Local dendrites with unique hyperspace $C(X)$

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**Abstract**

For a continuum $X$ we denote by $C(X)$ the hyperspace of subcontinua of $X$, metrized by the Hausdorff metric. Let $\mathcal{D}$ be the class of dendrites whose set of end points is closed and let $\mathcal{LD}$ be the class of local dendrites $X$ such that every point of $X$ has a neighborhood which is in $\mathcal{D}$. In this paper we study the structure of the classes $\mathcal{D}$ and $\mathcal{LD}$. As an application, we show that if $X \in \mathcal{LD}$ is different from an arc and a simple closed curve, and $Y$ is a continuum such that the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, then $X$ is homeomorphic to $Y$.

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1. Introduction

A continuum is a nonempty compact, connected, metric space. All concepts not defined here will be taken as in [35]. Given a continuum $X$, we consider the family:

$$2^X = \{ A \subset X : A \text{ is nonempty and closed} \}.$$  

The topology on $2^X$ will be induced by the Hausdorff metric $H$ [35, Definition 0.1, p. 1]. We also consider the following subspaces of $2^X$:

$$C(X) = \{ A \in 2^X : A \text{ is connected} \}, \quad F_1(X) = \{ \{ p \} : p \in X \}$$

and, if $p \in X$,

$$C(p, X) = \{ A \in C(X) : p \in A \}.$$  

Both $2^X$ and $C(X)$ are called hyperspaces of $X$. It is known that $2^X$ and $C(X)$ are arcwise connected continua [27, Corollary 14.10, p. 114]. It is also known that $F_1(X)$ is homeomorphic to $X$.

A continuum $X$ has unique hyperspace $C(X)$ if for every continuum $Y$ such that the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, then $X$ and $Y$ are homeomorphic.

The topic of this paper is inserted in the following general problem.

**Problem.** Find conditions, on the continuum $X$, in order that $X$ has unique hyperspace $C(X)$.
A finite graph is a continuum that can be written as the union of finitely many arcs, each two of which are either disjoint or intersect only in one or both of their end points. A tree is a finite graph that contains no simple closed curves. Let
\[ \mathcal{G} = \{ X : X \text{ is a finite graph} \}. \]

It has been proved that:

(a) If \( X \in \mathcal{G} \) is different from an arc and a simple closed curve, then \( X \) has unique hyperspace \( C(X) \) \([2, \text{Theorem 1, p. 38}]\).

If \( X \) is either an arc or a simple closed curve and \( Y \) is a continuum such that \( C(X) \) is homeomorphic to \( C(Y) \), then \( Y \) is either an arc or a simple closed curve \([2, \text{Lemma 11, p. 38}]\).

A dendrite is a locally connected continuum which contains no simple closed curves. Let
\[ \mathcal{D} = \{ X : X \text{ is a dendrite whose set of end points is closed} \}. \]

It has been proved that:

(b) If \( X \in \mathcal{D} \) is not an arc, then \( X \) has unique hyperspace \( C(X) \) \([15, \text{Theorem 10, p. 804}]\).

Moreover, if \( X \) is either an arc or if \( X \) is a dendrite such that \( X \notin \mathcal{D} \), then \( X \) does not have unique hyperspace \( C(X) \) \([7, \text{Theorem 5.2, p. 466}]\).

A local dendrite is a continuum such that every of its points has a neighborhood which is a dendrite. Let
\[ \mathcal{L} = \{ X : X \text{ is a local dendrite} \} \]

and
\[ \mathcal{LD} = \{ X \in \mathcal{L} : \text{each point of } X \text{ has a neighborhood which is in } \mathcal{D} \}. \]

One important part of this paper is dedicated to study the structure of the classes \( \mathcal{D} \) and \( \mathcal{LD} \). As an application, another important part of this paper is dedicated to prove the following result:

(c) If \( X \in \mathcal{LD} \) is different from an arc and a simple closed curve, then \( X \) has unique hyperspace \( C(X) \).

Since \( \mathcal{D} \subsetneq \mathcal{LD} \) and \( \mathcal{G} \subsetneq \mathcal{LD} \), (c) generalizes both (a) and (b).

The paper is divided in five sections. After this section, in Section 2 we present the definitions and the fundamental results that we use in the paper. In Section 3 we study the structure of the classes \( \mathcal{D} \) and \( \mathcal{LD} \). In Theorem 3.20 we present a useful characterization of the elements of \( \mathcal{LD} \), in terms of the dimension of \( C(X) \) at certain subcontinua of \( X \). As an application of this result we prove, in Theorem 3.21, that if \( X \in \mathcal{LD} \) and \( Y \in \mathcal{L} \) are such that \( C(X) \) is homeomorphic to \( C(Y) \), then \( Y \in \mathcal{LD} \). This is an important first step for the proof of (c). In Section 4 we consider, for every continuum \( X \), a class \( \Omega(X) \) of special subsets of \( X \) in such a way that if \( X \) and \( Y \) are two continua so that \( C(X) \) is homeomorphic to \( C(Y) \), then \( \Omega(X) \) is homeomorphic to \( \Omega(Y) \). After presenting some additional properties of the class \( \Omega(X) \), we prove in Theorem 4.10 that if \( X \in \mathcal{LD} \) is different from an arc, then the closure of \( \Omega(X) \) in \( C(X) \) is homeomorphic to \( X \). This is an important second step for the proof of (c). Finally in Section 5 we prove (c).

Results related to the subject of this paper can be found in \([1–7, 14–18, 20–26, 30–33]\).

2. Definitions and fundamental results

The symbol \( \mathbb{N} \) will denote the set of positive integers. Let \( Z \) be a metric space and \( A \) be a subset of \( Z \). We denote by \( |A| \) the cardinality of \( A \), and by \( \text{diam}(A) \) the diameter of \( A \). The interior, the closure and the boundary of \( A \) in \( Z \), will be denoted by \( \text{Int}_Z(A) \), \( \text{Cl}_Z(A) \) and \( \text{Bd}_Z(A) \), respectively. If \( \varepsilon > 0 \), we will use the set \( N(\varepsilon, A) = \bigcup_{p \in A} B_Z(p, \varepsilon) \) where, for \( p \in Z \), \( B_Z(p, \varepsilon) \) denotes the open \( \varepsilon \)-ball in \( Z \) centered at \( p \). If \( p \in Z \), then \( \text{dim}_p(Z) \) stands for the dimension of \( Z \) at \( p \) \([37, \text{p. 5}]\). The following two results are proved in \([15, \text{Lemma 5, p. 803}]\) and \([28, \text{Theorem 4.4, p. 28}]\), respectively.

**Theorem 2.1.** Let \( X \) be a locally connected continuum and \( A \subset C(X) \). If \( \text{dim}_A(C(X)) < \infty \), then \( \text{dim}_p(C(X)) < \infty \), for every \( p \in A \).

**Theorem 2.2.** Let \( X \) be a continuum. Then \( X \) is locally connected if and only if \( C(X) \) is locally connected.
Theorem 2.3. Let $X$ be a continuum and $p \in X$. If $X$ is locally connected at each point of an open set containing $p$, then $C(p, X)$ is a Hilbert cube if and only if $p$ is in the interior (relative to $X$) of a finite graph in $X$.

For $n \in \mathbb{N} \setminus \{1, 2\}$ an $n$-od is a continuum $Y$ which contains a subcontinuum $Z$ such that $Y - Z = \bigcup_{i=1}^{n} Z_i$, where $Z_i \neq \emptyset$ for each $i \in \{1, 2, \ldots, n\}$ and $Cl_Y(Z_i) \cap Z_j = \emptyset$ whenever $i \neq j$. The following result is proved in [2, Lemma 8, p. 37].

Theorem 2.4. Let $X$ be a continuum and $n \in \mathbb{N} \setminus \{1, 2\}$. If $K \in C(X)$ and $T$ is an $n$-od in $X$ such that, for some $\varepsilon > 0$, $T \in B_{C(X)}(K, \frac{\varepsilon}{2})$, then there is an $n$-cell $\Gamma$ in $C(X)$ such that $T \in \Gamma \subset B_{C(X)}(K, \varepsilon)$.

If $V$ is a 2-cell in a space $X$, then $\partial V$ denotes the manifold boundary of $V$. Note that if $V$ is a 2-cell and $h : [0, 1]^2 \to V$ is a homeomorphism, then $\partial V = h(Bd_{[0, 1]^2}(0, 1)^2))$ [11, Theorem 17A.8, p. 449]. If $A$ and $B$ are 2-cells such that $A \subset B$, then $A - \partial A$ is open in $B$ [37, 19.34, p. 123]. If $A$ is an arc with end points $p$ and $q$ then, by [27, Example 5.11, p. 33], $C(A)$ is a 2-cell such that $\partial C(A) = C(p, A) \cup C(q, A) \cup F_1(A)$.

Let $Z$ be a dendrite. It is known that every subcontinuum of $Z$ is a dendrite [36, Corollary 10.6, p. 167]. It is also known that every points $x$ and $y$ in $Z$, can be joined by a unique arc contained in $Z$. We denote such an arc by $[x, y]$, and consider that $[x, x] = \{x\}$. We define $(x, y) = [x, y] - \{x, y\}, [x, y] = [x, y] - \{y\}$ and $(x, y) = [x, y] - \{x\}$. If $Z$ is a local dendrite and $x, y \in Z$, then the symbol $[x, y]$ will represent an arc in $Z$ with end points $x$ and $y$. The sets $(x, y), [x, y]$ and $(x, y)$ are defined as above.

