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Best Approximation and Unique Extension of Lipschitz Functions

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1. INTRODUCTION

In the sequel, X will always denote a metric space with the metric d, x_0 a fixed point from X, and Y a subset of X such that $x_0 \in Y$. If f is a real-valued function defined on X, denote

$$\|f\|_Y = \sup\{|f(x) - f(y)|/d(x, y): x, y \in Y, x \neq y\}. \tag{1.1}$$

A Lipschitz function on X is a function $f: X \to R$ such that $\|f\|_X < \infty$. Denote by $\mathrm{Lip}_0 X$ the Banach space of all Lipschitz functions on X which vanish at x_0, with the norm $\|f\| = \|f\|_X$. Put also

$$Y^\perp = \{f: f \in \mathrm{Lip}_0 X, f|_Y = 0\}. \tag{1.2}$$

A Lipschitz extension of a function $f \in \mathrm{Lip}_0 Y$ is a function $F \in \mathrm{Lip}_0 X$ such that $F|_Y = f$ and $\|F\|_X = \|f\|_Y$. It is known (see, e.g., [2]) that every $f \in \mathrm{Lip}_0 Y$ has a Lipschitz extension in $\mathrm{Lip}_0 X$.

For a subset Y of X and $x \in X$ we put

$$d(x, Y) = \inf\{d(x, y): y \in Y\}. \tag{1.3}$$

Now, let E be a normed linear space, G a nonempty subset of E, x an element from E, and

$$P_G(x) = \{y \in G: \|x - y\| = d(x, G)\}. \tag{1.4}$$

An element from $P_G(x)$ is called a best approximation to x from G. If M is a subset of E we say that G is M-proximinal if $P_G(x) \neq \varnothing$, for all $x \in M$. If $P_G(x)$ contains exactly one element for every $x \in M$, then G is called M-chebyshevian. If the set G is E-proximinal (respectively E-chebyshevian) then we say, simply, that G is proximinal (respectively chebyshevian).

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We say that a linear subspace Z of E has the property (U) if every continuous linear functional on Z has a unique Hahn–Banach extension to E (i.e., linear and norm preserving) [6]. Let us denote by E^* the conjugate space of E and by $Z^{\perp}$ the annihilator of the subspace Z in E^*, i.e.,

$$Z^{\perp} = \{\varphi \in E^*; \varphi|_Z = 0\}. \tag{1.5}$$

Phelps [6] showed that the subspace Z of E has property (U) if and only if its annihilator $Z^{\perp}$ is chebyshevian. This result can be extended to Lipschitz functions:

THEOREM 1 ([5, Lemma 2]). *Let X be a metric space, x_0 in X, and $Y \subseteq X$ such that $x_0 \in Y$. The space $Y^{\perp}$ is chebyshevian for $f \in \mathrm{Lip}_0 X$ if and only if $f|_Y \in \mathrm{Lip}_0 Y$ has a unique Lipschitz extension in $\mathrm{Lip}_0 X$.*

We also need the following lemma.

LEMMA 1. *Every best approximation to $f \in \mathrm{Lip}_0 X$ from $Y^{\perp}$ is of the form $f - F$, where F is a Lipschitz extension of $f|_Y$ to X.*

Proof. Suppose F is a Lipschitz extension of $f|_Y$ to X. Then, by [5, Theorem 2 and Lemma 1], we get

$$\|f - (f - F)\|_X = \|F\|_X = \|f\|_Y = d(f, Y^{\perp}).$$

Conversely, if $g \in Y^{\perp}$ is a best approximation to f, then $\|f - g\|_X = d(f, Y^{\perp}) = \|f\|_Y$ and $(f - g)|_Y = f|_Y$. Therefore $F = f - g$ is a Lipschitz extension of $f|_Y$.

2. MAIN THEOREM

A metric space X is called uniformly discrete if there exists a number $\delta > 0$, such that $d(x, y) \geqslant \delta$ for all $x, y \in X$ with $x \neq y$. The following theorem appears in [5], in the hypothesis that Y has an accumulation point in X. The main result is:

THEOREM 2. *Let X, x_0, and Y be as in Theorem 1. Suppose, further, that Y is nonuniformly discrete. If every $f \in \mathrm{Lip}_0 Y$ has a unique Lipschitz extension, then $\overline{Y} = X$ (or equivalently $Y^{\perp} = \{0\}$).*

