$. The relation $\frown$ is symmetric. Moreover, $a \frown b$ implies $(a, b) = 0$, and hence a and b span a line. This line is a.c., so $a \frown b$ means that $a_\#$ and $b_\#$ are conjunct.

Suppose $a_1, a_2, b \in \Gamma$ and $a_1 \frown b \frown a_2$. We have

$$b \in a_1 \star C \cap C \star a_1 \cap a_2 \star C \cap C \star a_2 \cap \Gamma = I \text{ (say)}.$$

The following possibilities arise:

$(a_1, a_2) \neq 0$: then $b = 0$;

$(a_1, a_2) = 0$: now $a_1 \star a_2 \in \Gamma$, for

$$(a_1 \star a_2) \star (a_1 \star a_2) = \tau(a_1, a_1 \star a_2)\, a_2 - ((a_1 \star a_2) \star a_2) \star a_1 =$$
$$\tau^2(a_1 \star a_1, a_2)\, a_2 - \tau(a_1, a_2)\, a_2 \star a_1 + ((a_2 \star a_2) \star a_1) \star a_1 = 0.$$

We have $(a_1 \star a_2) \star a_1 = (a_1 \star a_2) \star a_2 = 0$, $a_1 \frown a_1 \star a_2 \frown a_2$. Distinguish two cases:

$a_1 \star a_2 \neq 0$: then $I = Ka_1 \star a_2 = Ka_2 \star a_1$;

$a_1 \star a_2 = 0$: it follows that $a_1, a_2 \in I$, so for independent a_1, a_2 I coincides with the a.c. line spanned by a_1, a_2.

We have proved:

(5.1) *A triangle of a.c. lines is degenerate.*

(5.2) *If two a.c. points are not conjunct, at most one a.c. point is conjunct with both, so a quadrangle of a.c. lines is degenerate.*

Suppose $a_1, \ldots, a_5 \in \Gamma$, $a_1 \frown a_2 \frown \ldots \frown a_5 \frown a_1$, but not

$$a_1 \frown a_3,\ a_2 \frown a_4,\ \ldots,\ a_5 \frown a_2.$$

We have

$$a_2 \in K a_1 \star a_3,\ \ a_5 \in K a_1 \star a_4, \quad \text{so} \quad a_3 \in K a_4 \star (a_1 \star a_3),\ \ a_4 \in K a_3 \star (a_1 \star a_4).$$

But $a_4 \star (a_1 \star a_3) + a_3 \star (a_1 \star a_4) = \tau^2(a_3, a_4)\, a_1 = 0$. This implies:

(5.3) *A pentagon of a.c. lines is degenerate.*

Suppose Γ contains an independent pair a, b with $a \frown b$. Take $c \in \Gamma$. The line spanned by a, b contains an $a_1 \neq 0$ with $(a_1, c) = 0$. So there is an $a_2 \neq 0$ such that $a \frown a_1 \frown a_2 \frown c$. It follows that we can find an element $d \in \Gamma$ with c, d independent and $c \frown d$. For every pair $a, c \in \Gamma$ there exist a_1, a_2, both $\neq 0$, for which $a \frown a_1 \frown a_2 \frown c$.

For the case $\tau = \mathrm{id}$, the preceding proofs were communicated to me by Professors F. van der Blij and T. A. Springer.

Hexagonic structures. Axial automorphisms

6. Let $\ulcorner V, E; I \urcorner$ be an incidence-structure, consistsing of a set V of *vertices*, a set E of *edges* and a relation I, symmetric between V and E. $p \in V$ and $q \in E$ are said to be incident if pIq. The elements $p, q \in V \cup E$ are called similar, $p \sim q$, when both are vertices or both are edges. A *chain* of length n is a sequence $p_0, p_1, \ldots, p_n \in V \cup E$ such that $p_{i-1} \nsim p_i$, $p_{i-1} I p_i$ $(i = 1, \ldots, n)$. p_0 and p_n are said to be joined by the chain. The chain is irreducible when all its elements are different.

(6.1) Definition. An *n-gonic structure* is an incidence-structure $\ulcorner V, E; I \urcorner$ meeting the following requirements (see Tits [2] § 11.1):
every pair $p, q \in V \cup E$ is joined by a chain of length $\leqslant n$;
a pair $p, q \in V \cup E$ is joined by at most one irreducible chain of length $< n$.

As before, let τ be a semi-automorphism of the split octaves of order 3. Take for V the set of a.c. points, for E the set of a.c. lines with respect to the triality $\varrho\tau$. Suppose $E \neq \emptyset$ (void set), then $\ulcorner V, E; \in \urcorner$ is a hexagonic structure; to be proved from the results of § 5 by straightforward verification. We shall denote this hexagonic structure by H_τ. The simplest example H_{id} corresponds to $\tau = \mathrm{id}$. In this case we have

$$\Gamma = \{a \in Q \mid a^2 = 0\} = \{a \in Q \mid (a, e) = 0\}, \ \text{and} \ a \frown b \Leftrightarrow ab = 0.$$

So $a_{\#}$ is a.c. iff $(a, e)=0$, the line l is a.c. iff l is a totally singular subalgebra of C. There is no distinction between a.c. and centripetal points, because $Q(b)=0 \Rightarrow b^2=(b, e)b$, so $b^2 \cdot b^2=(b, e)^2 b^2=0 \Rightarrow (b, e)=0$ or $b^2=0$.