A metric space $Y$, with metric $d$, is uniformly locally arcwise connected if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in Y$ with $d(x, y) < \delta$, there is an arc $A$ in $Y$ with end points $x$ and $y$, so that $\text{diam}(A) < \varepsilon$. It is known that locally connected continua are uniformly locally arcwise connected [19, Lemma 3-29, p. 129]. In particular dendrites are uniformly locally arcwise connected.

The following result, which will be useful in Section 3, was presented in [15, Lemma 3, p. 802] without proof. For the readers convenience, we prove it in this paper.

Theorem 2.5. Let $X$ be a dendrite, $p \in X$, $M \in C(X)$ and $[a, b]$ be a nondegenerate arc in $X$ such that $p \in (a, b) \subset [a, b] \subset M$. Suppose that a sequence $(M_n)_{n \in \mathbb{N}}$ in $C(X)$ converges, in the Hausdorff metric, to $M$. Then there is an $N \in \mathbb{N}$ such that $p \in M_n$, for every $n > N$.

Proof. Let $\varepsilon > 0$ be such that the sets $B_X(a, \varepsilon), B_X(b, \varepsilon)$ and $B_X(p, \varepsilon)$ are mutually disjoint. Let $d$ be the metric of $X$. Since $X$ is uniformly locally arcwise connected, there exists $\varepsilon > 0$ such that for every $x, y \in X$ with $\varepsilon \neq y$ and $d(x, y) < \delta$, there is an arc $A$ in $X$ with end points $x$ and $y$, so that $\text{diam}(A) < \varepsilon$. Since $(M_n)_{n \in \mathbb{N}}$ converges to $M$, there is $N \in \mathbb{N}$ such that $H(M_n, M) < \delta$, for every $n \geq N$. Now assume that $n > N$. Since $a, b \in [a, b] \subset M \subset N(\delta, M_n)$, there exist $a_1 \in M_n \cap B_X(a, \delta)$ and $b_1 \in M_n \cap B_X(b, \delta)$. Then $d(a_1, a_1) < \delta$ and $d(b, b_1) < \delta$, so there exist an arc $A$ with end points $a$ and $a_1$, and an arc $B$ with end points $b$ and $b_1$, so that $\text{diam}(A) < \varepsilon$ and $\text{diam}(B) < \varepsilon$. Note that $A \subset B_X(a, \varepsilon)$ and $B \subset B_X(b, \varepsilon)$. Let $a_2 \in A$ be so that $[a_1, a_2] \cap [a, b] = [a_2]$. We define $b_2 \in B$ so that $[b_1, b_2] \cap [a, b] = [b_2]$. Note that $C = [a_1, a_2] \cup [a_2, b_2] \cup [b_2, b_1]$ is an arc in $X$ that contains $p$, and whose end points $a_1$ and $b_1$ are in $M_n$. Thus, $C \subset M_n$ and then $p \in M_n$. 

Let $Y$ be a continuum, $p \in Y$ and $\beta$ be a cardinal number. We say that $p$ has order less than or equal to $\beta$ in $Y$, written $\text{ord}(p, Y) \leq \beta$, provided that $p$ has a basis $\mathcal{B}$ of neighborhoods in $Y$ such that, for each $U \in \mathcal{B}$, we have $|Bd_Y(U)| \leq \beta$. We say that $p$ has order equal to $\beta$ in $Y$, written $\text{ord}(p, Y) = \beta$, if $\text{ord}(p, Y) \leq \beta$ and $\text{ord}(p, Y) \geq \alpha$, for each cardinal number $\alpha$ such that $\alpha < \beta$.

Given a continuum $Y$, we consider the following sets:

$$E(Y) = \{p \in Y: \text{ord}(p, Y) = 1\}, \quad O(Y) = \{p \in Y: \text{ord}(p, Y) = 2\}$$

and

$$R(Y) = \{p \in Y: \text{ord}(p, Y) \geq 3\}.$$

The elements of $E(Y)$, $O(Y)$ and $R(Y)$ are called end points, ordinary points and ramification points of $Y$, respectively. We also consider the following subset of $Y$:

$$E_0(Y) = \{p \in Y: \text{there exists a convergent sequence } (e_n)_{n \in \mathbb{N}} \text{ in } E(Y) - \{p\} \text{ whose limit is } p\}.$$

Let $Z$ be a dendrite. Then $p \in E(Z)$ if and only if $p$ is an end point of every arc in $Z$ that contains $p$ (see [29, Theorem 15, p. 320] and [36, 10.44, p. 188]). From this fact and the following result of continua of $Z$ is a dendrite, it follows that if $A \in C(Z)$, then $E(Z) \cap A \subset E(A)$.

Let $Z$ be a dendrite. If $p \in Z$ and $\text{ord}(p, Z)$ is finite, then it is equal to the number of components of $Z - \{p\}$ [38, (11.4)iv, p. 88]. If $\text{ord}(p, Z)$ is infinite, then it is countable and the diameters of components of $Z - \{p\}$ tend to zero [38, (2.6), p. 92]. In this case we say that $p$ is a $l$-essential point of $Z$. We say that $p$ is a $R$-essential point of $Z$, if there exist a nondegenerate arc $A$ in $Z$ and a convergent sequence $(p_n)_{n \in \mathbb{N}}$ of different points in $A$, whose limit is $p$, and such that $p_n \in R(Z)$ for each $n \in \mathbb{N}$. 

Let $X$ be a local dendrite and $p \in X$. We say that $p$ is a I-essential point of $X$ (respectively, a II-essential point of $X$) if there exists a dendrite $Z$ which is a neighborhood of $p$ in $X$, such that $p$ is a I-essential point of $Z$ (respectively, a II-essential point of $Z$). We say that $p$ is an essential point of $X$ if $p$ is either a I-essential or a II-essential point of $X$.

The following result, easy to prove, is the equivalent version of [34, Properties 1.2(e), p. 43] for local dendrites.

**Theorem 2.6.** Let $X$ be a local dendrite and $p \in X$. Then $p$ is an essential point of $X$ if and only if $p \notin \text{Int}_X(T)$, for every tree $T$ contained in $X$.

**Theorem 2.7.** Let $X$ be a local dendrite and $A \in C(X)$. If there exists $p \in A$ such that $p$ is an essential point of $X$, then $\dim_A(C(X)) = \infty$.

**Proof.** By Theorem 2.6, $p$ is not in the interior, relative to $X$, of a finite graph in $X$. Then, by Theorem 2.3, $C(p, X)$ is a Hilbert cube. Hence, $\dim_A(C(p, X)) = \infty$. Since $A \subset C(p, X) \subset C(X)$, it follows from [37, Corollary 3.3, p. 16], that $\dim_A(C(p, X)) \leq \dim_A(C(X))$. Thus, $\dim_A(C(X)) = \infty$. $\square$

If $a, b \in \mathbb{R}^2$, then $\overline{ab}$ denotes the straight line segment joining $a$ and $b$. We will need three special dendrites $F_\omega$, $W$ and $W_0$, constructed in $\mathbb{R}^2$. The first one is the dendrite

$$F_\omega = \bigcup_{n \in \mathbb{N}} p_n,$$

where $p = (0, 0)$ and $p_n = (\frac{1}{n}, \frac{1}{n^2})$, for each $n \in \mathbb{N}$. We say that $p$ is the vertex of $F_\omega$. The second one is the dendrite

$$W = \overline{cb_1} \cup \left( \bigcup_{n \in \mathbb{N}} a_nb_n \right),$$

where $c = (-1, 0)$, $a_n = (\frac{1}{n}, \frac{1}{n^2})$ and $b_n = (\frac{1}{n}, 0)$, for each $n \in \mathbb{N}$. The third one is the dendrite

$$W_0 = \overline{b_1} \cup \left( \bigcup_{n \in \mathbb{N}} a_nb_n \right),$$

where $b = (0, 0)$ and, for each $n \in \mathbb{N}$, $a_n$ and $b_n$ are defined as before. Note that $W_0 = W - [c, b]$.

3. The class $\mathcal{D}$

We recall that $\mathcal{D}$ is the class of dendrites whose set of end points is closed. Let $X \in \mathcal{D}$. By [8, Theorem 3.3, p. 4], the order of every point of $X$ is finite. The following result follows from the proof of [8, Proposition 3.4, p. 4].

**Theorem 3.1.** Let $X \in \mathcal{D}$ and $p \in X$. If $p$ is the limit of a sequence $\{p_n\}_{n \in \mathbb{N}}$ of distinct ramification points of $X$ such that $p \neq p_1$, then $p$ is both the limit of a sequence of distinct ramification points of $X$, all in the arc $[p, p_1]$, and the limit of a sequence of end points of $X$, all different from $p$.

If $X$ is a dendrite then, by [10, Corollary 4, p. 298], $X$ satisfies the following property:

(S) for every $e \in X$ and every sequence $\{p_n\}_{n \in \mathbb{N}}$ in $X$ that converges to $p$, the sequence of arcs $\{[e, p_n]\}_{n \in \mathbb{N}}$ converges, in the Hausdorff metric, to the arc $[e, p]$.

**Theorem 3.2.** Let $X \in \mathcal{D}$ and $e \in X$. If $e$ is the limit of a sequence $\{e_n\}_{n \in \mathbb{N}}$ of distinct end points of $X$ such that $e \neq e_1$, then there exist an increasing sequence $\{m_n\}_{n \in \mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{p_m\}_{m \in \mathbb{N}}$ of distinct ramification points of $X$, all in the arc $[e, e_1]$, such that $e$ is the limit of $\{p_m\}_{m \in \mathbb{N}}$ and

$$[e, p_1] \cup \left( \bigcup_{m \in \mathbb{N}} [e_{m_n}, p_m] \right)$$

is a subcontinuum of $X$, homeomorphic to the dendrite $W_0$ defined in (2.3).