Proof. Since Y is nonuniformly discrete, for every $n \in N$, there exist $x_n, y_n \in Y$, $x_n \neq y_n$ such that $d(x_n, y_n) < 1/n$. Defining $f_n: X \to R$ by

$$f_n(x) = d(x, x_n) - d(x, y_n) - d(x_0, x_n) + d(x_0, y_n), \qquad n = 1, 2, 3, \ldots.$$

we have

$$\begin{aligned} f_n(x_0) &= 0, \qquad n = 1, 2, 3,\ldots, \\ -2d(x_n, y_n) \leqslant f_n(x_n) &= -d(x_n, y_n) - d(x_0, x_n) + d(x_0, y_n) \\ &\leqslant 0, \qquad n = 1, 2, 3,\ldots, \\ 0 \leqslant f_n(y_n) &= d(x_n, y_n) - d(x_0, x_n) + d(x_0, y_n) \\ &\leqslant 2d(x_n, y_n), \qquad n = 1, 2, 3,\ldots, \\ \| f_n \|_X &= \sup\{| d(x, x_n) - d(x, y_n) - d(y, x_n) + d(y, y_n)|/d(x, y): \\ & \qquad x, y \in Y, x \neq y\} \leqslant 2, \qquad n = 1, 2, 3,\ldots, \end{aligned}$$

so that $f_n \in \mathrm{Lip}_0\, X$ for $n = 1, 2, 3,\ldots$.

Let $a_n = d(x_0, y_n) - d(x_0, x_n)$, and suppose that the set $I = \{n \in N: a_n \leqslant 0\}$ is infinite, say $I = \{n_j: j \in N\}$. Then, we have $f_{n_j}(x_0) = 0, f_{n_j}(x_{n_j}) < 0$, $f_{n_j}(y_{n_j}) \geqslant 0, j = 1, 2, 3,\ldots$. Now, we consider the sequence $\{\psi_j\}$ of functions $\psi_j: f_{n_j}(X) \to [0, 1]$ defined by

$$\begin{aligned} \psi_j(t) &= 1, & & t < f_{n_j}(x_{n_j}), \\ &= t/f_{n_j}(x_{n_j}), & & f_{n_j}(x_{n_j}) \leqslant t < 0 = f_{n_j}(x_0), \\ &= 0, & & t \geqslant 0, \end{aligned}$$

for $j = 1, 2, 3,\ldots$. Putting $q_j = \psi_j \circ f_{n_j}$, we have

$$\| q_j \|_Y \geqslant | \psi_j(f_{n_j}(x_{n_j})) - \psi_j(f_{n_j}(y_{n_j}))|/d(x_{n_j}, y_{n_j}) \geqslant n_j .$$

By [5, Corollary 2] it follows that

$$\begin{aligned} d(x, Y) &\leqslant (\sup\{\psi_j(f_{n_j}(y)): y \in Y\} - \inf\{\psi_j(f_{n_j}(y)): y \in Y\})/(2 \| q_j \|_Y) \\ &= 1/(2 \| q_j \|_Y) \leqslant 1/n_j \to 0, \end{aligned}$$

so that $x \in \overline{Y}$, for all $x \in X$, that is $\overline{Y} = X$.

By Theorems 1 and 2, we have

Corollary 1. *Suppose that Y is nonuniformly discrete. Then $Y^\perp$ is chebyshevian in* $\mathrm{Lip}_0\, X$ *if and only if* $Y^\perp = \{0\}$.

We can also prove the following result.

Theorem 3. *Let X, x_0, and Y be as in Theorem* 1. *If* $(Y^\perp)_\perp^\perp$ *has the property* (U) *then every* $f \in \mathrm{Lip}_0\, Y$ *has a unique Lipschitz extension* $F \in \mathrm{Lip}_0\, X$.

Proof. Follows from [8, Corollary 3.1.b)] and the above Theorem 1.

Corollary 2. *Let X, x_0, and Y be as in Theorem* 1. *Suppose that Y is nonuniformly discrete. If* $(Y^\perp)_\perp^\perp$ *has the property* (U), *then* $\overline{Y} = X$ (*or equivalently* $Y^\perp = \{0\}$).

3. Examples

(a) Let $X = [0, 1]$, $d(x, y) = |x - y|$, $x_0 = 0$, and $Y = \{0, 1\}$. Then every $f \in \mathrm{Lip}_0\{0, 1\}$ has a unique Lipschitz extension $F \in \mathrm{Lip}_0[0, 1]$, namely, $F(x) = f(1)x$. This example shows that the supposition that Y is non-uniformly discrete is essential in Theorem 2.