Let $\mathscr{G}_\tau$ be the group of H_τ, consisting of all permutations of the sets of a.c. points and a.c. lines preserving incidence. It is clear that a $\star$-automorphism of C induces such a permutation. The following theorem I owe to Professor T. A. Springer:

(6.2) *The homomorphism* $\mathscr{G}_C \to \mathscr{G}_{\mathrm{id}}$ *is onto.*

Proof: A $\pi \in \mathscr{G}_{\mathrm{id}}$ maps a conjunct pair of a.c. points on a conjunct pair of a.c. points, and an orthogonal pair on an orthogonal pair. So π permutes the totally singular subspaces of Γ. By CHOW's theorem, π is induced by a semi-similarity t of $C_0=\{a \in C \mid (a, e)=0\}$ with respect to Q. However, if $\chi(K)=2$, this argument does not work, since the restriction of Q to C_0 has defect 1. For the present we suppose $\chi(K)\neq 2$. Let r be the multiplier of t, then $(\det t)^2=r^7$, so $r \in K^{*2}$. Now t can be chosen to have $r=1$. We extend t to C by $te=e$. If $t \notin \Gamma O_8^+(K, Q)$, replace it by $x \to \overline{tx}$. Then π is induced by $t \in \Gamma O_8^+(K, Q)$. According to (2.2), we can find t', t'' such that $t'xy=tx\cdot t''y$ $(x, y \in C)$. From $te=e$ we derive

$$t'x=t''x=tx\cdot t''e.$$

If $a, b \in \Gamma$, we have $ab=0 \Leftrightarrow t^{-1}a\cdot t^{-1}b=0 \Leftrightarrow t'(t^{-1}a\cdot t^{-1}b)=a(b\cdot t''e)=0$. Take $b \in \Gamma$, $b\neq 0$ and $a_1, a_2 \in C\bar{b} \cap \Gamma=Cb \cap bC$ such that a_1, a_2, b are independent. Then $a_1 b=a_2 b=0$, $a_1 a_2\neq 0$ (see § 5). So $b\cdot t''e \in a_1 C \cap a_2 C=Kb$. It follows that $t''e=\varkappa e$, $\varkappa \in K$, and $t'=t''=\varkappa t$. So t is a semi-automorphism.

We assume theorem (6.2) to hold if $\chi(K)=2$ as well. The only result dependent on it will be (8.7).

As the a.c. points of H_{id} span C_0 (see above), $t \in \mathscr{G}_C$ induces the identity iff the restriction $t \mid C_0$ is a homothetic transformation. It is easily verified that this implies $t=\mathrm{id}$. So $\mathscr{G}_C=\mathscr{G}_{\mathrm{id}}$. The subgroup of (linear) automorphisms is an algebraic group of type G_2 (see TITS [2], § 8).

A theorem similar to (6.2) seems to hold for $\mathscr{G}_{C^*} \to \mathscr{G}_\tau$.

7. In the sequel we shall characterize the hexagonic structure H_{id} by means of some geometrical properties, which we proceed to derive.

Let $H=\ulcorner V, E; I\urcorner$ be a hexagonic structure. If $p, q \in V \cup E$ we define the *distance* $d(p, q)$ to be the length of the shortest chain joining p and q. We have

$$d(p, q)=d(q, p),$$
$$d(p, q) \leqslant 6 \text{ and } d(p, q)=0 \Leftrightarrow p=q,$$
$$d(p, r) \leqslant d(p, q)+d(q, r).$$

p and q are incident iff $d(p, q)=1$, *joined* iff $d(p, q)=0$ or 2. In the cases $d(p, q)=2$ and 4 we shall denote the middle of the shortest chain joining p and q by $p \mapsto q$ and $p \wedge q$ respectively.

It follows from the definition of H that two vertices are joined by at most one edge, and that every 2-, 3-, 4-, 5-gon is degenerate. A sharper consequence is the following:

(7.1) Lemma. *Let $p_1, p_2, p_3 \in V \cup E$. Then*

(a) $d(p_1, p_2) = d(p_1, p_3) = 2,\ d(p_2, p_3) = 2 \Rightarrow p_1 \vdash\!\dashv p_2 = p_1 \vdash\!\dashv p_3,$

(b) $4, \qquad 2 \Rightarrow p_1 \wedge p_2 = p_1 \wedge p_3,$

(c) $3, \qquad 2 \Rightarrow d(p_1, p_2 \vdash\!\dashv p_3) = 2,$

(d) $3, \qquad 4 \Rightarrow d(p_1, p_2 \wedge p_3) = 1.$

Proof: (a) The chain $p_2, p_1 \vdash\!\dashv p_2, \ p_1, p_1 \vdash\!\dashv p_3, p_3$ has length < 6 and $> d(p_2, p_3)$, hence cannot be irreducible ∎

(b) Consider the chains $p_1, p_1 \vdash\!\dashv (p_1 \wedge p_2), p_1 \wedge p_2, (p_1 \wedge p_2) \vdash\!\dashv p_2, p_2, p_2 \vdash\!\dashv p_3$ and $p_1, p_1 \vdash\!\dashv (p_1 \wedge p_3), p_1 \wedge p_3, (p_1 \wedge p_3) \vdash\!\dashv p_3, p_3, p_2 \vdash\!\dashv p_3$.
These are different, hence must be reducible.
So $(p_1 \wedge p_2) \vdash\!\dashv p_2 = p_2 \vdash\!\dashv p_3 = (p_1 \wedge p_3) \vdash\!\dashv p_3$.
The chains $p_1, \ldots, (p_1 \wedge p_2) \vdash\!\dashv p_2$ and $p_1, \ldots, (p_1 \wedge p_3) \vdash\!\dashv p_3$ are irreducible.
So $p_1 \wedge p_2 = p_1 \wedge p_3$ ∎

