Characters of Finite Quasigroups VI: Critical Examples and Doubletons

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Examples of association schemes coming from symmetric group actions on doubletons are shown to have character tables which are not character tables of quasigroups. The examples are critical in separating the character theory of quasigroups from the general theory of association schemes.

1. CRITICAL EXAMPLES

The combinatorial character theory of quasigroups developed in earlier papers of this series [1–5] assigns a character table $\Psi$ to a non-empty, finite quasigroup $Q$ [1, Defn. 3.3]. The character table is defined and normalized so that, if the quasigroup is a group, then the quasigroup character table is exactly the group character table. Quasigroup character tables satisfy the orthogonality relations [1, Th. 3.4] satisfied by group character tables. However, it is easy to produce quasigroup character tables which are not group character tables. The most obvious examples such as those of [1, §4] and [2; §§6, 8] have 'character degrees' (i.e. entries in the first column) which are not positive integers. A slightly less direct example is the table of [3, (7.1)]. This is the character table of the many loops and quasigroups of order 5 which have a doubly transitive multiplication group. Since the unique group of order 5 does not have a doubly transitive multiplication group, the table is a loop character table which is not a group character table. In similar vein, the table

\[
\begin{bmatrix}
1 & 1 & 1 \\
\sqrt{2} & \sqrt{2}\cos 72^\circ & -\sqrt{2}\cos 36^\circ \\
\sqrt{2} & -\sqrt{2}\cos 36^\circ & \sqrt{2}\cos 72^\circ
\end{bmatrix}
\]

(1.1)

of [3, (7.3)] is a quasigroup character table which is not a loop character table. Certainly it is the character table of the quasigroup $(\mathbb{Z}_5, -)$. On the other hand, Albert's classification [6] of loops of order 5 shows that a non-abelian loop of that order must have a doubly transitive multiplication group. Critical examples such as [3, (7.1)] and [3, (7.3)] are useful in demarcating the domains of the respective theories. Thus [3, (7.1)] separates loop character theory from group character theory, while (1.1) shows that loop character theory cannot encompass quasigroup character theory.

The character table $\Psi$ of the quasigroup $Q$ is obtained from the association scheme $(Q, \Gamma)$, where the partition $\Gamma = \{C_1 = \bar{Q}, C_2, \ldots, C_n\}$ is the set of orbits of the multiplication group $G$ of $Q$ in its diagonal action on $Q^2$. The respective set of incidence matrices $\{a_1 = 1, a_2, \ldots, a_s\}$ of the elements of the $\Gamma$ forms a basis for a commutative subalgebra of the complex algebra that also has a basis $\{e_1, e_2, \ldots, e_s\}$ of orthogonal idempotents. If the basis changes are

\[
a_i = \sum_{j=1}^s \xi_{ij}e_j \quad \text{and} \quad e_i = \sum_{j=1}^s \eta_{ij}a_j,
\]

(1.2)
then the $s \times s$ matrix $\Psi = (\psi_{ij})$ is defined by

$$\psi_{ij} = (f_i)^{1/2} \xi_{ji} n^{-1} = n(f_i)^{-1/2} \eta_{ij},$$

(1.3)

where $n = |Q|$, $nn_j = |C_j|$, and $f_i = tr e_i$ [1, Defn. 3.3]. In the theory of association schemes, the transposes of the matrices $\Xi = (\xi_{ji})$ and $nH = (n\eta_{ij})$ are respectively referred to as the first and second eigenmatrices of the association scheme $(Q, \Gamma)$. More generally, (1.3) holds for an arbitrary (commutative, not necessarily symmetric) association scheme $(Q, \Gamma)$ described in the notation of [4, Defn. 4.1], and thus may be used similarly to define a character table $\Psi = (\psi_{ij})$ for the association scheme. Such a matrix $\Psi$ will be called a scheme character table, or more specifically the character table of the scheme $(Q, \Gamma)$. (One should be careful to distinguish this usage from that of 'character table' to refer to the first eigenmatrix of an association scheme.)