(b) Let $f \in \mathrm{Lip}_0[0, 1]$ and let Y be the set of points $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Then, we have:

THEOREM 4. *The following conditions are equivalent*:

(1°) $Y^\perp$ *is f-chebyshevian,*

(2°) $\|f\|_Y = |[x_k, x_{k+1}; f]|$, $k = 0, 1, 2,..., n$,

where $[x_k, x_{k+1}; f] = (f(x_{k+1}) - f(x_k))/(x_{k+1} - x_k)$.

Proof. (1°) ⇒ (2°) Obviously,

$$L(x) = [x_k, x_{k+1}; f](x - x_k) + f(x_k), \qquad x \in (x_k, x_{k+1}),\ k = 0, 1,..., n, \tag{3.1}$$

is a Lipschitz extension of $f|_Y$ and $\|f\|_Y \geqslant |[x_k, x_{k+1}; f]|$, $k = 0, 1,..., n$. Suppose that k_0, $0 \leqslant k_0 < n$ is such that $\|f\|_Y > |[x_{k_0}, x_{k_0+1}; f]|$. We have to consider the following cases:

(i) $f(x_{k_0}) < f(x_{k_0+1})$,

(ii) $f(x_{k_0}) > f(x_{k_0+1})$,

(iii) $f(x_{k_0}) = f(x_{k_0+1})$.

If condition (i) holds, put $z_1 = x_{k_0} + (f(x_{k_0+1}) - f(x_{k_0}))/\|f\|_Y$ and define the function F_1: $[0, 1] \to R$ by

$$\begin{aligned} F_1(x) &= L(x), && x \in [0, 1] - [x_{k_0}, x_{k_0-1}], \\ &= f(x_{k_0}) + \|f\|_Y (x - x_{k_0}), && x \in (x_{k_0}, z_1), \\ &= f(x_{k_0}), && x \in [z_1, x_{k_0+1}). \end{aligned} \tag{3.2}$$

It is easy to see that F_1 is a Lipschitz extension of $f|_Y$, distinct from L, and then, by Theorem 1, $Y^\perp$ is not f-chebyshevian.

In case (ii) the proof proceeds similarly. If condition (iii) holds, put $z_2 = (2x_{k_0} + x_{k_0+1})/3$ and define

$$\begin{aligned} F_2(x) &= L(x), && x \in [0, 1] - [x_{k_0}, x_{k_0+1}], \\ &= f(x_{k_0}) + \|f\|_Y (x - x_{k_0}), && x \in (x_{k_0}, z_2], \\ &= f(x_{k_0+1}) - \|f\|_Y (x - x_{k_0+1}), && x \in (z_2, x_{k_0 -1}). \end{aligned} \tag{3.3}$$

Then F_2 is a Lipschitz extension of $f|_Y$, different from L. By Theorem 1, $Y^\perp$ is not f-chebyshevian.

(2°) ⇒ (1°) If $|[x_k, x_{k+1}; f]| = \|f\|_Y$ for $k = 0, 1, 2,..., n$, then the function L defined by (3.1) is the only Lipschitz extension of $f|_Y$.

A consequence of Theorem 4 is:

COROLLARY 3. *Let Y be the set of points $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$, $f \in \mathrm{Lip}_0[0, 1]$ and*

$$K = \{h: h \in \mathrm{Lip}_0\,[0, 1], h(x_k) = f(x_k), k = 0, 1, 2,..., n+1\}. \quad (3.4)$$

Then $Y^\perp$ is K-chebyshevian if and only if $Y^\perp$ is f-chebyshevian.

(c) Let $C^1[0, 1]$ be the space of all continuously differentiable functions on $[0, 1]$ and let Y be the set of points $0 = x_0 < x_1 < \cdots < x_{n+1} = 1$. Put

$$Z = C^1[0, 1] \cap \mathrm{Lip}_0[0, 1], \qquad W = C^1[0, 1] \cap Y^\perp. \quad (3.5)$$

For $f \in Z$, we have

$$\|f\|_{[0,1]} = \max\{|f'(x)|: x \in [0, 1]\}. \quad (3.6)$$

Let us define the function set S by

$$S = \{h: h \in Z, [x_k, x_{k+1}; h][x_{k+1}, x_{k+2}; h] \neq -\|h\|_Y^2, k = 0, 1, 2,..., n-1\}. \quad (3.7)$$

We need the following two lemmas:

LEMMA 2. *Let $[p, q] \subset R$, $f(x) = ax + b$, $a, b \in R$, $a > 0$, and $M > a$. Then there exists a function $g \in C^1[p, q]$ such that $f(p) = g(p)$, $f(q) = g(q)$, $f'(p) = M$ $(f'(q) = M)$, $f'(q) = g'(q)$ $(f'(p) = g'(p))$ and* $\max\{|g'(x)|$: *$x \in [p, q]\} = M$.*