(c) Let $p_1, q_2, q_2 \vdash\!\dashv p_2, p_2$ and $p_1, q_3, q_3 \vdash\!\dashv p_3, p_3$ be the irreducible chains joining p_1 with p_2 and p_3.
Lengthen both by $p_2 \vdash\!\dashv p_3$. The resulting chains are reducible, so $q_2 \vdash\!\dashv p_2 = p_2 \vdash\!\dashv p_3$ ∎

(d) Take q_2, q_3 as in (c) and consider the chains
$p_1, q_2, q_2 \vdash\!\dashv p_2, p_2, p_2 \vdash\!\dashv (p_2 \wedge p_3), p_2 \wedge p_3$ and
$p_1, q_3, q_3 \vdash\!\dashv p_3, p_3, p_3 \vdash\!\dashv (p_2 \wedge p_3), p_2 \wedge p_3$. These are reducible, so
$q_2 \vdash\!\dashv p_2 = p_2 \vdash\!\dashv (p_2 \wedge p_3),\ q_3 \vdash\!\dashv p_3 = p_3 \vdash\!\dashv (p_2 \wedge p_3)$. Suppose that
$p_1, q_2, q_2 \vdash\!\dashv p_2, p_2 \wedge p_3$ and $p_1, q_3, q_3 \vdash\!\dashv p_3, p_2 \wedge p_3$ are irreducible, then $q_2 = q_3$, and $q_2 = q_3 = p_2 \wedge p_3$. So the chains must be reducible, which implies again $q_2 = q_3 = p_2 \wedge p_3$ ∎

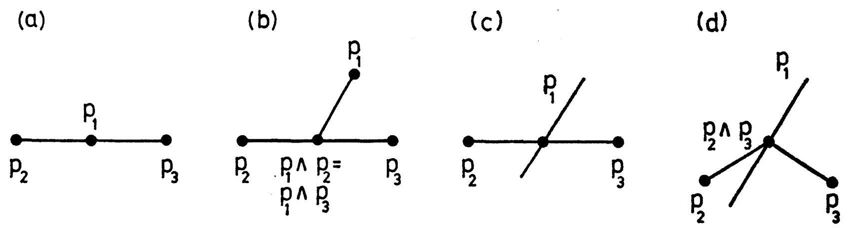

Fig. 1

For obvious reasons $p, q \in V \cup E$ are said to be *orthogonal*, $p \perp q$, if $d(p, q) \leqslant 4$. By the *orthoplement* of a subset $S \subset V \cup E$ is meant the set $S^{\perp} = \{p \in V \cup E \mid p \perp s \text{ for all } s \in S\}$. The *pencil* of p is the set $P(p) = \{q \in V \cup E \mid d(p, q) = 1\}$. We define
$P^n(p) = \{q \in V \cup E \mid d(p, q) = n, n-2, \ldots\} = \bigcup P(q)$ where $q \in P^{n-1}(p)$.

If $d(p, q) = 6$, to each $p_1 \in P(p)$ there is exactly one $q_1 \in P(q)$ with

$d(p_1, q_1)=4$. This correspondence $P(p) \leftrightarrow P(q)$ is called a *perspectivity* (Tits [2] § 11.2). It follows that $P(p)$ and $P(q)$ contain the same number of elements.

In H a closed irreducible chain of length 12 is called a hexagon. It is easily verified that if $p_0, p_1, \ldots, p_{12}=p_0$ is a hexagon we have $d(p_i, p_j)= =\min([i-j], [j-i])$, where $[i]$ is defined by $[i] \equiv i \bmod 12$, $0 \leqslant [i] < 12$.

(7.2) *Let $p_0, p_1, \ldots, p_{11}$ be a hexagon contained in H. For each $q \in V \cup E$, we have $d(q, p_i)=6$ for at least one i.*

Proof: Take i such that $d(q, p_i)$ is maximal. Suppose $d(q, p_i)=d<6$. Then p_{i-1} and p_{i+1} are not both contained in the shortest chain joining q and p_i, so $d(q, p_{i-1})=d+1$ or $d(q, p_{i+1})=d+1$ ▌

From (7.2) we infer that, if H contains a hexagon, each pencil contains at least 2 elements. A hexagon is an example of a hexagonic structure. When we want to avoid this and other freak cases, such as degenerate or improper hexagonic structures (see Tits [2] § 11.2), we shall suppose that each pencil contains at least 3 elements.

Let $\mathscr{G}$ be the group of *automorphisms* of H (pairs of permutations of V and E preserving I and consequently d). The pencil $P(q)$, $q \in V \cup E$ is said to be an *axis* of $\pi \in \mathscr{G}$ if $P(q)^{\perp}$ is kept fixed by π elementwise.

