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ω-Limit Sets for Axiom A Diffeomorphisms

RUFUS BOWEN

Department of Mathematics, University of California, Berkeley, California 94720

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One calls $\underline{x} = \{x_i\}_{i=a}^{b}$ ($a = -\infty$ or $b = +\infty$ is permitted) an *ε-pseudo-orbit* for a homeomorphism $f: X \to X$ if the $x_i \in X$ and satisfy

$$d(fx_i, x_{i+1}) < \epsilon \qquad \text{for all} \quad i \in [a, b-1).$$

A point $x \in X$ *δ-traces* $\underline{x}$ if

$$d(f^i x, x_i) \leqslant \delta \qquad \text{for all} \quad i \in [a, b].$$

Knowing that pseudo-orbits can be traced is useful information for understanding the dynamics of f. This idea will be used here to study the ω-limit sets of Axiom A diffeomorphisms. We will also relate pseudo-orbits to known stability properties of these diffeomorphisms.

1. ω-LIMIT SETS

A homeomorphism $f: X \to X$ is an *abstract ω-limit set* if there is a homeomorphism $g: Y \to Y$ on a compact metric space and a $y \in Y$ so that $g \mid \omega(y)$ is topologically conjugate to $f: X \to X$. Here

$$\omega(y) = \{z \in Y: g^{n_k} y \to z \text{ some } n_k \to +\infty\}.$$

THEOREM 1. *A homeomorphism $f: X \to X$ of a compact metric space is an abstract ω-limit set iff there is no open subset U of X with*

$$\text{(a) } U \neq \varnothing, X$$

and

$$\text{(b) } f(\overline{U}) \subset \operatorname{int} U.$$

Proof. If $f: X \to X$ is an abstract ω-limit set, then we may actually pick $g: Y \to Y$ with $X \subset Y$, $X = \omega(y)$ with $y \in Y$, and $f = g \mid \omega(y)$. Suppose U

was an open subset in X satisfying (a) and (b). Let $2\epsilon = d(X\backslash U, f(\bar{U})) > 0$. Pick $\delta < \epsilon$ so that

$$d(y_1, y_2) < \delta \Rightarrow d(gy_1, gy_2) < \epsilon.$$

Now there is an N so that $d(g^n y, X) < \delta$ for all $n \geqslant N$; otherwise one could find a point in $\omega(y)\backslash X$. Pick $M \geqslant N$ with $d(g^M y, f(\bar{U})) < \epsilon$. Then $d(g^M y, x) < \delta$ for some $x \in X$ and then $d(x, f(\bar{U})) < 2\epsilon$ and so $x \in \bar{U}$. Then

$$d(g^{M+1}y, f(\bar{U})) \leqslant d(g^{M+1}y, fx) < \epsilon.$$

Inductively $d(g^m y, f(\bar{U})) < \epsilon$ for all $m \geqslant M$. This implies

$$(X\backslash U) \cap \omega(y) = \varnothing,$$

a contradiction.

Now assume that $f: X \rightarrow X$ has no open subset U satisfying both (a) and (b).

Claim 1. Let $\epsilon > 0$. For any x', $x'' \in X$ there is an $I = [a, b]$ and an ϵ-pseudo-orbit $\{x_i\}_{i=a}^{b}$ with $x_a = x'$ and $x_b = x''$.

Proof. Fix x' and let V be the set of all x'' for which the claim is true. Then V is open. Furthermore, if $\{x_i\}_{i=a}^{b}$ is an ϵ-pseudo-orbit with $x_a = x'$ and x_b near $z \in \bar{V}$, then $\{x_i\}_{i=a}^{b+1}$ is an ϵ-pseudo-orbit where $x_{b+1} = f(z)$; hence $f(\bar{V}) \subset V$. We must now have $V = \varnothing$ or X; $V = X$ as $x' \in V$.

Claim 2. Let ϵ, $\epsilon' > 0$. For any $x \in X$ there is an ϵ-pseudo-orbit $\{x_i\}_{i=a}^{b}$ with $x_a = x_b = x$ and the set $\{x_i : a \leqslant i \leqslant b\}$ ϵ'-dense in X.

Proof. Let $y_0 = x, y_1, ..., y_m = x$ be ϵ'-dense in X. Then one can find ϵ-pseudo-orbits $\{x_i\}_{i=a_j}^{b_j}$ with $x_{a_j} = y_j$ and $x_{b_j} = y_{j+1}$ for $0 \leqslant j < m$. Reindexing we may assume $b_j = a_{j+1}$. Then $\{x_i\}_{i=a_0}^{b_{m-1}}$ is what we want.