**Proof.** Since $X \in \mathcal{D}$, the set $E(X)$ is closed in $X$. Then $e \in E(X)$. We can assume, without loss of generality, that $e \neq e_1$ for every $n \in \mathbb{N}$. Recall that $p \in E(X)$ if and only if $p$ is an end point of every arc in $X$ that contains $p$. Hence, $e_n \notin [e, e_1]$, for every $n \in \mathbb{N} - \{1\}$. Let $p_1 \in [e, e_1]$ be so that $[e_2, p_1] \cap [e, e_1] = [p_1]$. Note that $p_1 \in R(X)$, so $[e, p_1] \neq [e]$. Let $n_1 = 2$. If for infinitely many indices $n$ we have $[e, p_1] \cap [e, e_n] = [e, p_1]$, then for such indices $n$ it follows that $[e, p_1] \subset [e, e_n]$. Taking the limit we obtain, by the property (S), that $[e, p_1] \subset [e, e] = [e]$, so $[e, p_1] = [e]$. Since this is a contradiction we have shown that:
1) there exists $n_2 > n_1$ such that $[e, p_1] \cap [e, e_n] \neq [e, p_1]$, for each $n \geq n_2$.

Now we claim that:

2) $[e, e_n] \cap [e_2, p_1] = \emptyset$, for each $n \geq n_2$.

To show 2) assume, on the contrary, that $[e, e_n] \cap [e_2, p_1] \neq \emptyset$, for some $n \geq n_2$. By compactness of the set $[e, e_n] \cap [e_2, p_1]$, there exists $b \in [e_2, p_1]$ such that $[e_n, b] \cap [e_2, p_1] = [b]$. Since any two points of $X$ can be joined by a unique arc in $X$, the arc $[e, e_n]$ coincides with the arc $[e, p_1] \cup [p_1, b] \cup [e_n, b]$. Then $b \in [e, e_n]$ and $[e, p_1] \cap [e, e_n] = [e, p_1]$. Since this contradicts claim 1), claim 2) is true.

Since $X$ is a dendrite for every $n \geq n_2$, by 1) and [36, Theorem 10.10, p. 169], $[e, p_1] \cap [e, e_n]$ is a proper subarc of $[e, p_1]$. Let $p_2 \in [e, p_1]$ be so that $[e_n, p_2] \cap [e, p_1] = \{p_2\}$. Then $p_2 \in R(X) - \{p_1\}$, $[e, e_n] = [e, p_2] \cup [e_n, p_2]$ and, since $[e, e_n] \cap [e_2, p_1] = \emptyset$, we have $[e_n, p_2] \cap [e_n, p_1] = \emptyset$.

Following the argument that lead us to 1), we can show that there exists $n_3 > n_2$ such that $[e, p_2] \cap [e, e_n] \neq [e, p_2]$, for each $n \geq n_3$. Using the arguments that lead us to 2), it follows that $[e, e_n] \cap [e_n, p_1] = \emptyset$ and $[e, e_n] \cap [e_n, p_2] = \emptyset$, for each $n \geq n_3$. Let $p_3 \in [e, p_2]$ be so that $[e_n, p_3] \cap [e, p_2] = \{p_3\}$. Then $p_3 \in R(X) - \{p_1, p_2\}$, $[e, e_n] = [e, p_3] \cup [e_n, p_3]$, $[e_n, p_3] \cap [e_n, p_1] = \emptyset$ and $[e_n, p_3] \cap [e_n, p_2] = \emptyset$.

Proceeding in this fashion we can obtain an increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ of natural numbers and a sequence $\{p_m\}_{m \in \mathbb{N}}$ in $R(X)$ with the following properties:

(a) $p_1 \in [e, e_1]$ and $[e_n, p_1] \cap [e, e_1] = \{p_1\}$;
(b) for $m \geq 2$, we have $p_m \in [e, p_{m-1}]$, $[e_n, p_m] \cap [e, p_{m-1}] = \{p_m\}$ and $[e_n, p_m] \cap [e_n, p_1] = \emptyset$, for every $i \in \{1, 2, \ldots, m-1\}$.

Hence, $\{p_m\}_{m \in \mathbb{N}}$ is a sequence of distinct ramification points of $X$, all in the arc $[e, e_1]$. Given $m \in \mathbb{N}$ we have $p_m \in [e_n, p_m] \subset [e, e_n]$. Since the sequence $\{e_n\}_{n \in \mathbb{N}}$ converges to $e$, by the property (S), the sequence of arcs $\{[e, e_n]\}_{n \in \mathbb{N}}$ also converges, in the Hausdorff metric, to $[e]$. This implies that the sequence of pairwise disjoint arcs $\{[e_n, p_m]\}_{m \in \mathbb{N}}$ also converges, in the Hausdorff metric, to $[e]$. Then, $e$ is the limit of $\{p_m\}_{m \in \mathbb{N}}$ and

$$[e, p_1] \cup \left( \bigcup_{m \in \mathbb{N}} [e_n, p_m] \right)$$

is a subcontinuum of $X$, homeomorphic to the dendrite $W_0_0$. □

For a continuum $Y$ we recall that $e \in E_0(Y)$ if and only if there exists a convergent sequence $\{e_n\}_{n \in \mathbb{N}}$ in $E(Y) - \{e\}$, whose limit is $e$. Combining Theorems 3.1 and 3.2 we obtain the following result.

**Theorem 3.3.** Let $X \in \mathcal{D}$ and $e \in X$. Then $e \in E_0(X)$ if and only if $e$ is the limit of a sequence of distinct ramification points of $X$.

Now recall that $\mathcal{L}$ is the class of local dendrites, and that $\mathcal{LD}$ is the class of local dendrites with the property that each of its points has a neighborhood in $\mathcal{D}$. For $X \in \mathcal{LD}$ in this section we present two characterizations of the essential points of $X$. We also present a characterization of the elements of $\mathcal{LD}$, in terms of the dimension of $C(X)$ at certain subcontinua of $X$ (see Theorem 3.20).

**Theorem 3.4.** Let $X \in \mathcal{L}$. Then $X \in \mathcal{LD}$ if and only if $X$ contains no copy of $F_{\infty}$ or of $W$ (defined in (2.1) and (2.2)).

**Proof.** In [8, Theorem 3.3, p. 4] it is proved that a dendrite is in $\mathcal{D}$ if and only if it contains no copy of $F_{\infty}$ or of $W$. The result follows from this fact. □

The next result is proved in [29, Theorem 4, p. 303].

**Theorem 3.5.** The following conditions are equivalent:

(a) $X$ is a local dendrite;
(b) $X$ is a locally connected continuum which contains at most a finite number of simple closed curves;
(c) $X$ is a continuum and there exists a finite number of dendrites $D_1, D_2, \ldots, D_l$ such that $X = D_1 \cup D_2 \cup \cdots \cup D_l$ and $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$.

**Corollary 3.6.** If $X \in \mathcal{L}$ and $Z \in C(X)$, then $Z \in \mathcal{L}$. If $X \in \mathcal{LD}$ and $Z \in C(X)$, then $Z \in \mathcal{LD}$. 
Proof. Assume first that $X \in \mathcal{L}$ and $Z \in \mathcal{C}(X)$. Then, by [29, Theorem 1, p. 303] and [29, Theorem 2, p. 283], $X$ is hereditarily locally connected. Thus, $Z$ is a locally connected continuum and, by Theorem 3.5, $Z$ is a local dendrite.

Assume now that $X \in \mathcal{L}$ and $Z \in \mathcal{C}(X)$. By the first part of the proof, $Z \in \mathcal{L}$ and, by Theorem 3.4, $Z \in \mathcal{L}$. ⊓⊔

The next result follows from Theorems 3.4 and 3.5.

**Theorem 3.7.** Let $X$ be a continuum that contains no copy of $F_\omega$ or of $W$. Then $X \in \mathcal{L}$ if and only if there exists a finite number of dendrites $D_1, D_2, \ldots, D_l$ in $\mathcal{D}$ such that $X = D_1 \cup D_2 \cup \cdots \cup D_l$ and $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$.

From now on if $X \in \mathcal{L}$ (respectively, if $X \in \mathcal{L}$) we will think that $X$ is nondegenerate and that

$$X = D_1 \cup D_2 \cup \cdots \cup D_l,$$

(3.1)

where $D_1, D_2, \ldots, D_l$ are nondegenerate dendrites (respectively, $D_1, D_2, \ldots, D_l$ are nondegenerate elements of $\mathcal{D}$) such that $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots, l\}$ with $i \neq j$. We will also consider that:

$$P_X = \bigcup \{D_i \cap D_j : i, j \in \{1, 2, \ldots, l\} \text{ and } i \neq j\}.$$

Note that $P_X$ is a finite subset of $X$. Note also that if $p \in P_X$, then $p \notin E(X)$ and if $p \in D_i - P_X$, then $\text{ord}(p, D_i) = \text{ord}(p, X)$.

**Theorem 3.8.** If $X \in \mathcal{L}$ and $i \in \{1, 2, \ldots, l\}$, then $D_i - P_X$ is open in $X$.

Proof. Let $p \in D_i - P_X$. Since $X$ is locally connected and $P_X$ is finite, there exists an open and connected subset $U$ of $X$ such that $p \in U \cap X - P_X$. Then $U$ is a connected subset of $X$ such that $C \cap D_i \neq \emptyset$ and $C \cap P_X = \emptyset$, so $U \subset D_i - P_X$. This shows that $D_i - P_X$ is open in $X$. ⊓⊔

**Theorem 3.9.** If $X \in \mathcal{L}$ and $p \in D_i \cap P_X$, then there exists a nondegenerate arc $A$ in $D_i$ such that $A \cap P_X = \{p\}$.