The aim of the current note is to present critical examples of scheme character tables which are demonstrably not quasigroup character tables. Immediate sporadic examples are offered by the character tables of the schemes given by strongly regular graphs with 26 vertices. As a consequence of the complete classification of these graphs due to M. Z. Rosenfeld and others [7, pp. 175–182], the automorphism group of a scheme with such a character table is known to be intransitive. Since the automorphism groups of the schemes given by quasigroups are transitive, no quasigroup can have one of these character tables. The main task of this paper is to present members of an infinite family of examples, such as

$$\begin{bmatrix}
1 & 1 & 1 \\
\sqrt{7} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} \\
2\sqrt{5} & \frac{1}{2\sqrt{5}} & \frac{1}{5\sqrt{5}}
\end{bmatrix},$$

(1.4)

of scheme character tables which are demonstrably not quasigroup character tables. This is done in Theorem 2.4 and Corollary 4.3 below. The tables are those of certain Johnson schemes $J(r, 2)$, given by the action of the symmetric group $S_r$ on the set of doubletons (2-element subsets) of the set of $r$ elements on which $S_r$ acts. In passing, the third section of the paper discusses certain basic aspects of isotopy and fusion [3, §3]. These are needed in the fourth section, but are also of independent interest. Moreover Theorem 4.1, used during the derivation of Corollary 4.3, gives a complete classification of sharply transitive subsets of the group (4.2) of linear polynomials under substitution acting on doubletons from a finite odd-order field.

The search for the critical examples was motivated by the following comment [8, p. 304; 9, p. 325] on the first two papers [1, 2] of this series:

'We think that most of their calculations are carried out in the context of association schemes without using any algebraic structures of quasigroups or loops'.

The examples thus become critical in demarcating the theory of association schemes from the character theory of quasigroups, suggesting that quasigroup character tables may possess special properties not exhibited by general scheme character tables. Earlier, awareness of the critical example (1.1) or [3, (7.3)] demarcating loop character theory from quasigroup character theory led to recognition of a property, viz. 'the character table of $Q$ determines the character table of $Q \times Q$'; that is true if $Q$ is a loop but which may be false if $Q$ is a quasigroup [4].

It is especially significant that (1.4) is not a quasigroup character table, since it is the character table of the scheme given by the action of the derived subgroup $O^2(2)$ of the isometry group acting on the 28-element set of anisotropic points of the 6-dimensional vector space over $GF(2)$ with quadratic form $x_1x_4 + x_2x_5 + x_3x_6$. (See [10, Table 4],
where 'character table' is used in the sense of 'first eigenmatrix'. In fact (1.4) is obtained by substituting \( r = 2 \) into the table

\[
\begin{bmatrix}
1 & 1 & 1 \\
\frac{\sqrt{(2^r+1)(2^r-1)}}{3} & \frac{\sqrt{2^{r+1}-1}}{3(2^r-1)} & -\frac{\sqrt{2^{r+1}-1}}{3(2^r-1)} \\
2\sqrt{\frac{2^r-1}{3}} & -\sqrt{\frac{2^r+1}{3(2^r-1)}} & \frac{2^r-2}{\sqrt{3(2^r-1)}}
\end{bmatrix}.
\]

For each positive integer \( r \), (1.5) is the character table of an association scheme [10, 11]. For \( r = 1 \), it is the character table of the symmetric group \( S_3 \). For \( r = 3 \), it is the character table of the simple non-associative Moufang loop \( M(2) \) of order 120 (cf. [12]). The natural question thus arose (cf. [11, Ch. II]) as to whether (1.5) is the character table of a quasigroup for each positive integer \( r \). Theorem 2.4 gives a negative answer.

2. Loop Transversals and Johnson Schemes

Let \( S \) be a subgroup of a group \( G \), and \( T \) a (right) transversal from \( G \) to \( S \), so that \( G = \bigcup_{t \in T} St \). For \( g \) in \( G \), define \( \bar{g} \) in \( T \) by \( g \in S\bar{g} \). A binary operation + on \( T \) is defined by \( t + u = \bar{u} \). For each \( u \) in \( T \), the mapping

\[ R_T(u) : T \rightarrow T; t \mapsto t + u \]

is a bijection of \( T \) [13, Th. 1.1; 14; 15, 2.2]. The transversal \( T \) is said to be a loop transversal if \((T, +, 1)\) is a loop, i.e. if each mapping \( L_T(u) : T \rightarrow T; t \mapsto u + t \) is also a bijection of \( T \). Recall that a set \( A \) of bijections of a set \( Q \) is said to be sharply transitive [16] (Baer used the term 'simply transitive' [13, p. 119]) if, for each pair \((q, r)\) in \( Q^2 \), there is a unique \( \alpha \) in \( A \) with \( q\alpha = r \). The following three well known propositions on loop transversals and sharply transitive sets are essential for the proof of Theorem 2.4 below.