(7.3) *Suppose H contains a hexagon. Then*

$$P(q)^{\perp}=\{p \in V \cup E \mid d(p, q) \leqslant 3\}.$$

Proof: Take $p \in P(q)^{\perp}$ and $q_1, q_2 \in P(q)$, $d(q_1, q_2)=2$. Distinguish the following cases:
$d(p, q_1) \leqslant 2$ or $d(p, q_2) \leqslant 2$. Then $d(p, q) \leqslant 3$.
$d(p, q_1)=d(p, q_2)=3$. Apply (7.1(c)).
$d(p, q_1)=d(p, q_2)=4$. From (7.1(a), (b)) we infer
$p \wedge q_1=p \wedge q_2 \in P(q)$, $d(p, q)=2$ ▌

(7.4) *Suppose each pencil contains at least 3 elements. Then an axial automorphism is determined completely by an axis and the image of one element outside the orthoplement of this axis.*

We shall need two steps:
Suppose H contains a hexagon. Let $P(a)$ be an axis of π, $p, q \notin P(a)^{\perp}$ and $d(p, q)=1$. Then πp determines πq uniquely.

Proof: Write $p=p_{d(a,p)}$, $q=p_{d(a,q)}$. If p_4 is lacking, take the element next to p_5 in the shortest chain joining p_5 and a. Let c be the element after p_4 in this chain. If p_6 is lacking, take $p_6 \in P(p_5)$, $p_6 \neq p_4$. Then $d(a, p_6)=6$. Select a $b \in P(p_6)$, $b \neq p_5$ and take c' such that $d(b, c')=2$, $d(a, c')=3$. We have $p_4=c \mapsto p_5$, $p_5=P(p_4) \cap \{c'\}^{\perp}=P(p_6) \cap \{p \mid d(c, p)=2\}$, $p_6=p_5 \mapsto (c' \wedge p_5)$. These relations are preserved by π, since $\pi c=c$, $\pi c'=c'$ ▌

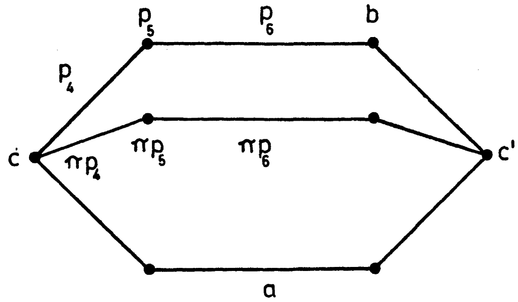

Fig. 2

Now the proof of (7.4) is easily completed by means of:

(7.5) *Suppose each pencil contains at least* 3 *elements. Then the complement of* $P(q)^\perp$, $q \in V \cup E$, *is connected, i.e. a pair* $p, p' \notin P(q)^\perp$ *can be joined by a chain of elements outside* $P(q)^\perp$.

Proof: If $d(p, q) < 6$, we can find $r \in P(p)$ such that $d(r,q) = d(p,q)+1$. So we may confine ourselves to the case $d(p, q) = d(p', q) = 6$. Let $p = p_0, p_1, \dots, p_d = p'$ be a chain of length $d = d(p, p')$. Now $d(p_i, q) \geqslant \geqslant \max(6-i, 6-d+i) \geqslant 6-\frac{1}{2}d$. So only the case $d=6$, $d(p_3, q)=3$ needs further consideration. Select $a \in P(p')$, $a \neq p_5$ and $p'' \in P(a)$, $p'' \neq p'$ such that $d(p'', q) = 6$. This is possible because $P(a)$ contains at least 3 elements. $d(p_1, a) = 6$, so $d(p_1, p'') = 5$. Let $p_1, p_2', \dots, p_6' = p''$ be the shortest chain joining p_1 and p''. We have $p_2 \neq p_2'$, since $p_2 = p_2' \Rightarrow p_4 = p_4' \in P(a)$, $d(p_1, a) \leqslant 4$ by (7.1). Suppose $d(p_3', q) = 3$, then $d(p_2, q) = d(p_2', q) = 4$, so $d(p_1, q) = 3$, contradictory to $d(p_1, q) = 5$ ▌

Remark: We need the property of $P(a)$ to contain 3 elements only for a with $d(a, q)$ odd.

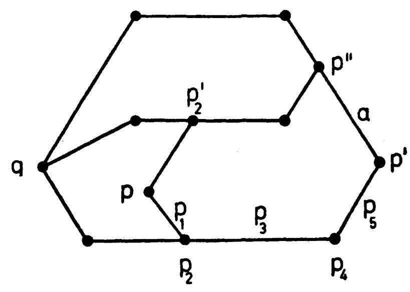

Fig. 3

Let π be an axial automorphism, $\pi \neq \mathrm{id}$, and $P(a)$ an axis of π. $P(a)^\perp$ is the set of fixed elements for π. It is easily verified that a is determined by $P(a)^\perp$. So, except for the identity, an automorphism has at most one axis.

For brevity we shall sometimes say: π has axis a, when we mean: π has axis $P(a)$.

8. Now we shall investigate the axial automorphisms of H_τ. Let $\pi \in \mathscr{G}_\tau$ have the a.c. line l for axis and consider an a.c. point $x_\#$, $x \in \Gamma$, outside the orthoplement of the axis. This means $(x, a) \neq 0$ for an $a \in l$. (The concepts of orthogonality in the linear space and in the incidence-structure are not to be confused: if $x_\#$, $y_\#$ denote a.c. points and l, m a.c. lines, we have $(x, y) = 0 \Leftrightarrow x_\# \perp y_\#$, $(x, l) = 0 \Leftrightarrow x_\# \perp l$, and $(l, m) = 0 \Rightarrow l \perp m$, but not inversely, since $(l, m) = 0$ iff $d(l, m) \leqslant 2$). Take $y \in l$ such that $(x, y) = 0$, and suppose there is an a.c. line m through x, not containing $x \star y$. Select $x' \in m$ independent of x, and $y' \in l$ such that $(x', y') = 0$. Now $\pi x_\#$ is conjunct to $x \star y_\#$ and orthogonal to $x' \star y'$, hence contained in the 2-dimensional intersection of the centric plane $(x \star y) \star C \cap C \star (x \star y)$ and the orthoplement of $x' \star y'$. It follows that $\pi x_\#$ is dependent on $x_\#$ and $y_\#$.