Proof. The proof of the lemma is obvious from Fig. 1:

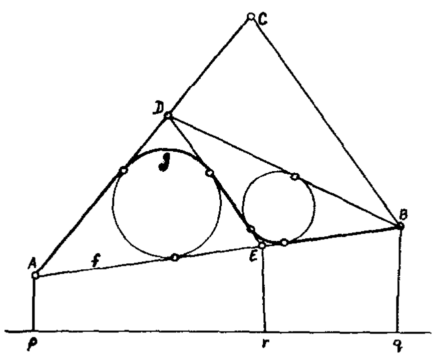

FIGURE 1

$$\text{(AC)} \quad s_1(x) = f(p) + M(x - p),$$
$$\text{(BC)} \quad s_2(x) = f(q) - M(x - q),$$
$$\text{(DE)} \quad s_3(x) = f(r) - M(x - r), \qquad r \in (p, q).$$

LEMMA 3. *If $h \in S$, then $h|_Y$ has at least one Lipschitz extension $H \in Z$.*

Proof. Let $h \in S$ and $k_0 \in N$, $0 \leqslant k_0 < n + 1$, such that

$$[x_{k_0}, x_{k_0+1}; h] = \| h \|_Y. \tag{3.8}$$

By the definition of S, we have

$$-\| h \|_Y < [x_{k_0-1}, x_{k_0}; h] \leqslant \| h \|_Y,$$
$$-\| h \|_Y < [x_{k_0+1}, x_{k_0+2}; h] \leqslant \| h \|_Y.$$

Applying Lemma 2 to the intervals $[x_{k_0-1}, x_{k_0}]$ and $[x_{k_0+1}, x_{k_0+2}]$, twice it follows that there exists a function H_1 in $C^1[x_{k_0-1}, x_{k_0+2}]$ such that $\max\{| H_1'(x)|: x \in [x_{k_0-1}, x_{k_0+2}]\} = \| h \|_Y$ and which interpolates the function h at the points $x_{k_0-1}, x_{k_0}, x_{k_0+1}, x_{k_0+2}$.

Applying Lemma 2 to the intervals $[x_i, x_{i+1}]$, $i = 0, 1,..., k_0 - 2, k_0 + 2,..., n$, we get a function $H \in Z$, which is a Lipschitz extension of $h|_Y$ to $[0, 1]$.

If $[x_{k_0}, x_{k_0+1}; h] = -\| h \|_Y$ we can proceed analogously.

THEOREM 5. *The subspace W is S proximinal and for each $h \in S$ the following equality holds*:

$$d(h, W) = d(h, Y^{\perp}). \tag{3.9}$$

Proof. Let $h \in S$. By Lemma 3, $h|_Y$ has a Lipschitz extension $H \in Z$. Then, $h - H \in W$, and this is a best approximation to h, from $Y^{\perp}$.

But then

$$d(h, Y^{\perp}) \leqslant d(h, W) \leqslant \| h - (h - H)\|_X = d(h, Y^{\perp}),$$

so that

$$\| h - (h - H)\|_X = d(h, W) = d(h, Y^{\perp}).$$

Remark 1. Let $f \in Z - S$; that is, there exists $0 \leqslant k_1 < n + 1$ such that

$$[x_{k_1-1}, x_{k_1}; f][x_{k_1}, x_{k_1+1}; f] = -\| f \|_Y^2.$$

In this case, it is possible that no Lipschitz extension to f exists in Z; e.g., for $f(x) = -4x^2 + 4x$, $Y = \{0, \frac{1}{2}, 1\}$ we have

$$[0, \tfrac{1}{2}; f][\tfrac{1}{2}, 1; f] = -4$$

and the only Lipschitz extension of $f|_Y$ is

$$F(x) = 2x, \qquad x \in [0, \tfrac{1}{2}),$$
$$= -2(x-1), \qquad x \in [\tfrac{1}{2}, 1],$$

which, obviously, does not belong to Z.

By Lemmas 2 and 3, every $h \in S$ has a best approximation in W, namely, $h - H$, where H is a Lipschitz extension of h, such that $H \in Z$. We can show that every best approximation is of this form (Lemma 1). It follows that W is chebyshevian for $h \in S$ if and only if $h|_Y$ has a unique Lipschitz extension in Z. A class of such functions is given by

$$S_1 = \{h: h \in S, h(x_k) = h(1)\, x_k, k = 0, 1, 2, \ldots, n+1\}. \tag{3.11}$$

THEOREM 6. *W is S_1-chebyshevian.*

Proof. If $h \in S_1$, then the unique Lipschitz extension of h in Z is $H(x) = h(1)x$. Therefore $h(x) - h(1)x$ is the only element of best approximation for h in W.