**Proposition 2.1** [13, p. 119]. (i) Let \( T \) be a transversal from a group \( G \) to a subgroup \( S \). Then the set \( \{R_T(u) \mid u \in T\} \) of bijections of \( T \) is sharply transitive iff \( T \) is a loop transversal.

(ii) Let \( A \) be a subset of a group \( G \) of bijections of a non-empty set \( Q \). Let \( S \) be the stabilizer in \( G \) of an element of \( Q \). Then \( A \) is sharply transitive iff it is a loop transversal from \( G \) to \( S \). Moreover, for each element \( g \) of \( G \), \( A \) is sharply transitive \( \iff \) \( gA \) is sharply transitive \( \iff \) \( Ag \) is sharply transitive. \( \square \)

**Proposition 2.2.** Let \( Q \) be a quasigroup with multiplication group \( G \), considered as a group of permutations of \( Q \). Then \( G \) contains a sharply transitive subset that includes the identity permutation.

**Proof.** If \( Q \) is empty, the result is immediate. Otherwise, let \( S \) be the stabilizer in \( G \) of an element \( e \) of \( Q \). A loop transversal \( T \) from \( G \) to \( S \) is provided by \( \{\rho(e, y) \mid y \in Q\} \), where [15, 226]

\[ \rho(e, y) = R(e \backslash e)^{-1}R(e \backslash y). \]

Then \( T \) is a sharply transitive subset of \( G \) by Proposition 2.1. Also \( \rho(e, e) = 1 \). \( \square \)
Proposition 2.3. Let $r$ be an integer greater than 2. Let $S$, be the symmetric group $r!$ of bijections of the set $r = \{1, \ldots, r\}$. Let $S^2$ denote the monomorphic image of $S$, under the permutation representation

$$S \to \binom{r}{2}; \sigma \mapsto \{(i, j) \mapsto (i\sigma, j\sigma)\}$$

of $S$, on unordered pairs of distinct elements of $r$. Then $S^2$ contains no sharply transitive subset if $r$ is even. \hfill \Box

A direct, combinatorial proof of Proposition 2.3 was given in [17, Th. 4.3]. Nomura [18, Cor.] extended the combinatorial method, and mentioned [18, Rem. 2] an alternative, character-theoretic proof of Proposition 2.3. (We are grateful to T. Grundhöfer for bringing [16] to our attention.)

The permutation representation of $S^2$ on $Q = (\{\})$ is multiplicity-free, so that the orbits of the diagonal action of $S^2$ on $Q^2$ give an association scheme, the Johnson scheme $J(r, 2)$ [9, III.2; 19; 20, §4.2] or 'triangular' association scheme [21, 22]. The character tables of these schemes may be computed from the Eberlein polynomials [9, 19, Th. III. 2.10; 20, Th. 4.6]. In particular, (1.4) is the scheme character table for $J(8, 2)$.

The first main result may now be given:

Theorem 2.4. Let $r$ be an even integer greater than 4. Then the character table of the Johnson scheme $J(r, 2)$ is not a quasigroup character table.

Proof. Suppose that $Q$ is a quasigroup whose character table is the character table of the scheme $J(r, 2)$, for even $r$ greater than 4. Let $G$ be the multiplication group of $Q$, giving association scheme $(Q, \Gamma)$ from the orbits of $G$ on $Q^2$. For the values of $r$ being considered, the Johnson scheme $J(r, 2)$ is uniquely determined by its character table unless $r = 8$ [22–24]. If $r = 8$, there are three exceptional association schemes with the same character table. They are described in [7, pp. 184–185]. Their relations are distance relations in strongly regular graphs. However, the automorphism groups of these graphs are not transitive on the 28 vertices: one has an orbit of length 8, while the other two have orbits of length 4. Since the transitive group $G$ is a subgroup of the group of automorphisms of the relational structure $(Q, \Gamma)$, it follows that $(Q, \Gamma)$ cannot coincide with any of these exceptional schemes. Thus $(Q, \Gamma)$ coincides with the Johnson scheme $J(r, 2)$ in each case.