Claim 3. Let $\epsilon > 0$. There is an ϵ-pseudo-orbit $\underset{\sim}{x} = \{x_i\}_{i=-\infty}^{\infty}$ with

$$d(fx_i, x_{i+1}) \rightarrow 0 \text{ as } |i| \rightarrow \infty$$

and $\omega(\underset{\sim}{x}) = \alpha(\underset{\sim}{x}) = X$. Here

$$\omega(\underset{\sim}{x}) = \{z \in X: x_{i_n} \rightarrow z \text{ some } i_n \rightarrow +\infty\},$$

$$\alpha(\underset{\sim}{x}) = \{z \in X: x_{i_n} \rightarrow z \text{ some } i_n \rightarrow -\infty\}.$$

Proof. Pick $x \in X$. Let $\{x_i\}_{i=a_m}^{b_m}$ be an ϵ-pseudo-orbit with $x_{a_m} = x_{b_m} = x$ which is $1/|m| + 1$ – dense in X. We may assume $b_m = a_{m+1}$ for every $m \in Z$. Then $\{x_i\}_{i=-\infty}^{\infty}$ works.

Claim 4. $f\colon X \to X$ is an abstract ω-limit set.

Proof. Let $\underline{x}$ be as in claim 3. For $i \in Z$ define

$$p_i = \begin{cases} (x_i, 1/(2i+1)) & \text{for } i \geqslant 0 \\ (x_i, 1/-2i) & \text{for } i < 0. \end{cases}$$

Let $Y = (X \times \{0\}) \cup \{p_i : i \in Z\} \subset X \times [0, 1]$ and define $g\colon Y \to Y$ by $g(x, 0) = (fx, 0), g(p_i) = p_{i+1}$. The properties of $\underline{x}$ imply that g is a homeomorphism and $\omega(p_0) = \alpha(p_0) = X \times \{0\}$.

Remark. The condition of f says that f admits no nontrivial filtrations (see [9]). The above shows that $f\colon X \to X$ is an abstract ω-limit iff it is an abstract α-limit set.

2. Axiom *A* Diffeomorphisms

Let $f\colon M \to M$ be a diffeomorphism of a compact manifold satisfying Smale's Axiom *A* (see [9]). For $\alpha > 0$ and $x \in M$ let

$$W_\alpha^s(x) = \{y \in M\colon d(f^n y, f^n x) \leqslant \alpha \,\forall\, n \geqslant 0\},$$
$$W_\alpha^u(x) = \{y \in M\colon d(f^{-n} y, f^{-n} x) \leqslant \alpha \,\forall\, n \geqslant 0\}.$$

Then there are constants $c > 0$ and $\lambda \in (0, 1)$ so that when $x \in \Omega$, the nonwandering set of x, one has (for small α)

$$y \in W_\alpha^s(x),\ n \geqslant 0 \Rightarrow d(f^n y, f^n x) \leqslant c\lambda^n d(y, x)$$

and

$$y \in W_\alpha^u(x),\ n \geqslant 0 \Rightarrow d(f^{-n} y, f^{-n} x) \leqslant c\lambda^n d(y, x).$$

This is part of the stable manifold theory of Hirsch and Pugh [5]. Furthermore, for each small $\alpha > 0$ there is a $\beta > 0$ so that

$$W_\alpha^s(x) \cap W_\alpha^u(y) \text{ consists of a single point and lies in } \Omega \text{ whenever } x, y \in \Omega \text{ with } d(x, y) \leqslant \beta.$$

This is a statement of canonical coordinates [9, p. 781].

Lemma. *Let f be an Axiom A diffeomorphism. For each $\delta > 0$ there is an $\epsilon > 0$ so that every ϵ-pseudo-orbit $\underline{x}$ of $f \mid \Omega$ is δ-traced by some $x \in \Omega$.*

Proof. Let $\alpha > 0$ be determined later and $\beta > 0$ as in canonical coordinates above; assume $\beta < \alpha$. Pick M large enough that $c\lambda^M\alpha < \beta/2$ and then $\epsilon > 0$ so small that

$$\text{if } \{y_i\}_{i=0}^{M} \text{ is an } \epsilon\text{-pseudo-orbit for } f \mid \Omega,$$
$$\text{then } d(f^j y_0, y_j) < \beta/2 \text{ for all } j \in [0, M].$$