Proof. Since $P_X$ is finite there exists $\varepsilon > 0$ such that $B_X(p, \varepsilon) \cap P_X = \{p\}$. Since $D_i$ is uniformly locally arcwise connected, if $d$ denotes the metric on $X$, there is $\delta > 0$ such that for every $x, y \in D_i$ with $x \neq y$ and $d(x, y) < \delta$, there is an arc $A$ in $D_i$ with end points $x$ and $y$, so that $\text{diam}(A) < \varepsilon$. Taking $x \in D_i$ such that $x \neq p$ and $d(x, p) < \delta$, it then follows that there exists an arc $A$ in $D_i$ with end points $x$ and $p$, so that $\text{diam}(A) < \varepsilon$. Then, $A \subset B_X(p, \varepsilon)$, so $A \cap P_X = \{p\}$. ⊓⊔

**Theorem 3.10.** If $X \in \mathcal{L}$, then $E_a(D_i) \cap P_X = \emptyset$, for every $i \in \{1, 2, \ldots, l\}$.

Proof. Assume, on the contrary, that there exists $e \in E_a(D_i) \cap P_X$. Then $e$ is the limit of a sequence $\{e_n\}_{n \in \mathbb{N}}$ in $E(D_i) - \{e\}$. If there exists an infinite subset $J$ of $\mathbb{N}$ such that $e_n = e_m$ for every $n, m \in J$, then $e = e_n$, for some $n \in \mathbb{N}$. Since this is a contradiction, we can assume that $e_n \neq e_m$ if $n \neq m$. By Theorem 3.2, there exist an increasing sequence $\{n_m\}_{m \in \mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{p_m\}_{m \in \mathbb{N}}$ of distinct ramification points of $D_i$, all in the arc $[e, e_1]$ of $D_i$, such that $e$ is the limit of $\{p_m\}_{m \in \mathbb{N}}$ and

$$F = [e, p_1] \cup \left( \bigcup_{m \in \mathbb{N}} [e_{n_m}, p_{m}] \right)$$

is a subcontinuum of $D_i$, homeomorphic to the dendrite $W_0$ defined in (2.3). Since $e \in P_X$, there exists $k \in \{1, 2, \ldots, l\} - \{i\}$ such that $e \in D_i \cap D_k$. Then $e \in D_k \cap P_X$, so, by Theorem 3.9, there exists a nondegenerate arc $A$ is $D_k$ such that $A \cap P_X = \{e\}$. Then $A \cup F$ is a copy of $W$. This implies that $X$ contains a copy of $W$. Since this contradicts Theorem 3.4, it follows that $E_a(D_i) \cap P_X = \emptyset$. ⊓⊔

As a consequence of the previous theorem, we have the following result.

**Theorem 3.11.** If $X \in \mathcal{L}$, then $E(X)$ is closed in $X$. In particular, $E_a(X) \subset E(X)$.

Proof. To show that $E(X)$ is closed in $X$, let $e$ be the limit of a sequence $\{e_n\}_{n \in \mathbb{N}}$ in $E(X)$. We can assume that $e_n \neq e$ for every $n \in \mathbb{N}$. If there exists an infinite subset $J$ of $\mathbb{N}$ such that $e_n = e_m$ for every $n, m \in J$, then $e \in E(X)$. Hence, without loss of generality, we can assume that $e_n \neq e_m$ if $n \neq m$. By (3.1), we can also assume that there exists $i \in \{1, 2, \ldots, l\}$ such that $e_n \in D_i$ for every $n \in \mathbb{N}$. Hence, $e \in D_i$. Since $P_X \subset X - E(X)$ we have $e_n \in D_i - P_X$ so $\text{ord}(e_n, D_i) = \text{ord}(e_n, X) = 1$, for all $n \in \mathbb{N}$. Since $D_i \in D$, the set $E(D_i)$ is closed in $D_i$. Hence, $e \in E(D_i)$ and, indeed, $e \in E_a(D_i)$ so, by Theorem 3.10, $e \notin P_X$. Hence, $\text{ord}(e, X) = \text{ord}(e, D_i) = 1$. This shows that $E(X)$ is closed in $X$. In particular, $E_a(X) \subset E(X)$. ⊓⊔

The following result is the equivalent version of Theorem 3.3, for the elements of $\mathcal{L}$. 

Theorem 3.12. Let \( X \in \mathcal{D} \) and \( e \in X \). Then the following assertions are equivalent:

(a) \( e \in E_d(X) \);
(b) \( e \) is the limit of a sequence of distinct ramification points of \( X \), all in one arc of \( X \) that contains \( e \);
(c) \( e \) is the limit of a sequence of distinct ramification points of \( X \).

Proof. To show that (a) implies (b), let us assume that \( e \in E_d(X) \). By (3.1), there is \( i \in \{1, 2, \ldots, l\} \) such that \( e \in D_i \). Since \( e \in E(X) \) and \( P_X \subset X - E(X) \), we have \( e \in D_i - P_X \). Then \( \text{ord}(e, D_i) = \text{ord}(e, X) = 1 \). Let \( \{e_n\}_{n \in \mathbb{N}} \) be a sequence in \( E(X) - \{e\} \) whose limit is \( e \). We can consider that \( e_n \neq e_m \) if \( n \neq m \). Since \( D_i - P_X \) is open in \( X \) (Theorem 3.8), we can also consider that \( e_n \in D_i - P_X, \) for every \( n \in \mathbb{N} \), so \( \text{ord}(e_n, D_i) = \text{ord}(e_n, X) = 1 \). Hence, \( e \) is the limit of a sequence of distinct end points of \( D_i \) and, by Theorem 3.2, \( e \) is also the limit of a sequence of distinct ramification points of \( D_i, \) all in one arc of \( D_i \) that contains \( e \). Since every ramification point of \( D_i \) is a ramification point of \( X \), this shows that (a) implies (b).

The assertion (b) implies (c) is obvious. To show that (c) implies (a), let us assume that \( e \) is the limit of a sequence \( \{p_n\}_{n \in \mathbb{N}} \) of distinct ramification points of \( X \). Then \( p_n \neq p_m \) if \( n \neq m \) and we can consider that \( p_n \neq e \) for every \( n \in \mathbb{N} \). Let us assume first that \( e \notin P_X \). By (3.1), there is \( i \in \{1, 2, \ldots, l\} \) such that \( e \in D_i \). Since \( D_i - P_X \) is open in \( X \) (Theorem 3.8), we can consider that \( p_n \in D_i - P_X, \) for every \( n \in \mathbb{N} \). Thus, \( \text{ord}(p_n, D_i) = \text{ord}(p_n, X) \geq 3 \) so each \( p_n \) is a ramification point of \( D_i \). By Theorem 3.1, \( e \) is the limit of a sequence \( \{e_n\}_{n \in \mathbb{N}} \) of end points of \( D_i \), all different from \( e \). We can then assume that \( e_n \neq e_m \) if \( n \neq m \). Since \( P_X \) is finite, we can also assume that \( e_n \notin P_X, \) for every \( n \in \mathbb{N} \). Thus, \( \text{ord}(e_n, X) = \text{ord}(e_n, D_i) = 1, \) for each \( n \in \mathbb{N} \) and then \( e \in E_d(X) \).

Let us consider now that \( e \in P_X \). By (3.1), we can assume that there is \( i \in \{1, 2, \ldots, l\} \) such that \( p_n \in D_i \) for each \( n \in \mathbb{N} \). Then \( e \in D_i \). If \( p_n \in P_X \) for infinitely many indices \( n \) then, since \( P_X \) is finite and \( p_n \neq e \) for every \( n \in \mathbb{N} \), there exists \( q \in P_X - \{e\} \) such that \( q = p_n \) for infinitely many indices \( n \). Thus, \( e = q \) and, since this is a contradiction, we can assume that \( p_n \notin P_X \) for every \( n \in \mathbb{N} \). Hence, \( \text{ord}(p_n, D_i) = \text{ord}(p_n, X) \geq 3, \) for each \( n \in \mathbb{N} \). By Theorem 3.1, \( e \in E_d(D_i) \). Hence, \( E_d(D_i) \cap P_X = \emptyset \). Since this contradicts Theorem 3.10, we conclude that the case \( e \in P_X \) is not possible. This shows that (c) implies (a).

Corollary 3.13. If \( X \in \mathcal{D} \), then \( O(X) \) is open in \( X \).

Proof. Let \( p \in O(X) \) and assume, on the contrary, that \( O(X) \) is not open. Then, for each \( n \in \mathbb{N} \), there exists \( p_n \in B_X(p, \frac{1}{n}) \) such that \( p_n \in X - O(X) = E(X) \cup R(X) \). Note that \( p \) is the limit of the sequence \( \{p_n\}_{n \in \mathbb{N}} \). If there exists an infinite subset \( J \) of \( \mathbb{N} \) such that \( p_n = p_m \) for every \( n, m \in J \), then we have \( p \in E(X) \cup R(X) \). Since this contradicts the fact that \( p \in O(X) \), we can consider that \( p_n \neq p_m \) if \( n \neq m \). If for infinitely many indices \( n \), we have \( p_n \in E(X) \) then, since \( E(X) \) is closed in \( X \), we have \( p \in E(X) \). Since this contradicts the fact that \( p \in O(X) \), it follows that \( p_n \in E(X) \) only for finitely many indices \( n \). Hence, we can assume that \( p_n \in R(X) \) for every \( n \in \mathbb{N} \). Then, by Theorems 3.11 and 3.12, \( p \in E_d(X) \subset E(X) \). Since this contradicts the fact that \( p \in O(X) \), we conclude that \( O(X) \) is open in \( X \).