Now by the hypothesis on $r$, the permutation representation $S^2$ is 2-closed [25, p. 134], so that the permutation representation of $G$ on $Q$ is similar to the permutation representation of a subgroup $H$ of $S^2$. By Proposition 2.2, $G$ contains a sharply transitive subset, whence $H$ does also, by the similarity. But this implies that $S^2$ contains a sharply transitive subset, a contradiction to Proposition 2.3. \hfill \Box

3. Isotopy and Fusion

An isotopy from a quasigroup $(P, +)$ to a quasigroup $(Q, \cdot)$ is a triple $(\alpha, \beta, \gamma)$ of bijections from $P$ to $Q$ such that

$$\forall x, y \in P, \quad x^\alpha \cdot y^\beta = (x + y)^\gamma. \quad (3.1)$$

If such a triple exists, the quasigroups $(P, +)$ and $(Q, \cdot)$ are said to be isotopic. Then the relation of being isotopic, also known as isotopy, is an equivalence relation
amongst quasigroups. This relation includes the relation of isomorphism, since a quasigroup isomorphism \( \theta \) is an isotopy \( (\theta, \theta, \theta) \). Note that isotopic groups are isomorphic. With isotopies as morphisms, quasigroups are the objects of a (large) category. In this category, the equation \((\alpha, \beta, \gamma) = (\gamma, \gamma, \gamma)(\gamma^{-1} \alpha, \gamma^{-1} \beta, 1_Q)\) holds. Thus to within isomorphism, it suffices to study principal isotopies, namely isotopies \((\alpha, \beta, 1_Q)\) with the identity mapping \(1_Q\) on a set \(Q\) as third component.

In general, the character tables of isotopic finite, non-empty quasigroups may bear very little relationship to each other. The following theorem gives cases where there is a strong relationship. The process of obtaining certain schemes and character tables from others by fusion is described in [3, §3]:

**Theorem 3.1.** (i) Isotopic loops have identical character tables. Moreover, principally isotopic loops have identical association schemes.

(ii) If a quasigroup is isotopic to a loop, then the quasigroup character table is obtained by fusion from the loop character table. Moreover, if the quasigroup is principally isotopic to the loop, then the quasigroup scheme is obtained by fusion from the loop scheme.

**Proof.** Statement (i) is an immediate consequence of (ii) and the symmetry of the isotopy relation. To prove (ii), it suffices to consider a principal isotopy \((\alpha, \beta, 1_Q)\) from a loop \((Q, +)\) with identity element \(e\) to a quasigroup \((Q, \cdot)\). Then specializing (3.1),

\[
\forall x, y \in Q, \quad x^\alpha \cdot y^\beta = x + y.
\]  
(3.2)

In the respective quasigroups \((Q, +)\) and \((Q, \cdot)\), denote the right multiplications by \(R_+(y), R.(y)\) and the left multiplications by \(L_+(x), L.(x)\). By (3.2),

\[
\beta L.(x^\alpha) = L_+(x) \quad \text{and} \quad \alpha R.(y^\beta) = R_+(y).
\]
(3.3)

Since \(R_+(e) = L_+(e) = 1_Q\), one has \(\beta = L.(e^\alpha)^{-1}\) and \(\alpha = R.(e^\beta)^{-1}\). Then \(\text{Mlt}(Q, +) = \langle L_+(x), R_+(x) \mid x \in Q \rangle = \langle L.(e^\alpha)^{-1}L.(x^\alpha), R.(e^\beta)^{-1}R.(x^\beta) \mid x \in Q \rangle \cong \langle L.(x), R.(x) \mid x \in Q \rangle = \text{Mlt}(Q, \cdot)\). By [3, Th. 3.1], it follows that the quasigroup scheme is obtained by fusion from the loop scheme, as required. 