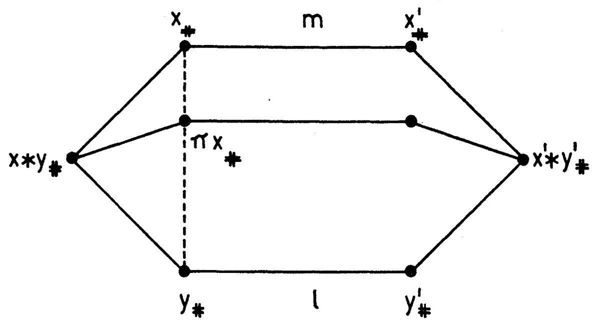

Fig. 4

In the case of H_{id}, according to (6.2), π is induced by a semi-linear transformation t of C. As the axis remains fixed pointwise, t is linear. So tx is a linear combination of x and y with for coefficients constants. If a_1, a_2 is a basis of l, we may choose $y = (x, a_1)a_2 - (x, a_2)a_1$. This induces us to investigate the transformations t of the form

(8.1) $$tx = x + \lambda\{(x, a_1)a_2 - (x, a_2)a_1\}.$$

Consider H_τ again. Let $a_0, a_1, a_2, a_3 \in \Gamma$, $a_0 \frown a_1 \frown a_2 \frown a_3$, but not $a_0 \frown a_2$ and $a_1 \frown a_3$. Define $\alpha_{02}, \alpha_{20}, \alpha_{13}, \alpha_{31}$ by

$$a_1 = \alpha_{02}(a_0 \star a_2) = \alpha_{20}(a_2 \star a_0),$$

$$a_2 = \alpha_{13}(a_1 \star a_3) = \alpha_{31}(a_3 \star a_1).$$

Then we have
$a_1 = \alpha_{02}(a_0 \star (\alpha_{13}(a_1 \star a_3))) = \alpha_{02}\, \tau\alpha_{13}\, (a_0 \star (a_1 \star a_3)) = \alpha_{02}\, \tau\alpha_{13}\, \tau^2(a_0, a_3)\, a_1$, and similarily $a_1 = \alpha_{20}\, \tau^2\alpha_{31}\, \tau(a_0, a_3)\, a_1$. So

$$\alpha_{02}\, \tau\alpha_{13}\, \tau^2(a_0, a_3) = \alpha_{20}\, \tau^2\alpha_{31}\, \tau(a_0, a_3) = 1.$$

It follows that

$$(\tau^2\alpha_{20})^{-1}\, \tau\alpha_{02} = (\tau^2\alpha_{13})^{-1}\, \tau\alpha_{31} = \gamma(a_1, a_2)$$

is independent of a_0 or a_3 and invariant under inversion of the order. So $\gamma(a_1, a_2)=\gamma(a_2, a_1)$. Let A be a linear transformation of the line l spanned by a_1, a_2. Write $Aa_1=A_{11}a_1+A_{21}a_2$, then $Aa_1\star a_3=\tau^2A_{11}\ (a_1\star a_3)=(\alpha_{13})^{-1}\ \tau^2A_{11}\ a_2,\ a_3\star Aa_1=(\alpha_{31})^{-1}\ \tau A_{11}\ a_2$. So $\gamma(Aa_1, a_2)=(\tau^2A_{11})^{-1}\ \tau A_{11}\ \gamma(a_1, a_2)$ and in the same way $\gamma(a_1, Aa_2)=(\tau^2A_{22})^{-1}\ \tau A_{22}\ \gamma(a_1, a_2)$. Combining we get $\gamma(Aa_1, Aa_2)=(\tau^2 \det A)^{-1}\ \tau\det A\ \gamma(a, a_2)$.

(8.2) *Let t be of the form* (8.1) *with $a_1, a_2 \in \Gamma$, independent, $a_1 \frown a_2$, and suppose for some $x \in \Gamma$ we have $tx\neq x$, $tx \in \Gamma$, then t satisfies*

$$(8.3) \qquad (\tau\lambda)^{-1}\ \tau^2\lambda=-\gamma(a_1, a_2).$$

Proof: It is easily verified that for A as above

$$tx=x+(\det A)^{-1}\ \lambda\{(x, Aa_1)Aa_2-(x, Aa_2)Aa_1\}.$$

It follows that A affects both sides of (8.3) in the same way. We may suppose $(x, a_1)=1$, $(x, a_2)=0$. Write $a_4=x$, $a_3=\alpha_{24}(a_2\star a_4)=\alpha_{42}(a_4\star a_2)$. Then $tx\star tx=(a_4+\lambda a_2)\star(a_4+\lambda a_2)=$
$\tau^2\lambda\ (a_2\star a_4)+\tau\lambda\ (a_4\star a_2)=((\alpha_{24})^{-1}\tau^2\lambda+(\alpha_{42})^{-1}\tau\lambda)\ a_3=$
$=(\tau^2\lambda\ \tau^2\alpha_{13}+\tau\lambda\ \tau\alpha_{31})\ a_3=0.$
So $(\tau\lambda)^{-1}\tau^2\lambda=-(\tau^2\alpha_{13})^{-1}\tau\alpha_{31}=-\gamma(a_1, a_2)$ ▮