Corollary 3.14. Let \( X \in \mathcal{D} \) and \( A \subset C(X) \). Then:

(a) \( E_d(A) \subset E_d(X) \);
(b) if \( A \cap E_d(X) = \emptyset \), then \( A \) is a finite graph.

Proof. By Corollary 3.6, \( A \in \mathcal{D} \). To prove (a) let \( e \in E_d(A) \). Then, by Theorem 3.12, \( e \) is the limit of a sequence of distinct ramification points of \( A \). Since a ramification point of \( A \) is also a ramification point of \( X \), \( e \) is the limit of a sequence of distinct ramification points of \( X \). Then, by Theorem 3.12, \( e \in E_d(X) \). This shows (a).

To show (b) let us assume that \( A \cap E_d(X) = \emptyset \). By Theorem 3.4, \( X \) contains no copy of \( F_\infty \). Hence, the order of every point of \( X \) is finite. In particular the order of every point of \( A \) is finite. If we assume that \( A \) is not a finite graph then, by [36, Theorem 9.10, p. 144], \( A \) contains infinitely many ramification points. Since \( A \) is compact, there is \( e \in A \) such that \( e \) is the limit of a sequence of distinct ramification points of \( A \). Then, by Theorem 3.12 and (a), \( e \in E_d(A) \subset E_d(X) \) so \( A \cap E_d(X) \neq \emptyset \). This contradiction shows that \( A \) is a finite graph.
From Theorem 3.4 if \( X \in \mathcal{L}\mathcal{D} \), then the order of every point in \( X \) is finite. Thus, \( X \) contains no \( l \)-essential points. This implies that every essential point of \( X \) is a II-essential point of \( X \) and, by Corollary 3.15, is also an element of \( E_0(X) \). Hence, \( E_0(X) \) is precisely the set of essential points of \( X \).

**Theorem 3.16.** Let \( X \in \mathcal{L}\mathcal{D} \) and \( A \in C(X) \). Then there exists \( p \in A \) such that \( p \) is an essential point of \( X \) if and only if \( \dim_A(C(X)) = \infty \).

**Proof.** Suppose \( p \in A \) is an essential point of \( X \). Since \( X \in \mathcal{L} \), by Theorem 2.7, \( \dim_A(C(X)) = \infty \). Now suppose that \( A \) does not contain essential points of \( X \). Then, by Corollary 3.15, \( A \cap E_0(X) = \emptyset \). Hence, by part (b) of Corollary 3.14, \( A \) is a finite graph. Let \( G \) be a finite graph in \( X \) such that \( A \subset \text{Int}_X(G) \) and \( G \cap E_0(X) = \emptyset \). By [12, 7.4, p. 278], \( \dim(G) < \infty \). Thus, if \( m = \dim_A(C(G)) \), then \( m < \infty \). Now we show that \( m = \dim_A(C(X)) \), so let \( \mathcal{V} \) be an open subset of \( C(X) \) such that \( A \subset \mathcal{V} \). Since \( A \subset \text{Int}_X(G) \), there exists \( \varepsilon > 0 \) such that \( N(\varepsilon, A) \subset \text{Int}_X(G) \). Thus, \( B_C(X)(\varepsilon, A) \subset C(G) \), so \( A \in \text{Int}_C(X)(C(G)) \). Let \( \mathcal{V} \) be an open subset of \( C(G) \) such that \( A \subset \mathcal{V} \subset \mathcal{W} \cap \text{Int}_C(X)(C(G)) \) and \( |BD_C(G)(\mathcal{V})| = m \). Since \( |BD_C(G)(\mathcal{V})| = |BD_C(G)(\mathcal{V})| \), it follows that \( \dim_A(C(X)) = m \), so \( \dim_A(C(X)) < \infty \). □

If \( D \) is a dendrite then, by [29, Theorem 8, p. 302], \( O(D) \) is dense in \( D \) and, by [36, Theorem 10.23, p. 174], \( R(D) \) is countable.

**Theorem 3.17.** If \( X \in \mathcal{L} \), then \( R(X) \) is countable and \( O(X) \) is dense in \( X \).

**Proof.** Recall that \( X = D_1 \cup D_2 \cup \cdots \cup D_l \), where \( D_1, D_2, \ldots, D_l \) are nondegenerate dendrites such that \( |D_i \cap D_j| < \infty \) for every \( i, j \in \{1, 2, \ldots, l\} \) with \( i \neq j \). It is not difficult to prove that

\[
R(X) = (R(X) \cap P_X) \cup \left( \bigcup_{i=1}^{l} (R(D_i) - P_X) \right).
\]

Since each set \( R(D_i) \) is countable and \( P_X \) is finite, by the above equality, \( R(X) \) is countable. To show that \( O(X) \) is dense in \( X \), let \( p \in X \) and \( U \) be an open set in \( X \) such that \( p \in U \). Let \( i \in \{1, 2, \ldots, l\} \) be such that \( p \in D_i \). Assume that \( p \notin P_X \). Since \( P_X \) is finite and \( X \) is locally connected, there exists an open and connected subset \( V \) of \( X \) such that \( p \in V \cap (X - P_X) \). Then \( V \subset D_i \) and, since \( O(D_i) \) is dense in \( D_i \), there is \( q \in V \cap O(D_i) \). Since \( V \cap P_X = \emptyset \), we have \( \text{ord}(q, X) = \text{ord}(q, D_i) = 2 \), so \( U \cap O(X) = \emptyset \). Now assume that \( p \in P_X \). Taking a sequence of distinct ordinary points of \( D_i \) that converges to \( p \), we infer that there exists \( q \in U \cap O(D_i) \) such that \( q \notin P_X \). Then \( \text{ord}(q, X) = \text{ord}(q, D_i) = 2 \), so \( U \cap O(X) = \emptyset \). □

The following result generalizes Theorem 2.5.

**Theorem 3.18.** Let \( X \in \mathcal{L} \), \( p \in X \), \( M \subset C(X) \) and \( [a, b] \) be a nondegenerate arc in \( X \) such that \( p \in (a, b) \subset [a, b] \subset M \) and \( M \cap P_X \subset \{p\} \). Suppose that a sequence \( \{M_n\}_{n \in \mathbb{N}} \) in \( C(X) \) converges, in the Hausdorff metric, to \( M \). Then there is \( N \in \mathbb{N} \) such that \( p \in M_n \) for every \( n > N \).

**Proof.** Let us assume that, for infinitely many indices \( n \), we have \( M_n \cap (P_X - \{p\}) \neq \emptyset \). Since \( P_X \) is finite, there exists \( q \in P_X - \{p\} \) such that \( q \in M_n \), for infinitely many indices \( n \). Thus, \( q \in M \), so \( M \cap P_X \not\subset \{p\} \). Since this is a contradiction, we have that \( M_n \cap (P_X - \{p\}) = \emptyset \), only for finitely many indices \( n \). Taking a subsequence of \( \{M_n\}_{n \in \mathbb{N}} \), if necessary, we can consider that \( M_n \cap P_X \subset \{p\} \), for every \( n \in \mathbb{N} \).

Assume first that \( p \in P_X \). If for infinitely many indices \( n \), we have \( M_n \cap P_X = \emptyset \), then \( M \cap P_X = \emptyset \). Since this is a contradiction, there exists \( N \in \mathbb{N} \) such that \( M_n \cap P_X \neq \emptyset \), for each \( n > N \). Thus, \( p \in M_n \), for every \( n > N \).

Assume now that \( p \notin P_X \). Then \( M \cap P_X = \emptyset \). If for infinitely many indices \( n \), we have \( M_n \cap P_X \neq \emptyset \), then \( M \cap P_X \neq \emptyset \). This implies that \( p \in P_X \), which is a contradiction. Hence, \( M \cap P_X \neq \emptyset \), only for finitely many indices \( n \). Taking a subsequence of \( \{M_n\}_{n \in \mathbb{N}} \), if necessary, we can consider that \( M_n \cap P_X = \emptyset \), for every \( n \in \mathbb{N} \). Since \( M \cap P_X = \emptyset \), there is \( i \in \{1, 2, \ldots, l\} \) such that \( M \subset D_i - P_X \). This implies that \( M_n \subset D_i - P_X \) for all \( n \), except finitely many of them. Taking again a subsequence of \( \{M_i\}_{n \in \mathbb{N}} \), if necessary, we can assume that \( M_n \subset D_i \) for all \( n \in \mathbb{N} \). Since \( D_i \) is a dendrite, by Theorem 2.5, there exists \( N \in \mathbb{N} \) such that \( p \in M_n \), for every \( n > N \). □

The following result is the equivalent version of [15, Lemma 4, p. 802] for the elements of \( \mathcal{L}\mathcal{D} \). It says that, for \( X \in \mathcal{L}\mathcal{D} \), arbitrarily close to \( X \) we can find a subset of \( X \) which is a finite graph.

**Theorem 3.19.** Let \( X \in \mathcal{L}\mathcal{D} \). Given \( \varepsilon > 0 \), there is \( G \subset C(X) \) such that \( H(G, X) < \varepsilon \) and \( G \cap E_0(X) = \emptyset \).