The scope of Theorem 3.1(ii) is very broad. Let \(Q = (Q, \cdot)\) be a finite, non-empty quasigroup, with element \(e\). Let \(S\) denote the stabilizer of \(e\) in the multiplication group \(G\) of \(Q\). Let \(T\) denote the loop transversal from \(G\) to \(S\{p(e, y) \mid y \in Q\}\) used in the proof of Proposition 2.2. The bijection \(\lambda: T \to Q; t \mapsto et\) may then be used to induce a loop structure \((Q, +, e)\) on \(Q\) isomorphic with the loop structure \((T, +, 1)\) on \(T\).

**Proposition 3.2.** The quasigroup \((Q, \cdot)\) is principally isotopic to the loop \((Q, +)\).

**Proof.** The bijection \(\lambda: T \to Q\) has \(\mu: Q \to T; y \mapsto \rho(e, y)\) as its two-sided inverse. Note that \(\tilde{g} = \rho(e, g)\) for \(g\) in \(G\). Then for \(x, y \in Q\), one has \(xR_+(y) = xR_+(\rho(e, x)) = \rho(e, x)\rho(e, y) \lambda = \rho(e, x\rho(e, y)) \lambda = x\rho(e, y)\), whence \(R_+(y) = \rho(e, y)\). By (2.2), this means that \(R_+(y) = R.(e^\alpha)^{-1}R.(yL(e)^{-1})\). Comparing (3.2) with the second equation of (3.3), a principal isotopy from \((Q, +)\) to \((Q, \cdot)\) is given by \((R.(e)^{-1}, L.(e)^{-1}, 1_Q)\).

4. **Odd Prime Power Cases**

For an even integer \(r\) greater than 2, it was observed in Proposition 2.3 that the permutation group \(S_r^2\) on doubletons has no sharply transitive subset. In general, little
is known about the existence of sharply transitive subsets of $S_r$ for odd $r$, unless $r$
 happens to be an odd prime power $q$. It is then convenient to identify the set $q$ with
the field $F = GF(q)$. Let $\zeta$ generate the cyclic group $F^*$ of non-zero field elements
under multiplication. In [17, Th. 4.2], it was shown that the $\frac{q(q-1)}{2}$-element subset
$T = \{ F \to F; x \mapsto \zeta^i x + c \mid c \in F, 0 < i \leq \frac{1}{2}(q-1) \}$
(4.1)
of $S_q$ forms a sharply transitive subset of $S_q^2$. Note that $\{ \zeta^i \mid 0 < i \leq \frac{1}{2}(q-1) \}$ is a
transversal from $F^*$ to its two-element subgroup $\{ \pm 1 \} = \{ \zeta^0, \zeta^{(q-1)/2} \}$. Let
$L_q = \{ x \mapsto mx + c \mid m \in F^*, c \in F \}$
(4.2)
denote the subgroup of $S_q$ comprising the action of linear polynomials, composed by
substitution. Let $L_q^2$ denote the corresponding subgroup of the permutation group $S_q^2$.
The set $T$ of (4.1) is a sharply transitive subset of $L_q^2$. The following theorem describes
all sharply transitive subsets of $L_q^2$.

**Theorem 4.1.** Let $q$ be an odd prime power. Let $U$ be a transversal from $F^*$ to
$\{ \pm 1 \}$. Then
$T = T(U) = \{ x \mapsto ux + c \mid u \in U, c \in F \}$
(4.3)
is a sharply transitive subset of $L_q^2$. Conversely, each sharply transitive subset $T$ of $L_q^2$
is of the form $T = T(U)$ for some transversal $U$ from $F^*$ to $\{ \pm 1 \}$. In particular, there are
$2^{(q-1)/2}$ distinct sharply transitive subsets of $L_q^2$.