(8.4) *Let t be of the form* (8.1) *with $a_1, a_2 \in \Gamma$, independent, $a_1 \frown a_2$ and satisfying* (8.3). *Then t induces an automorphism of H_τ.*

Proof: As t^{-1} is of the form (8.1) with $-\lambda$ instead of λ, it suffices to show $x \in \Gamma \Rightarrow tx \in \Gamma$ and $x \frown y \Rightarrow tx \frown ty$. The first follows from the proof of (8.2). For the second, we need only consider the case $tx\neq x$, $ty\neq y$. Take a_3, a_4 as before. Now if $(y, a_2)=0$, y is dependent on a_3 and a_4. If $(y, a_2)\neq 0$, we may suppose $(y, a_1)=0$, $(y, a_2)=1$. Write $a_5=y$, $a_4=\alpha_{35}(a_3\star a)$. Now $(a_3, ta_5)=0$. We have $a_3\star ta_5=a_3\star(a_5-\lambda a_1)=(\alpha_{35})^{-1}a_4-(\alpha_{31})^{-1}\tau\lambda\ a_2$. Now $-(\alpha_{31})^{-1}\tau\lambda=(\alpha_{31})^{-1}\lambda\ \tau^2\gamma(a_1, a_2)=(\tau\alpha_{13})^{-1}\lambda=\lambda\ \tau^2\alpha_{24}=(\tau\alpha_{35})^{-1}\lambda$. So $a_3\star ta_5=(\alpha_{35})^{-1}ta_4$. This implies $ta_4 \frown ta_5$.

Combining (8.2) and (8.4) we get (see also TITS [2] § 6.4):

(8.5) *Let l be an a.c. line, a_1, a_2 a basis of l, $x_\#$ and $x_\#'$ a.c. points of H_τ with $x'=x+\lambda\{(x, a_1)a_2-(x, a_2)a_1\}$. Then there exists a $\pi \in \mathscr{G}_\tau$ with axis l such that $\pi x_\#=x_\#'$.*

Proof: Define t by (8.1) with y instead of x for all $y \in C$. Then (8.2) applies, so (8.3) is satisfied. It follows that (8.4) also applies. $\pi=t_\#$ ▮

Let L be the subfield of K of fixed elements for τ. K is a cyclic extension of L of order 3. Condition (8.3) means that $N_{K/L}(\gamma(a_1, a_2))=-1$. If this is fulfilled, λ is determined up to a factor from L^* (see e.g. BOURBAKI [1] Algèbre, chap. V § 10, no 5 and § 11, no 5). Suppose that there are at least 3 a.c. lines through each a.c. point, then it follows that for each axis there exist axial automorphisms of the form (8.1) and no others.

The automorphisms with the same a.c. line for axis constitute a group isomorphic to the additive group of L.

For H_{id} we have $\gamma(a_1, a_2) = -1$. Moreover, it may be verified directly that every linear transformation of the form (8.1) with $a_1, a_2 \in \Gamma$, independent and $a_1 \frown a_2$ is an automorphism of C. We indicate the lines of the computation. The line spanned by a_1, a_2 is a totally singular subalgebra D of C. Its orthoplement D' is a 6-dimensional subalgebra with radical D. D' contains a split quaternion algebra A. So $C = A \oplus Ax$ with suitable x (see VAN DER BLIJ and SPRINGER [1] § 2), and $D' = A \oplus Bx$, $D = Bx$, where B is a right ideal of A, satisfying $\bar{c}c' = 0$ for all $c, c' \in B$. The rest is straightforward, utilizing the multiplication rule $(a_1 + b_1 x)(a_2 + b_2 x) = (a_1 a_2 - Q(x)\bar{b}_2 b_1) + (b_2 a_1 + b_1 \bar{a}_2)x \quad (a_1, b_1, a_2, b_2 \in A)$. A similar treatment of H_τ would require an elaboration of the structure of C as a ring with respect to the $\star$-multiplication.

Now we restate the result (8.5) in terms of distance:

(8.6) *Let l, m, m' be a.c. lines of H_τ satisfying $d(l, m) = d(l, m') = 4$, $d(m, m') = 2$ and $d(l, m \mapsto m') = 3$. Then there is a $\pi \in \mathscr{G}_\tau$ having l for axis such that $\pi m = m'$.*

Proof: Take $x_\# \in m$, $x_\# \neq m \mapsto m'$, and $x_\#' \in m'$ such that $x_\#$ and $x_\#'$ are collinear with the point $y_\# \in l$ having $d(y_\#, m \mapsto m') = 2$. Apply (8.5) ▌

Concerning axial automorphisms with an a.c. point for axis we have the following result:

(8.7) *For H_{id} the only axial automorphism having an a.c. point for axis is the identity, except if $\chi(K) = 3$. In this case, for every a.c. point $a_\#$ and pair of a.c. points $x_\#, x_\#'$ on an a.c. line l, such that $d(a_\#, l) = 3$, $d(a_\#, x_\#) = d(a_\#, x_\#') = 4$, there is a $\pi \in \mathscr{G}_{\mathrm{id}}$ with axis $a_\#$ such that $\pi x_\# = x_\#'$.*