Remark 2. J. Favard and recently de Boor [1] considered a problem analogous to that in Example (c).

(d) Finally, let X be a metric space of finite diameter (i.e., $\sup\{d(x, y): x, y \in X\} < \infty$), x_0 a fixed element in X, and Y a subset of X such that $x_0 \in Y$. Let $f \in \mathrm{Lip}_0 X$ and let $G(f)$ be the set of best approximation to f from $Y^\perp$. We can define on $\mathrm{Lip}_0 X$ the uniform norm $\|\cdot\|_u$: $\mathrm{Lip}_0 X \to R$ by

$$\|f\|_u = \sup\{|f(x)|: x \in X\}, \qquad f \in \mathrm{Lip}_0 X. \tag{3.12}$$

Obviously, the set $G(f) \subset Y^\perp$ is closed, convex, and bounded, for every $f \in \mathrm{Lip}_0 X$. We consider the following problems: Find $g_*, g^* \in G(f)$ such that

$$\|f - g_*\|_u = \inf\{\|f - g\|_u: g \in G(f)\}, \tag{3.13}$$

and

$$\|f - g^*\|_u = \sup\{\|f - g\|_u: g \in G(f)\}; \tag{3.14}$$

i.e., find the nearest and the farthest point to f in $G(f)$, in the uniform norm.

Since every element in $G(f)$ is of the form $f - F$, where F is a Lipschitz extension of $f|_Y$ it follows that the problems (3.13) and (3.14) are equivalent to the following problems: Find two Lipschitz extensions F_* and F^* of $f|_Y$ such that

$$\|F_*\|_u = \inf\{\|F\|_u: F \text{ is a Lipschitz extension of } f|_Y\} \tag{3.13'}$$

and

$$\|F^*\|_u = \sup\{\|F\|_u: F \text{ is a Lipschitz extension of } f|_Y\}. \tag{3.14'}$$

THEOREM 6. *The infimum* (3.13) *is attained for every* $g_* = f - F_*$ *such that* F_* *is a Lipschitz extension of* $f|_Y$ *and* $\|F_*\|_u = \|f|_Y\|_u$. *The set of these extensions is nonempty.*

Proof. If F is a Lipschitz extension of $f|_Y$ then

$$\|F\|_u \geqslant \sup\{|F(y)|: y \in Y\} = \sup\{|f(y)|: y \in Y\} = \|f|_Y\|_u .$$

Therefore, if $\|F_*\|_u = \|f|_Y\|_u$ then $\inf\{\|F\|_u: F$ is a Lipschitz extension of $f|_Y\} = \|F_*\|_u = \|f|_Y\|_u$. Now, if F is a Lipschitz extension of $f|_Y$, we define a new Lipschitz function F_* by

$$\begin{aligned} F_*(x) &= \|f|_Y\|_u && \text{if} \quad F(x) > \|f|_Y\|_u , \\ &= F(x) && \text{if} \quad -\|f|_Y\|_u \leqslant F(x) \leqslant \|f|_Y\|_u , \\ &= -\|f|_Y\|_u && \text{if} \quad F(x) < -\|f|_Y\|_u . \end{aligned} \tag{3.15}$$

It is easy to see that F_* is a Lipschitz extension of $f|_Y$ such that $\|F_*\|_u = \|f\|_u$.

THEOREM 7. *The supremum* (3.14) *is attained for* $f - F_1{}^*$ *or* $f - F_2{}^*$ *or for both of these functions, where*

$$F_1{}^*(x) = \inf\{[f(y) + \|f\|_Y\, d(x, y)]: y \in Y\}, \tag{3.16}$$

and

$$F_2{}^*(x) = \sup\{[f(y) - \|f\|_Y\, d(x, y)]: y \in Y\}. \tag{3.17}$$

Proof. By [2], $F_1{}^*$ and $F_2{}^*$ are Lipschitz extensions of $f|_Y$ and obviously, for every Lipschitz extension F of $f|_Y$ we have

$$F_2{}^*(x) \leqslant F(x) \leqslant F_1{}^*(x), \qquad x \in X.$$

From these inequalities, it follows that

$$\|F\|_u \leqslant \max(\|F_1{}^*\|_u , \|F_2{}^*\|_u).$$

Remark 3. Dunham [3] has considered a problem similar to the problem in (d) in the case when $G(f)$ has the betweenness property (see [3] for definition). In (d) the set $G(f)$, being convex, has the betweenness property. We found explicitly the nearest and the farthest points of f in $G(f)$.

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