**Proof.** Since \( X = D_1 \cup D_2 \cup \cdots \cup D_l \) and, for each \( i \in \{1, 2, \ldots, l\} \), we have \( D_i \in \mathcal{D} \), by [15, Lemma 4, p. 802], there is \( G_i \in C(D_i) \) such that \( H(D_i, G_i) < \varepsilon \) and \( G_i \cap E_0(D_i) = \emptyset \). Note that \( G_i \cap E_0(X) = \emptyset \). Let \( i, j \in \{1, 2, \ldots, l\} \) be so that \( i < j \) and
$D_1 \cap D_j \neq \emptyset$. Fix $p_i \in G_i$, $p_j \in G_j$ and an arc $A_{ij}$, with end points $p_i$ and $p_j$, such that $A_{ij} \subset D_i \cup D_j$. Since $p_i, p_j \notin E_a(X)$, we have $A_{ij} \cap E_a(X) = \emptyset$. Note that $G_i \cup G_j \cup A_{ij}$ is a subcontinuum of $X$ such that

$$H(D_i \cup D_j, G_i \cup G_j \cup A_{ij}) < \varepsilon \quad \text{and} \quad (G_i \cup G_j \cup A_{ij}) \cap E_a(X) = \emptyset.$$  

Let

$$G = \left( \bigcup_{i=1}^l G_i \right) \cup \{ A_{ij}; \ i, j \in \{ 1, 2, \ldots, l \}, \ i < j \text{ and } D_i \cap D_j \neq \emptyset \}.$$  

Then $G \in C(X)$, $H(G, X) < \varepsilon$ and $G \cap E_a(X) = \emptyset$. □

The following theorem characterizes the elements of the class $\mathcal{L} \mathcal{D}$. It is the equivalent version of [15, Theorem 8, p. 802] for the elements of $\mathcal{L} \mathcal{D}$.

**Theorem 3.20.** Suppose $X \in \mathcal{L}$. Then $X \in \mathcal{L} \mathcal{D}$ if and only if $X$ satisfies the following property:

\[(\star) \quad \text{for each } Z \in C(X) \text{ there is a sequence } \{ A_n \}_{n \in \mathbb{N}} \text{ in } C(X), \text{ whose limit in the Hausdorff metric is } Z, \text{ and } \dim A_n(C(X)) < \infty \text{ for each } n \in \mathbb{N}.\]

**Proof.** Assume first that $X \in \mathcal{L} \mathcal{D}$ and let $Z \in C(X)$. By Corollary 3.6, $Z \in \mathcal{L} \mathcal{D}$. Given $n \in \mathbb{N}$, by Theorem 3.19, there is $B_n \in C(Z)$ such that $H(B_n, Z) < \frac{1}{2^n}$ and $B_n \cap E_a(Z) = \emptyset$. By part (b) of Corollary 3.14, $B_n$ is a finite graph contained in $Z$, so there is a subcontinuum $A_n$ of $B_n$ such that $A_n \cap E_a(X) = \emptyset$ and $H(A_n, B_n) < \frac{1}{2^n}$. By Corollary 3.15, $A_n$ has no essential points of $X$ so, by Theorem 3.16, $\dim A_n(C(X)) < \infty$. Hence, $X$ satisfies $(\star)$.

Assume now that $X$ satisfies $(\star)$ and that $X \notin \mathcal{L} \mathcal{D}$. We claim that:

1) there exist a nondegenerate arc $[a, b]$ in $X$ and an essential point $p$ of $X$ such that $p \in (a, b) \subset [a, b]$, $[a, b] \cap P_X \subset \{ p \}$ and $\dim_{(a,b)}(C(X)) = \infty$.

To show this note that, by Theorem 3.4, $X$ contains either a copy of $W$ or a copy of $F_\omega$ (defined in (2.1) and (2.2)). Assume first that $X$ contains a copy of $W$. Then there exist a nondegenerate arc $[a, b]$ in $X$ and a point $p \in (a, b) \subset [a, b]$ such that $p$ is the limit of a sequence of different ramification points of $X$, all in the subarc $(p, b)$ of $[a, b]$. If $p \notin P_X$, then we can consider the arc $[a, b]$ so that $[a, b] \cap P_X = \emptyset$. If $p \in P_X$ then, since $P_X$ is finite, we can assume that $[a, b]$ is such that $[a, b] \cap P_X = \{ p \}$. In each case it follows that $p$ is an $l$-essential point of $X$ such that $p \in [a, b]$ so, by Theorem 2.7, $\dim_{(a,b)}(C(X)) = \infty$.

Assume now that $X$ contains a copy $F$ of $F_\omega$ with vertex $p$. Then $\text{ord}(p, X) = \infty$, so $p$ is an $l$-essential point of $X$. Let us consider that $p \notin P_X$. Since $P_X$ is finite, we can find a copy $F_0$ of $F_\omega$, with vertex $p$, such that $F_0 \subset F$ and $F_0 \cap P_X = \emptyset$. Let $[a, b]$ be a nondegenerate arc in $F_0$ such that $p \in (a, b) \subset [a, b]$. Then $[a, b] \cap P_X = \emptyset$. If $p \in P_X$ then, since $P_X$ is finite, there exists a nondegenerate arc $[a, b]$ in $F$ such that $p \in (a, b) \subset [a, b]$ and $[a, b] \cap P_X = \{ p \}$. In each case, since $p$ is an essential point of $X$ such that $p \in [a, b]$ and $\dim_{(a,b)}(C(X)) = \infty$. This completes the proof of 1).

Let $[a, b]$ be a nondegenerate arc in $X$ that satisfies 1). Since $X$ satisfies $(\star)$, there is a sequence $\{ M_n \}_{n \in \mathbb{N}}$ in $C(X)$, whose limit in the Hausdorff metric is $[a, b]$, and $\dim_{M_n}(C(X)) < \infty$ for each $n \in \mathbb{N}$. By Theorem 2.7, for every $n \in \mathbb{N}$ and each $q \in M_n$, $q$ is not an essential point of $X$. Thus, $p \notin M_n$ for every $n \in \mathbb{N}$. However, by Theorem 3.18, there is $N \in \mathbb{N}$ such that $p \in M_n$ for every $n > N$. This contradiction shows that $X \notin \mathcal{L} \mathcal{D}$. □

As an application of Theorem 3.20, we present the following result, which is an important part of the proof of the main theorem of this paper.

**Theorem 3.21.** Let $X \in \mathcal{L} \mathcal{D}$ and $Y \in \mathcal{L}$ be such that $C(X)$ is homeomorphic to $C(Y)$. Then $Y \in \mathcal{L} \mathcal{D}$.

**Proof.** Let $h : C(X) \to C(Y)$ be a homeomorphism. Let $Z \in C(Y)$ and $A \in C(X)$ be so that $h(A) = Z$. Since $X \in \mathcal{L} \mathcal{D}$, by Theorem 3.20, there is a sequence $\{ A_n \}_{n \in \mathbb{N}}$ in $C(X)$, whose limit in the Hausdorff metric is $A$, and $\dim_{A_n}(C(X)) < \infty$ for each $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, let $Z_n = h(A_n)$. Then $\{ Z_n \}_{n \in \mathbb{N}}$ is a sequence in $C(Y)$, whose limit in the Hausdorff metric is $h(A) = Z$, and $\dim_{Z_n}(C(Y)) < \infty$ for every $n \in \mathbb{N}$. This implies that $Y$ satisfies $(\star)$ of Theorem 3.20 so, by the same theorem, $Y \in \mathcal{L} \mathcal{D}$.

4. The class $\Omega(X)$

We recall that if $V$ is a 2-cell, then $\partial V$ represents the manifold boundary of $V$. Given a continuum $X$, we consider the
following subset of $C(X)$:

$$\Omega(X) = \{ A \in C(X) : \text{there exists a 2-cell } V \text{ in } C(X) \text{ such that } A \in \text{Int}_{C(X)}(V) \cap \partial V \}.$$ 

In connection to the problem of finding conditions on a continuum $X$, in order that $X$ has unique hyperspace, the class $\Omega(X)$ plays the role described in the following result. Such result was proved in [15, Lemma 2, p. 801] for the elements of $\mathbb{D}$, but it is valid in general, with the same proof.

**Theorem 4.1.** Let $X$ and $Y$ be continua such that $C(X)$ is homeomorphic to $C(Y)$. Then $\Omega(X)$ is homeomorphic to $\Omega(Y)$.

In this section we will prove that if $X \in \mathbb{D}$ is different from an arc, then $C_{C(X)}(\Omega(X))$ is homeomorphic to $X$ (see Theorem 4.10). To do this we require some previous results. The first two are proved in [15, Proposition 2, p. 799] and [15, Lemma 6, p. 803], respectively.

**Theorem 4.2.** Let $X$ be a continuum. If $A \in \Omega(X)$, then $\dim A(C(X)) \leq 2$.

**Theorem 4.3.** Let $X$ be a locally connected continuum and $p \in X$. If $\dim_{|p|}(C(X)) < \infty$, then $\{ p \} \in C_{C(X)}(\Omega(X))$.

**Theorem 4.4.** Let $X$ be a locally connected continuum that satisfies $(\star)$ of Theorem 3.20. Let $Y$ be a continuum such that $C(X)$ is homeomorphic to $C(Y)$. Then $F_1(Y) \subset C_{C(Y)}(\Omega(Y))$.

**Proof.** By Theorem 2.2, $Y$ is locally connected. Let $h : C(X) \to C(Y)$ be a homeomorphism, $\{ p \} \in F_1(Y)$ and $A \in C(X)$ be such that $h(A) = \{ p \}$. By $(\star)$ of Theorem 3.20, there is a sequence $\{ A_n \}_n$ in $C(X)$, whose limit in the Hausdorff metric is $A$, and $\dim_{A_n}(C(X)) < \infty$ for each $n \in \mathbb{N}$. Then $\{ h(A_n) \}_n$ is a sequence in $C(Y)$, whose limit in the Hausdorff metric is $h(A) = \{ p \}$, and $\dim_{h(A_n)}(C(Y)) < \infty$ for each $n \in \mathbb{N}$. Let $\{ p_n \}_n$ be a convergent sequence in $Y$, whose limit is $p$, and $p_n \in h(A_n)$ for every $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, since $Y$ is locally connected, $p_n \in h(A_n)$ and $\dim_{h(A_n)}(C(Y)) < \infty$, by Theorem 2.1, we have $\dim_{p_n}(C(Y)) < \infty$ and then, by Theorem 4.3, $\{ p_n \} \in C_{C(Y)}(\Omega(Y))$. Thus, $\{ p \} \in C_{C(Y)}(\Omega(Y))$. □

Combining Theorems 3.20 and 4.4, we obtain the following result which is the equivalent version of [15, Lemma 7, p. 803], for the elements of $\mathbb{D}$.