Consider first an ϵ-pseudo-orbit $\{x_i\}_{i=0}^{rM}$ with $r > 0$. Define x'_{kM} for $k \in [0, r]$ recursively by $x_0' = x_0$ and

$$x_{(k+1)M} = W_\alpha^u(f^M x'_{kM}) \cap W_\alpha^s(x_{(k+1)M}) \in \Omega.$$

This makes sense: $d(f^M x'_{kM}, f^M x_{kM}) \leqslant c\lambda^M\alpha < \beta/2$ since

$$x'_{kM} \in W_\alpha^s(x_{kM})$$

and $d(f^M x_{kM}, x_{(k+1)M}) < \beta/2$ by the choice of ϵ; so $d(f^M x'_{kM}, x_{(k+1)M}) < \beta$ and we can apply canonical coordinates. Now let $x = f^{-rM}x'_{rM}$. For $i \in [0, rM]$ pick s with $i \in [sM, (s+1)M)$. Then

$$d(f^i x, f^{i-sM}x'_{sM}) \leqslant \sum_{t=s+1}^{r} d(f^{i-tM}x'_{tM}, f^{i-tM+M}x'_{(t-1)M})$$
$$\leqslant \sum_{t=s+1}^{r} c\alpha\lambda^{tM-i} \leqslant \frac{c\alpha\lambda}{1-\lambda},$$

where we use $x'_{tM} \in W_\alpha^u(f^M x'_{(t-1)M})$. As

$$x_{sM} \in W_\alpha^s(x_{sM}), d(f^{i-sM}(x'_{sM}), f^{i-sM}x_{sM}) \leqslant \alpha;$$

by the choice of ϵ one has

$$d(f^{i-sM}x_{sM}, x_i) < \beta/2.$$

By the triangle inequality

$$d(f^i x, x_i) < \alpha + \beta/2 + c\alpha\lambda/(1-\lambda).$$

For small α, this is less than δ.

Now any ϵ-pseudo-orbit $\{x_i\}_{i=0}^{n}$ extends to $I = [0, rM]$ when $rM \geqslant n$ by letting $x_i = f^{i-n}x_n$ for $i \in (n, rM]$. An x which δ-traces this extended pseudo-orbit will δ-trace the original one on $[0, n]$. If $\{x_i\}_{i=a}^{b}$ is an ϵ-pseudo-orbit, then $\{x_{j+a}\}_{j=0}^{b-a}$ is also and x δ-tracing this one yields $f^{-a}x$ which δ-traces the first. Thus every ϵ-pseudo-orbit on a finite interval is δ-traced. Finally, if $\{x_i\}_{i=-\infty}^{\infty}$ is an ϵ-pseudo-orbit, then let $x^{(m)}$ δ-trace $\{x_i\}_{t=-m}^{m}$ and let x be the limit of some subsequence of the $x^{(m)}$'s. Then x δ-traces $\{x_i\}_{i=-\infty}^{\infty}$.

Remark. The above result was stated by Sinai [8, p. 38] for the case of an Anosov diffeomorphism and the proof in that case is a variation of something in [1]. The above proof is really in [2, p. 30] and [3, p. 381]. The above lemma is closely related to the specification property [3] which is relevant to the entropy theory of f.

THEOREM 2. *Let $f: M \to M$ be an Axiom A diffeomorphism and $X \subset \Omega$ a compact f-invariant set which is an abstract ω-limit set. Then $X = \omega(x)$ for some $x \in \Omega$.*

Proof. Let $\underline{x}$ be an ϵ-pseudo-orbit for $f \mid X$ as constructed in the proof of theorem 1 with $\omega(\underline{x}) = X$. For $\delta > 0$, provided ϵ is small enough, one can find $x \in \Omega$ δ-tracing $\underline{x}$ by the lemma. Because $d(fx_i, x_{i+1}) \to 0$ as $i \to \infty$, for any $N > 0$ one can find a K so that

$$d(f^{i+j}x, f^j x_i) < 2\delta \qquad \text{for all} \quad j \in [-N, N]$$

whenever $i \geqslant K$. Now 2δ is an expansive constant for $f \mid \Omega$ when δ is small (i.e., when $x \neq y$ are in Ω, there is an $n \in Z$ with $d(f^n x, f^n y) > 2\delta$); this is what the statement $W^s_{2\delta}(x) \cap W^u_{2\delta}(x) = \{x\}$ says. From expansiveness, for any $\alpha > 0$ there is an $N = N(\alpha)$ so that

$$y, z \in \Omega, \quad d(f^j y, f^j z) < 2\delta \;\forall j \in [-N, N]$$
$$\Rightarrow d(y, z) < \alpha.$$

One sees now that $d(f^i x, x_i) \to 0$ as $i \to \infty$. From this follows $\omega(x) = \omega(\underline{x}) = X$.