Proof: Let the a.c. point $a_\#$ be an axis of $\pi \in \mathscr{G}_{\mathrm{id}}$. We can find a normal basis $x_0, y_0, x_1, x_2, x_3, y_1, y_2, y_3$ of C (§ 1, end) with $x_1 = a$. The points $x_{1\#}, \ldots, y_{3\#}$ and the joining lines constitute a hexagon of H_{id}. According to (6.2), π is induced by $t \in \mathscr{G}_C$. The plane spanned by x_1, y_2, y_3 is fixed pointwise, so t is linear, and $tx_1 = \lambda x_1$, $ty_2 = \lambda y_2$, $ty_3 = \lambda y_3$. Now $tx_1 = ty_2 \cdot ty_3$ implies $\lambda = 1$. The lines joining $y_{2\#}, x_{3\#}$ and $y_{3\#}, x_{2\#}$ are a.c. A straightforward computation shows that t has the form

$$\begin{aligned}
&tx_1 = x_1,\ ty_2 = y_2,\ ty_3 = y_3,\\
&tx_2 = x_2 + \alpha y_3,\ tx_3 = x_3 - \alpha y_2,\\
&ty_1 = y_1 + \alpha(x_0 - y_0) + \alpha^2 x_1,\\
&tx_0 = x_0 + \alpha x_1,\ ty_0 = y_0 - \alpha x_1.
\end{aligned}$$

It can be verified that t is an automorphism. Now the pencil

$$P((\lambda_1 x_1 + \lambda_2 y_2 + \lambda_3 y_3)_\#)$$

remains fixed elementwise. We may suppose $\lambda_2 \neq 0$. In the plane spanned by y_1, x_2, y_3 we have

$$(\lambda_1 x_1 + y_2 + \lambda_3 y_3,\ \mu_1 y_1 + \mu_2 x_2 + \mu_3 y_3) = \lambda_1 \mu_1 + \mu_2.$$

The line $\lambda_1\mu_1 + \mu_2 = 0$ is not a.c., since it does not contain x_2. Consider

$$z = (\lambda_1 x_1 + y_2 + \lambda_3 y_3)(\mu_1 y_1 - \lambda_1 \mu_1 x_2 + \mu_3 y_3).$$

Now $(t-1)z = \alpha\mu_1(\lambda_1 x_1 + y_2 + \lambda_3 y_3) - 3\alpha\lambda_1\mu_1 x_1$ has to be proportional to $\lambda_1 x_1 + y_2 + \lambda_3 y_3$ for all λ_1, μ_1. So $\chi(K) = 3$ or $\alpha = 0$. If $3 = 0$, t is axial for all $\alpha \in K$, for the line containing z may arbitrarily be chosen in the pencil $P((\lambda_1 x_1 + y_2 + \lambda_3 y_3)_{\#})$ ∎

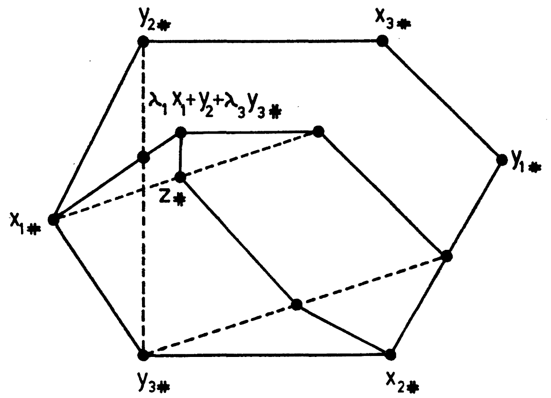

Fig. 5

The different behaviour if $\chi(K) = 3$ is related with several other peculiarities of the algebraic groups of type G_2 in this case (see Tits [2] § 10).

Similar results to (8.7) seem to be obtainable for H_τ, $\tau \neq \text{id}$. Instead of 3α the trace $\alpha + \tau\alpha + \tau^2\alpha$ will occur.

9. In order to distinguish H_{id} from H_τ, $\tau \neq \text{id}$, by geometrical means we consider for an a.c. point $a_{\#}$ the pencil $P(a_{\#})$ of a.c. lines containing $a_{\#}$. If $\tau = \text{id}$, in the centric plane $aC \cap Ca$ every line containing a is a.c. In the general case, however, we have the following theorem:

(9.1) *Suppose H_τ contains a hexagon. Then the pencil $P(a_{\#})$ of an a.c. point $a_{\#}$ possesses in the centric plane $a \star C \cap C \star a$ the structure of a projective line over L, embedded in the pencil of all lines of $a \star C \cap C \star a$ containing $a_{\#}$.*

Again L designates the subfield of K of invariant elements for τ. The proof of (9.1) requires an algebraic preliminary, the proof of which I owe to Professor T. A. Springer:

(9.2) *Suppose $a \star a = b \star b = (a, b) = 0$, $a \star b \neq 0$, so $b \star a = \alpha(a \star b)$. Then $N_{K/L}(\alpha) = -1$.*

Proof: Choose $x \in C$ satisfying $(x, a \star b) = 1$. Then $(x, b \star a) = \alpha$. We have

$$((a \star x) \star b) \star a = -((b \star x) \star a) \star a = -(b \star x, a)a = -a,$$
$$((a \star x) \star b) \star b = \tau(a \star x, b)\, b = \tau^2\alpha\, b.$$