**Theorem 4.5.** Let $X \in \mathbb{D}$ and let $Y$ be a continuum such that $C(X)$ is homeomorphic to $C(Y)$. Then $F_1(Y) \subset C_{C(Y)}(\Omega(Y))$.

**Theorem 4.6.** Let $X \in \mathbb{D}$ and $A \in \Omega(X)$, then

$$A \cap E_0(X) = \emptyset \quad \text{and} \quad A \cap R(X) = \emptyset.$$ 

**Proof.** Let $A \in \Omega(X)$. If $A \cap E_0(X) \neq \emptyset$ then, by Corollary 3.15, there exists $p \in A$ such that $p$ is an essential point of $X$. Then, by Theorem 3.16, $\dim_A(C(X)) = \infty$. Since this contradicts Theorem 4.2, we obtain that $A \cap E_0(X)$ is empty.

Now assume that there is $p \in A \cap R(X)$, since $A \in \Omega(X)$, there exists a 2-cell $V$ in $C(X)$ such that $A \in \text{Int}_{C(X)}(V) \cap \partial V$. Let $\varepsilon > 0$ be such that $BC_{C(X)}(A, \varepsilon) \subset V$. Since $3 \leq \text{ord}(p, X) < \infty$, there exist $n \in \mathbb{N} - \{ 1, 2 \}$ and an $n$-od $T$ in $X$ such that $T \in BC_{C(X)}(A, \frac{1}{2})$. By Theorem 2.4, there is an $n$-cell $\Gamma$ such that $T \subset \Gamma \subset BC_{C(X)}(A, \varepsilon)$. This implies that the 2-cell $V$ contains the $n$-cell $\Gamma$, a contradiction. Therefore, $A \cap R(X) = \emptyset$. □

**Theorem 4.7.** Let $X$ be a continuum, $A \in C(X) - F_1(X)$ and $[p, q]$ be a nondegenerate arc in $X$, with end points $p$ and $q$, such that $\{ p, q \} = [p, q] - \{ p, q \}$ is open in $X$ and $A \subset (p, q)$. Then $A \notin \Omega(X)$.

**Proof.** Let us assume that $A \in \Omega(X)$, since $A$ is a nondegenerate subcontinuum of an arc, there exist $a, b \in A$ such that $a \neq b$ and $A = [a, b]$. We can consider that, in the natural order, $a \prec [p, q]$ from $p$ to $q$, we have $p < a < b < q$. Since $A \in \Omega(X)$, there exists a 2-cell $V$ in $C(X)$ such that $A \in \text{Int}_{C(X)}(V) \cap \partial V$. Let $\delta > 0$ be such that $BC_{C(X)}(A, \delta) \subset V$. Fix $c_1, c_2, d_1, d_2 \in [p, q]$ so that

$$p \leq c_1 < a < c_2 < d_2 < b < d_1 \leq q.$$ 

Then $[c_1, d_1] \in BC_{C(X)}(A, \delta)$ and $[c_2, d_2] \in BC_{C(X)}(A, \delta)$. It is not difficult to prove that

$$D = \{ [c, d] \subset [p, q] : c_1 \leq c \leq c_2 \text{ and } d_2 \leq d \leq d_1 \}$$ 

is a 2-cell in $C(X)$ such that

$$A \in D - \partial D \subset D \subset BC_{C(X)}(A, \delta) \subset V.$$ 

Since both $D$ and $V$ are 2-cells and $D \subset V$, the set $D - \partial D$ is open in $V$ [37, 19.34, p. 123]. Thus, $A$ is in the manifold interior of $V$, so $A \notin \partial V$. Since this is a contradiction, we conclude that $A \notin \Omega(X)$.
For a continuum X, a free arc in Y is an arc A in Y, joining two different points p and q of Y, such that the set A − {p, q} is open in Y. We recall that if p ∈ Y, then C(p, Y) = {A ∈ C(Y): p ∈ A}. If A is an arc in Y, joining two different points p and q, and B is a simple closed curve in Y, then both C(A) and C(B) are 2-cells in C(Y). Using Theorem 4.7, it follows that

\[ \Omega(A) = \partial C(A) = C(p, A) \cup C(q, A) \cup F_1(A) \] and \[ \Omega(B) = F_1(B). \]

Note that \( \Omega(A) \) is homeomorphic to \( \Omega(B) \).

**Theorem 4.8.** Let \( X \in \Sigma \) be different from an arc and \( A \in C(X) \). Then \( A \in \Omega(X) \) if and only if \( A \) satisfies exactly one of the following conditions:

(a) \( A = \{p\} \), for some \( p \in X - (R(X) \cup E_0(X)) \);
(b) \( A \) is nondegenerate arc \( \{e, p\} \) in \( X \), such that \( e \in E(X) - E_0(X) \) and \( \{e, p\} \subset O(X) \).

Moreover \( A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X) \) if and only if \( A \) satisfies exactly one of the following conditions:

(c) \( A = \{r\} \), for some \( r \in R(X) \cup E_0(X) \);
(d) \( A = \{e, r\} \), for some \( e \in E(X) - E_0(X) \) and \( r \in R(X) \) so that \( \{e, r\} \subset O(X) \).

**Proof.** If \( X \) is a simple closed curve, then

\[ \Omega(X) = F_1(X) = \text{Cl}_{C(X)}(\Omega(X)). \]

Thus, in this situation, the result is true. Consider then that \( X \) is not a simple closed curve. To show the first part of the theorem, assume that \( A \in \Omega(X) \). By Theorem 3.11, we have

\[ X = E_0(X) \cup (E(X) - E_0(X)) \cup O(X) \cup R(X). \]

By Theorem 4.6, \( A \cap E_0(X) = \emptyset = A \cap R(X) \). Thus, \( A \subset (E(X) - E_0(X)) \cup O(X) \). Since \( A \cap E_0(X) = \emptyset \), by part (b) of Corollary 3.14, \( A \) is a finite graph in \( X \) and, since \( A \cap R(X) = \emptyset \), either \( A \) is a one-point-set or a nondegenerate arc in \( X \) without ramification points of \( X \). Assume first that \( A = \{p\} \), for some \( p \in X \). Then \( p \in (E(X) - E_0(X)) \cup O(X) \), \( A \) satisfies (a).

Assume now that \( A \) is a nondegenerate arc \( \{a, b\} \) in \( X \). Consider that \( A \subset O(X) \). By Corollary 3.13, \( O(X) \) is open in \( X \). Hence, there is an arc \( \{p, q\} \in O(X) \) such that \( \{p, q\} = [p, q] - \{p, q\} \) is open in \( X \) and \( A \subset (p, q) \). Then, by Theorem 4.7, \( A \not\subset \Omega(X) \). Since this is a contradiction and \( A \subset (E(X) - E_0(X)) \cup O(X) \), it follows that \( A \cap (E(X) - E_0(X)) \neq \emptyset \). Since \( X \) is not an arc, \( A \) is of the form described in (b).

Let us assume now that \( A \in C(X) \) satisfies (a). Then \( A = \{p\} \), for some \( p \in (E(X) - E_0(X)) \cup O(X) \). If \( p \in O(X) \) then, since \( O(X) \) is open in \( X \), there is a nondegenerate free arc \( \{a, b\} \) in \( O(X) \) such that \( p \in (a, b) \subset \{a, b\} \). Let \( V = C([a, b]) \). Then \( V \) is a 2-cell in \( C(X) \) such that \( A \in \partial V \). Let \( \varepsilon > 0 \) be such that \( B_{X}(p, \varepsilon) \subset (a, b) \). It is easy to see that \( B_{C(X)}(A, \varepsilon) \subset V \), so \( A \in \text{Int}_{C(X)}(V) \). Hence, \( A \in \Omega(X) \).

If \( p \in E(X) - E_0(X) \), then there exists \( t \in X \) such that \( \{p, t\} \subset O(X) \). Then \( \{p, t\} \) is a free arc in \( X \). Let \( V = C([p, t]) \). Then \( V \) is a 2-cell in \( C(X) \) such that \( A \in \partial V \). Let \( \varepsilon > 0 \) be such that \( B_{X}(t, \varepsilon) \subset (p, t) \). It is easy to see that \( B_{C(X)}(A, \varepsilon) \subset V \), so \( A \in \text{Int}_{C(X)}(V) \). Hence, \( A \in \Omega(X) \).

Assume now that \( A \in C(X) \) satisfies (b). Then \( A = \{e, p\} \), for some \( e \in E(X) - E_0(X) \) and \( \{e, p\} \subset O(X) \). Since \( O(X) \) is open in \( X \), there exists \( q \in X \) such that \( A \subset \{e, q\} \subset \{e, q\} \) and \( \{e, q\} \subset O(X) \). Then \( \{e, q\} \) is a free arc in \( X \). Let \( V = C([e, q]) \). Then \( V \) is a 2-cell in \( C(X) \) and, since \( A \subset C([e, q]) \), we have \( A \in \partial V \). Let \( \varepsilon > 0 \) be such that \( N(A, \varepsilon) \subset \{e, q\} \). It is easy to see that \( B_{C(X)}(A, \varepsilon) \subset V \), so \( A \in \text{Int}_{C(X)}(V) \). Hence, \( A \in \Omega(X) \). This concludes the proof of the first part of the theorem.