**Proof.** Define the moment map
$$\begin{pmatrix} F \\ 2 \end{pmatrix} \to F \times F^{*2}; \{ a, b \} \mapsto (\frac{1}{2}(a + b), \frac{1}{2}(a - b)^2).$$
(4.4)
The moment map is a bijection, having
$$F \times F^{*2} \to \begin{pmatrix} F \\ 2 \end{pmatrix}; (\mu, \sigma^2) \mapsto \{ \mu + \sigma, \mu - \sigma \}$$
(4.5)
as its two-sided inverse. Intuitively one may regard a doubleton $\{a, b\}$ as a probability
distribution on $F$ assigning weight $\frac{1}{2}$ to each of $a$ and $b$. The element $\mu$ of $F$ is then
thought of as the mean of the distribution, and $\sigma^2$ as its variance. Rather than studying
the action of $S_q^2$ on $\binom{F}{2}$, it is computationally more convenient to study the action of $S_q^2$
on $F \times F^{*2}$. These actions are similar via the moment map and its inverse. In
particular, a familiar calculation from elementary probability theory shows that an
element $x \mapsto mx + c$ of $L_q^2$ acts on $F \times F^{*2}$ as
$$(\mu, \sigma^2) \mapsto (m\mu + c, m^2\sigma^2).$$
(4.6)
Now suppose given $U$ and $T(U)$ as in (4.3). For an ordered pair $((\mu, \sigma^2), (v, \tau^2))$ of
elements of $F \times F^{*2}$, there is a unique element $x \mapsto ux + c$ of $T(U)$ taking $(\mu, \sigma^2)$ to
$(v, \tau^2)$. By the second component of (4.6), one must have $u^2\sigma^2 = \tau^2$, whence $u$
is determined as the unique element of $U$ which squares to $\tau^2/\sigma^2$. The first component of
(4.6) then yields the equation $u\mu + c = v$, having the unique solution $c = v - u\mu$. This
shows that the $T(U)$ are sharply transitive subsets of $L_q^2$, as claimed.

Conversely, suppose given a sharply transitive subset $T$ of $L_q^2$. Now (4.6) specializes to
$$\begin{pmatrix} 0, 1 \end{pmatrix} \mapsto (c, m^2).$$
(4.7)
As $x \mapsto mx + c$ ranges over the $q(q-1)/2$ elements of $T$, the pair $(c, m^2)$ must range
over the $q(q-1)/2$ elements $F \times F^{*2}$. It thus remains to check that $f: x \mapsto mx + c$ and
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\[ g: x \mapsto -mx + d \]
cannot both lie in \( T \). Write \( f: x \mapsto m(x - a) \) and \( g: x \mapsto m(b - x) \). Then \( f \) and \( g \) both map \( (\frac{1}{2}(a + b), 1) \) in \( F \times F* \) to \( (\frac{1}{2}m(b - a), m^2) \), a contradiction to the sharp transitivity. \( \Box \)

**Remark.** Theorem 4.1 admits a graphical interpretation. For any set \( V \) of permutations on a finite set \( X \), let \( \Gamma(V) \) denote the undirected graph with vertex set \( V \) and edge set

\[ \{ \{\alpha, \beta\} \subseteq V \mid d(\alpha, \beta) < |X| \} \]

where

\[ d(\alpha, \beta) = |\{ x \in X \mid x\alpha \neq x\beta \}| \]

is the Hamming metric on the symmetric group \( X! \) on \( X \) [26, §6B3]. A subset of \( V \) is sharply transitive iff it induces a subgraph of \( \Gamma(V) \) consisting of \( |X| \) isolated vertices. Then in the context of Theorem 4.1, \( \Gamma(L_q^2) \) decomposes as a disjoint union of subgraphs \( K(u) \) indexed by the elements \( u \) of the transversal \( U \). Each \( K(u) \) is a complete bipartite graph on the sets

\[ \{ x \mapsto ux + c \mid c \in F \} \]

and

\[ \{ x \mapsto -ux + c \mid c \in F \} \]

of vertices of \( \Gamma(L_q^2) \).

For an odd prime power \( q \), the existence of the sharply transitive subsets (4.3) prevents one from using the argument of the proof of Theorem 2.4 in an attempt to show that no quasigroup \( (Q, .) \) may have the character table of the Johnson scheme \( J(q, 2) \). Thus it is conceivable that there might be a quasigroup structure \( (Q, .) \) on the set \( Q = (F) \) such that, for some \( e \) in \( Q \), the sharply transitive set \( \{ \rho(e, y) \mid y \in Q \} \) of Proposition 2.2 is of the form (4.3). The following proposition, however, shows that this cannot happen for \( q > 3 \).