Write $(a \star x) \star b = u$, so $u \star a = -a$, $u \star b = \tau^2\alpha\, b$.
Now $(u, a) = ((a \star x) \star b, a) = -((b \star x) \star a, a) = 0$, $(u, b) = 0$.
Compare the following expressions $(b \star a) \star u = -(u \star a) \star b = a \star b$,
$(b \star a) \star u = \tau^2\alpha\, (a \star b) \star u = -\tau^2\alpha\, (u \star b) \star a = -\tau^2\alpha\, \tau\alpha\, (b \star a) =$
$= -\tau^2\alpha\, \tau\alpha\, \alpha\, (a \star b)$ ▮

Proof of (9.1): According to a conclusion from (7.2), $P(a_{\#})$ contains at least 2 a.c. lines l_1, l_2. Take $a_1 \in l_1$, $a_2 \in l_2$ such that a is not dependent on a_1, a_2. Then $a_1 \star a_2 \neq 0$, and from (9.2) we deduce $a_2 \star a_1 = \alpha(a_1 \star a_2)$, $N_{K/L}(-\alpha) = 1$. Now we have

$$(\alpha_1 a_1 + \alpha_2 a_2) \star (\alpha_1 a_1 + \alpha_2 a_2) = (\tau^2\alpha_1\, \tau\alpha_2 + \alpha\, \tau^2\alpha_2\, \tau\alpha_1)\, (a_1 \star a_2).$$

So $\alpha_1 a_1 + \alpha_2 a_2 \in \Gamma$ iff $\alpha_1 = 0$ or $\alpha_2 = 0$ or

$$\alpha_1, \alpha_2 \neq 0 \text{ and } (\tau(\alpha_2^{-1}\alpha_1))^{-1}\, \tau^2(\alpha_2^{-1}\alpha_1) = -\alpha.$$

We have remarked already (§ 8, after (8.5)) that this equation is solvable and that the solutions form a coset mod L^* in K^*. Finally $\alpha_1 a_1 + \alpha_2 a_2 \in \Gamma$ implies that the line spanned by a and $\alpha_1 a_1 + \alpha_2 a_2$ is a.c. ▮

Consider a centric plane $a \star C \cap C \star a$ of H_τ. If $\tau \neq \mathrm{id}$, the set of a.c. points contained in $a \star C \cap C \star a$ is not a projective plane, as it is in the case $\tau = \mathrm{id}$. The point of intersection of a pair of non-a.c. lines need not be a.c. We want to express this property in terms of the elements of the hexagonic structure and its group.

Suppose $a_1, a_2, a_3 \in \Gamma$ such that $d(a_{1\#}, a_{2\#}) = 4$, $d((a_1 \star a_2)_{\#}, a_{3\#}) = 6$. We ask for an axial automorphism π with an a.c. line for axis satisfying $\pi a_{1\#} = a_{2\#}$, $\pi a_{3\#} = a_{3\#}$. The axis of π has to intersect the non-a.c. line l spanned by a_1, a_2 in an a.c. point. This point has to be orthogonal to

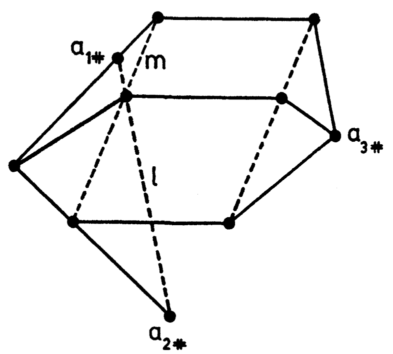

Fig. 6

$a_{3\#}$, hence must belong to the non-a.c. line m, which is the intersection of the orthoplement of a_3 with $(a_1 \star a_2) \star C \cap C \star (a_1 \star a_2)$. If $\tau \neq \mathrm{id}$, it is easy to choose a_1, a_2, a_3 so as to ensure that $l \cap m$ is not a.c. If $\tau = \mathrm{id}$, each point $b_\# \in l \cap m$ is necessarily a.c., and we may take the line spanned by b and $a_3 \star b$ for axis, except if $l \neq m$ and $b_\# = a_{1\#}$ or $b_\# = a_{2\#}$. We have shown:

(9.3) *Suppose* $a_{1\#}, a_{2\#}, a_{3\#}$ *are a.c. points of* H_{id}, $d(a_{1\#}, a_{2\#}) = 4$ *and* $d(a_{1\#}, a_{3\#}) = d(a_{2\#}, a_{3\#}) = d((a_1 \star a_2)_\#, a_{3\#}) = 6$. *Then there is a* $\pi \in \mathscr{G}_{\mathrm{id}}$ *with an a.c. line for axis such that* $\pi a_{1\#} = a_{2\#}$, $\pi a_{3\#} = a_{3\#}$.

Finally we remark that, for each automorphism σ of order 3 of K, there is a hexagonic structure H_τ containing a hexagon with $\tau = \sigma$ on K. To prove this, take a normal basis of C, and put

$$\tau \sum_0^3 (\xi_i x_i + \eta_i y_i) = \sum_0^3 (\sigma\xi_i \cdot x_i + \sigma\eta_i \cdot y_i).$$

Then $x_1, y_2, x_3, y_1, x_2, y_3$ determine a hexagon.

(To be continued)