We remark that if f satisfies the no-cycle property (see [10]), then any abstract ω-limit set in M actually is in Ω and so the above theorem applies to it. Whether or not f has cycles, all actual ω-limit sets for f are contained in Ω. If X, Y are abstract ω-limit sets lying inside a single basic set Ω_i of Ω (see [9]), then using the transitivity of $f \mid \Omega_i$ one can get $x \in \Omega_i$ with $\alpha(x) = Y$ and $\omega(x) = X$.

We next see how to use pseudo-orbits to recover the fundamental neighborhoods of [4].

THEOREM 3[4]. *Let $f: M \to M$ be an Axiom A diffeomorphism. Then Ω has a neighborhood U so that $\bigcap_{n \in Z} f^n U = \Omega$.*

Proof. Let $\delta > 0$ be small and pick $\epsilon > 0$ as in the lemma. Pick $\gamma < \epsilon/2$ so that

$$x, y \in M, \quad d(x, y) < \gamma \Rightarrow d(fx, fy) < \epsilon/2.$$

Let $U = \{z \in M: d(z, \Omega) < \gamma\}$. If $z \in \bigcap_{n \in Z} f^n U$, pick $x_i \in \Omega$ so that

$$d(f^i z, x_i) < \gamma.$$

Then

$$d(fx_i, x_{i+1}) < d(fx_i, f^{i+1}z) + d(f^{i+1}z, x_{i+1}) < \epsilon;$$

that is, $\{x_i\}_{i=-\infty}^{\infty}$ is an ϵ-pseudo-orbit. Let $x \in \Omega$ δ-trace it. Then

$$d(f^i x, f^i z) < \gamma + \delta \leqslant 2\delta \qquad \text{for all} \quad i$$

and so

$$z \in W^s_{2\delta}(x) \cap W^u_{2\delta}(x) = \{x\},$$

i.e., $z = x \in \Omega$.

Pseudo-orbits can be viewed as orbits which arise when one adds stochastic perturbations to f or alternately if f is only approximately well-defined. That is, after applying f to the point x_i one allows a small perturbation of fx_i to x_{i+1}. In the Anosov case the average behavior of ϵ-pseudo-orbits tends in some sense to a certain Gibbs measure as $\epsilon \to 0$ [8, p. 37–39](this Gibbs measure is the smooth invariant measure if one exists). One expects similar behavior for Axiom A diffeomorphisms without cycles. Here, for any neighborhood U of Ω, there is an $\epsilon > 0$ so that every ϵ-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ eventually lies in $U(x_i \in U$ for large $i)$, and then is δ-traced by some point $x \in \Omega$. Most pseudo-orbits should eventually be near attractors; a paper of Ruelle [7] makes it seem hopeful that the result of [8] mentioned above can be extended to attractors.

Pseudo-orbits are related to stability theorems. Let U be a small neighborhood of Ω and $g: M \to M$ be a homeomorphism C^0-near the Axiom A diffeomorphism f. Then for $y \in \Lambda(g) = \bigcap_{n \in Z} f^n \bar{U}$ one picks $x_i(y) \in \Omega$ near $f^i y$. Then $\{x_i(y)\}_{i=-\infty}^{\infty}$ will be an ϵ-pseudo-orbit for $f \mid \Omega$; this is just like the proof of Corollary 2. Let $h(y) \in \Omega$ δ-trace $\{x_i(y)\}_{i=-\infty}^{\infty}$. Because $f \mid \Omega$ is expansive, $h(y)$ will be uniquely determined provided δ is small enough. From expansiveness will also follow that $h: \Lambda(g) \to \Omega$ is continuous. It is easy to check that $fh = hg$. Nitecki [6] showed that h is a surjection. Walters [11] did this first for Anosov f; in this case one can conclude that h is surjective because it is C^0-near the identity and defined on all of the manifold M.

In case g is C^1-near f, g is also Axiom A. Because the stable and unstable manifolds of g vary C^1-continuously as g varies C^1-continuously [5], the same ϵ's and δ's above can be used for g also. One then gets $h^*: \Omega = \Lambda(f) \to \Lambda(g)$. Now $x' = hh^*(x) \in \Omega$ has the property that

$$d(f^i(x), f^i(x')) \text{ is small for all } i;$$

by expansiveness $x' = x$. Similarly $h^*h = \mathrm{id}_{\Lambda(g)}$ and we conclude that $h: \Lambda(g) \to \Omega$ is a homeomorphism. This is the local part of Ω-stability [10], [5].

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