**Proposition 4.2.** Let \( q > 3 \) be an odd prime power. Let \( (Q, .) \) be a quasigroup structure on the set \( Q = (F) \) with the character table of \( J(q, 2) \). Then for no \( e \) in \( Q \) can the sharply transitive set \( \{ \rho(e, y) \mid y \in Q \} \) be of the form \( T = T(U) \) as in (4.3).

**Proof.** For the values of \( q \) being considered, the Johnson scheme \( J(q, 2) \) is uniquely determined by its character table [22–24]. Thus the non-diagonal orbits of the multiplication group \( G \) of \( (Q, .) \) on \( Q^2 \) are \( C_2 = \{(A, B) \mid |A \cap B| = 1\} \) and \( C_3 = \{(A, B) \mid |A \cap B| = 0\} \). Suppose that, for some \( e \) in \( Q \), the set \( \{ \rho(e, y) \mid y \in Q \} \) is of the form \( T(U) \). Without loss of generality (cf. Proposition 2.1(ii)), one may take \( e = \{ \pm 1 \} \).

In \( F* \) one has \( U \cap \{ \pm 1 \} = \{1\} \), since \( \rho(e, e) = 1 \). The bijection \( \lambda: T \mapsto Q; t \mapsto et \) gives a loop structure \( (Q, +, e) \) on \( Q \) isomorphic with the loop structure \( (T, +, 1) \) on \( T \). By Proposition 3.2, the quasigroup \( (Q, .) \) is principally isotopic to the loop \( (Q, +) \). By Theorem 3.1(ii), the quasigroup scheme is obtained by fusion from the loop scheme. Thus the \( (Q, .) \)-classes \( C_2 \) and \( C_3 \) are unions of \( (Q, +)-\)classes.

For a non-zero element \( a \) of \( F \), define \( \hat{a} \) in \( U \) by \( a \in \{ \pm 1 \} \hat{a} \). Consider an element \( ax + b \) of \( L_q \). Its representative in \( T = T(U) \) is

\[ \overline{ax + b} = \hat{a}x + b. \]  

(4.8)

For elements \( ux + d, wx + e \) of \( T \), it follows that

\[ (ux + d) + (wx + e) = (uv)ux + (wd + e) \]  

(4.9)

in the loop \( (T, +, 1) \). Then

\[ (ux + d)R_T(wx + e)L_T(wx + e)^{-1} = ux + [dw + (1 - v)e]. \]  

(4.10)
Now $R_T(wx + e)L_T(wx + e)^{-1}$ is an element of the multiplication group of the loop $(T, +, 1)$ that stabilizes 1. Thus for elements $t, s$ of $T$, the pairs $(1, t)$ and $(1, R_r(s)L_r(s)^{-1})$ both lie in the same $(T, +)$-class. Take $t = ux + (u - 1)$ for some element $u$ of $U$ distinct from 1. Take $s = ux + u$. One obtains the pairs $(1, ux)$—using (4.10)—lying in the same $(T, +)$-class. Mapping via the bijection $A.: T \rightarrow Q$, one obtains the pairs $({\pm 1}, {-1,2u-1})$ and $({\pm 1}, {\pm u})$ lying in the same $(Q, +)$-class. However, the first pair lies in the $(Q, +)$-class $C_2$, while the second pair lies in the $(Q, +)$-class $C_3$. This is a contradiction, since $C_2$ is a union of $(Q, +)$-classes. □

**Corollary 4.3.** The character table of the Johnson scheme $J(5, 2)$ is not a quasigroup character table.

**Proof.** For $q = 5$, a computer search shows that the only sharply transitive subsets of $S^2_q$ that contain $1$ are those lying entirely in $L^2_q$, and thus of the form $T = T(U)$. Proposition 4.2 then shows that there can be no quasigroup with the character table of $J(5, 2)$. □

**Acknowledgement**

We are indebted to a referee for many helpful suggestions, in particular for pointing out sporadic examples such as those mentioned in Section 1 and the graphical interpretation of Theorem 4.1.

**References**

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*Received 2 May 1989 and accepted in revised form 3 January 1990*

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