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Transformation of Series

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Abstract. Nonlinear transformation of series are evaluated using the A_n polynomials defined in the decomposition method.

Given a specific function $f(u)$ with u given by a convergent series, a convenient evaluation of $f(u)$ can be made by using the A_n polynomials which have been used to represent nonlinearities in differential and partial differential equations in the decomposition method.

For an analytic function $f(u)$, we can write (using decomposition)

$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, \dots, u_n)$$

where the A_n are given by $A_n = \sum_{n=0}^{\infty} c(\nu, n) f^{(\nu)}(u)|_{u=u_0}$.

The $f^{(\nu)}(u)_{u=u_0}$, or simply $f^{(\nu)}(u_0)$, is the νth derivative of $f(u)$ evaluated at $u = u_0$. To determine $c(\nu, n)$, we form the sum of possible products of ν components u_i with $i = 0, 1, 2, \dots$ which sum to n, and divide by the factorial of the number of repetitions of the components.

For convenience, we list

$$\begin{aligned}
A_0 &= f(u_0) \\
A_1 &= u_1 f^{(1)}(u_0) \\
A_2 &= u_2 f^{(1)}(u_0) + \left(\frac{u_1^2}{2!}\right) f^{(1)}(u_0) \\
A_3 &= u_3 f^{(1)}(u_0) + u_1\ u_2 f^{(2)} u_0 + \left(\frac{u_1^3}{3!}\right) f^{(3)}\ (u_0) \\
&\vdots
\end{aligned}$$

For differential equations involving a nonlinearity $f(u)$, the decomposition method determines components $u_0, u_1, \dots$ such that the n-term approximation

$$\phi_n = \sum_{i=0}^{n-1} u_i$$

Suppose we are now given the convergent series $u = \sum_{n=0}^{\infty} c_n\ x^n$ and $f(u)$. We wish to find the resulting transformed series. Since $u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} c_n\ x^n$, we have $u_n = c_n\ x^n$. Consequently,

$$A_n = (u_0, u_1, \dots, u_n) = x^n A^n(c_0, c_1, \dots, c_n) \tag{1}$$

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so that

$$A_0(u_0) = f(u_0) = f(c_0)\ x\ A_0(c_0),$$
$$A_1(u_0, u_1) = u_1 f^{(1)}(u_0) = x\ c_1\ f^{(1)}(c_0) = x\ A_1(c_0, c_1),$$
$$A_2(u_0, u_1, u_2) = u_2 f^{(1)}(u_0) + \left(\frac{u_1^2}{2!}\right) f^{(2)}(u_0),$$
$$= x^2 c_2\ f^{(1)}(u_0) + x^2 \left(\frac{c_1^2}{2!}\right) f^{(2)}(u_0) = x^2 A_2(c_0, c_1, c_2),$$

etc. Now we can state the theorem:

THEOREM.

$$f(u) = f\left(\sum_{n=0}^{\infty} c_n\, x^n\right) = \sum_{n=0}^{\infty} A_n(c_0, \dots, c_n) x^n$$

The following very simple examples are given to demonstrate use and to make comparisons:

EXAMPLE 1:

$$u = e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

We know, of course, that

$$u^2 = e^{2x} = \sum_{m=0}^{\infty} \frac{(2x)^m}{m!}$$

however, we obtain the result using the theorem which can be used also for less transparent cases. Since $e^x = \sum_{m=0}^{\infty} x^m/m! = \sum_{m=0}^{\infty} c_m x^m$, we have $c_m = 1/m!$. Hence, from Equation (1), $(e^x)^2 = \sum_{m=0}^{\infty} A_m\ x^m$ where we can evaluate the $A_n(u_0 \dots u_n)$, replace u_i by c_i, and use $c_m = 1/m!$ so that

$$A_0(u_0) = A_0(c_0) = 1$$
$$A_1(u_0, u_1) = A_1(c_0, c_1) = 2$$
$$A_2(u_0, u_1, u_2) = A_2(c_0, c_1, c_2) = 2$$
$$A_3(u_0 \dots u_3) = A_3(c_0 \dots c_3) = 4/3$$

Therefore, $(e^x)^2 = \sum_{m=0}^{\infty} A_m\ x^m = 1 + 2x + 2x^2 + (4/3)x^3 + \cdots = \sum_{m=0}^{\infty} (2x)^m/m!$ which is the expected result. This is, of course, a trivial example to make the method transparent and so checks can readily be made.

EXAMPLE 2: Let $u = \sin t = t - t^3/3! + t^5/3! - \cdots = \sum_{n=0}^{\infty} (-1)^n\ t^{2n+1}/(2n+1)!$. We have $c_0 = 0$, $c_1 = 0$, $c_2 = 0$, $c_3 = -1/3! \dots$. Thus, $A_0 + A_1\ t + A_2\ t^2 + A_3\ t^3 + \cdots = t^2 + \cdots$ or $u^2 = t^2 - 2t^4/3! + \cdots$.

EXAMPLE 3:

$$u(x) = \tan^{-1}\ x$$

and

$$f(u) = u^2$$
$$u(x) = \tan^{-1}\ x = x - x^3/3 + x^5/5 - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)} + \cdots.$$

We have $u_0 = x$, $u_1 = -x^3/3$, $u_2 = x^5/5$

$$A_0(u_0) = u_0^2 = x^2$$
$$A_1(u_0, u_1) = 2u_0 u_1 = -2x^4/3$$
$$A_2(u_0, u_1, u_2) = u_1^2 + 2u_0 u_2 = x^6/9 + 2x^6/5 = 23x^6/45$$
$$\vdots$$
$$(\tan^{-1} x)^2 = x^2 - \frac{2}{3}x^4 + \frac{23}{45}x^6 + \cdots$$

Let $x = 1/5$ then $\tan^{-1}(1/5) = (1/5) - (1/5)^3/3 + (1/5)(1/5)^5 - \cdots \cong 0.2$. As expected, $[\tan^{-1}(1/5)]^2 = (1/5)^2 - (2/3)(1/5)^4 + (23/45)(1/5^6)\cdots \cong 0.04$.

EXAMPLE 4:

$$
\begin{aligned}
u(x) &= \sum_{n=0}^{\infty} \frac{u^{(n)}(a)}{n!}(x-a)^n = u(a) + u'(a)\,(x-a) + \cdots \\
u &= f^{(0)}\,(a) + f^{(1)}(a)\,(x-a) + \cdots = u_0 + u_1 + \cdots \\
f(u) &= \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \sum_{\nu=1}^{n} c(\nu,n) f^{(\nu)}(u_0) \\
A_0 &= f(u_0) \\
A_1 &= c(1,1)\, f^{(1)}(u_0) = u_1\, f^{(1)}(u_0) \\
A_2 &= c(1,2)\, f^{(1)}(u_0) + c(2,2)\, f^{(2)}(u_0) \\
&= u_2\, f^{(1)}(u_0) + (1/2)\, u_1^2\, f^{(2)}(u_0) \\
f(u) &= f(u(a)) + u'(a)\,(x-a)\, f'(u(a)) + \cdots .
\end{aligned}
$$

For multi-dimensional series,

$$
\begin{aligned}
u = u(x_1,\ldots,x_n) &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} c_{m_1,\ldots,m_n} x_1^{m_1} \ldots x_n^{m_n} \\
f(u) &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} A_{m_1,m_2,\ldots,m_n} x_1^{m_1} \ldots x_n^{m_n}
\end{aligned}
$$

where the (Adomian) polynomials $A_{m_1,\ldots,m_n}$ are functions of the coefficients $c_{\mu_1,\ldots,\mu_n}$.

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