To show the second part of the theorem, assume that

\[ A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X). \]

Then there is a sequence \( \{A_n\}_{n \in \mathbb{N}} \) in \( \Omega(X) \), whose limit in the Hausdorff metric is \( A \). If for infinitely many indices \( n \) we have \( A_n \in F_1(X) \), then \( A \in F_1(X) \). Thus, \( A = \{p\} \) for some \( p \in X \). Since \( A \not\subset \Omega(X) \), by (a), we have \( p \in R(X) \cup E_0(X) \). Hence, \( A \) satisfies (c). Assume now that \( A_n \in F_1(X) \) only for finitely many indices \( n \). Taking a subsequence of \( \{A_n\}_{n \in \mathbb{N}} \), if necessary, we can consider that \( A_n \not\subset F_1(X) \), for every \( n \in \mathbb{N} \). Thus, given \( n \in \mathbb{N} \), by (b), \( A_n = \{e_n, p_n\} \) for some \( e_n \in E(X) - E_0(X) \) and \( \{e_n, p_n\} \subset O(X) \). We claim that:

1) if \( Y \in C(X) \cap \Sigma \) is such that \( A_n \subset Y \), for infinitely many indices \( n \), then \( A \) satisfies either (c) or (d).

Let \( Y \) be as assumed. Taking a subsequence of \( \{A_n\}_{n \in \mathbb{N}} \), if necessary, we can consider that \( A_n \subset Y \) for every \( n \in \mathbb{N} \). Then \( A \subset Y \). Note that \( e_n \in E(Y) - E_0(Y) \) and \( \{e_n, p_n\} \subset O(Y) \), for every \( n \in \mathbb{N} \). Taking subsequences of \( \{e_n\}_{n \in \mathbb{N}} \) and \( \{p_n\}_{n \in \mathbb{N}} \), if necessary, we can also assume that \( e_n \to e \) and \( p_n \to p \), for some \( e \in E(Y) \) and \( p \in Y \). Let us consider that the set
There is a nondegenerate arc described on p. 2072, the sequence of arcs \( \{ e_n \}_{n \in \mathbb{N}} \) converges to \( e \) by property (S) described on p. 2072, the sequence of arcs \( \{ e_n \}_{n \in \mathbb{N}} \) converges in the Hausdorff metric, to \( e \). Since \( \{ e_n \}_{n \in \mathbb{N}} \subseteq \{ e \}_{n \in \mathbb{N}} \), for every \( k \in \mathbb{N} \), it follows that the sequence of arcs \( \{ e_n \}_{n \in \mathbb{N}} \) converges in the Hausdorff metric, to \( e \). Since the sequence \( \{ e_n \}_{n \in \mathbb{N}} \) also converges, in the Hausdorff metric, to \( A \), we conclude that \( A = \{ e \} \), so \( A \) satisfies (c).

Let us assume now that the set \( \{ e_n : n \in \mathbb{N} \} - \{ e \} \) is finite. Then there is \( N \in \mathbb{N} \) such that \( e_n = e \), for every \( n \geq N \). Hence, \( e \in E(X) - E_0(X) \) and \( \lambda = [e, p_n] \), for each \( n \geq N \). Since the sequence \( \{ p_n \}_{n \geq N} \) converges to \( p \), by property (S) described on p. 2072, the sequence of arcs \( \{ A_n \}_{n \in \mathbb{N}} \) converges in the Hausdorff metric, to \( \{ e, p \} \). This implies that \( A = \{ e, p \} \), if \( p = e \), then \( A \) satisfies (a) so \( A \in \mathcal{Q} \). This is a contradiction, so \( p \neq e \). Since \( (e, p_n) \subseteq O(X) \), for every \( n \geq N \), it follows that \( (e, p) \subset O(X) \). If \( p \in O(X) \), then \( A \) satisfies (b), so \( A \in \mathcal{Q} \). This contradiction shows that \( p \neq O(X) \). If \( p \in E(X) \) then \( X = Y = A \), so \( X \) is an arc. This is a contradiction, so \( p \notin E(X) \). Hence, \( p \in R(X) \) and then \( A \) satisfies (d). This completes the proof of 1.

Recall that, by Theorem 3.7, \( X = D_1 \cup D_2 \cup \cdots \cup D_l \), where each \( D_i \in \mathcal{D} \) and \( \{ D_i \cap D_j \} \leq 0 \) for every \( i, j \in \{ 1, 2, \ldots, l \} \) with \( i \neq j \). Let \( t_i \in \{ 1, 2, \ldots, l \} \) be such that \( e_n \in D_{t_i} \), for infinitely many indices \( n \). Taking a subsequence of \( \{ A_n \}_{n \in \mathbb{N}} \), if necessary, we can consider that \( e_n \in D_{t_i} \), for every \( n \in \mathbb{N} \). If \( A_n \subseteq \{ e \} \), for infinitely many indices \( n \) then, by 1), \( A \) satisfies either (c) or (d).

Assume then that \( A_n \subseteq D_{t_i} \) only for finitely many indices \( n \). Taking a subsequence of \( \{ A_n \}_{n \in \mathbb{N}} \), if necessary, we can consider that \( A_n \subseteq D_{t_i} \), for all \( n \in \mathbb{N} \). Since \( A_1 \subseteq D_{t_i} \), there exist \( i_2 \in \{ 1, 2, \ldots, l \} \) such that \( A_1 = \{ e_1, q_1 \} \cap \{ q_1, p_1 \} \subseteq D_{i_2} \) and \( \{ q_1, p_1 \} \cap \{ D_i - \{ q_1 \} \} \neq \emptyset \). We claim that:

2) \( D_{i_1} = \{ e_1, q_1 \} \) and \( e_n = e_1, \) for every \( n \in \mathbb{N} \).

To show 2) note that, since \( e_1 \in E(X) \) and \( q_1 \in P_X \subset X - E(X) \), we have \( e_1 \neq q_1 \). Moreover \( e_1 \in E(D_{i_1}) \) and, since \( q_1 \in O(X) \), it follows that \( q_1 \in E(D_{i_1}) \cap E(D_{i_2}) \). If there exists \( t \in D_{i_1} - \{ e_1, q_1 \} \) then, since \( D_{i_1} \) is an arc with one common end point, any arc in \( D_{i_1} \) from \( t \) to a point in \( \{ e_1, q_1 \} \) intersects \( \{ e_1, q_1 \} \). Then, some point of \( \{ e_1, q_1 \} \) is a ramification point of \( D_{i_1} \). This implies that some point of \( \{ e_1, p_1 \} \) is a ramification point of \( X \). This contradicts the fact that \( (e_1, p_1) \subset O(X) \), so \( D_{i_1} = \{ e_1, q_1 \} \). Now let \( n \in \mathbb{N} \). Since \( e_n \in E(X) \) \( \cap D_{i_1} \) and \( q_1 \notin E(X) \), we have \( e_n \notin E(D_{i_1} \cap \{ q_1 \} \). Then \( e = e_1 \), since \( D_{i_1} = \{ e_1, q_1 \} \). This shows 2).

Now we claim that:

3) \( D_{i_1} \cap D_{i_2} = \{ q_1 \} \).

To show 3) assume, on the contrary, that there exists \( t \in D_{i_1} \cap D_{i_2} \) such that \( t \neq q_1 \). Then \( t \in D_{i_2} \cap P_X \) so, by Theorem 3.9, there is a nondegenerate arc \( C \) in \( D_{i_2} \) such that \( C \cap P_X = \{ t \} \). Thus, \( C \cap D_{i_1} = \{ t \} \). Since \( D_{i_1} = \{ e_1, q_1 \} \) \( \cap D_{i_1} = \{ t \} \), if \( t = e_1 \), then \( e_1 \notin E(X) \) and, if \( t \in (e_1, q_1) \), then \( t \in R(X) \). In the first case we contradict the fact that \( e_1 \in E(X) \) and, in the second case, the fact that \( (e_1, q_1) \subset O(X) \). Hence, \( D_{i_1} \cap D_{i_2} = \{ q_1 \} \). This shows 3).

Now we are going to prove that:

4) there is no \( i \in \{ 1, 2, \ldots, l \} \) \( - \{ i_1, i_2 \} \) such that \( q_1 \in D_{i_1} \cap D_{i_2} \cap D_{i_1} \) and there is no \( i \in \{ 1, 2, \ldots, l \} \) \( - \{ i_1 \} \) such that \( (D_{i_1} \cap {q_1}) \cap D_{i_2} \neq \emptyset \).

To show the first part of 4), let \( i \in \{ 1, 2, \ldots, l \} \) \( - \{ i_1, i_2 \} \) be such that \( q_1 \in D_{i_1} \cap D_{i_2} \cap D_{i_1} \). Then \( q_1 \in D_{i_1} \cap P_X \) so, by Theorem 3.9, there is a nondegenerate arc \( D \) in \( D_{i_1} \) such that \( D \cap P_X = \{ q_1 \} \). This implies that \( q_1 \in R(X) \) and, since this contradicts the fact that \( q_1 \in O(X) \), the first part of 4) is true. To show the second part of 4), let \( i \in \{ 1, 2, \ldots, l \} \) \( - \{ i_1 \} \) be such that \( (D_{i_1} \cap \{ q_1 \}) \cap D_{i_2} \neq \emptyset \). Let \( t \in (D_{i_1} \cap \{ q_1 \}) \cap D_{i_2} \). Then \( t \in P_X \subset X - E(X) \), so \( t \neq e_1 \). Since \( t \in D_{i_1} \cap P_X \), by Theorem 3.9, there is a nondegenerate arc \( D' \) in \( D_{i_1} \) such that \( D' \cap P_X = \{ t \} \). This implies that \( t \in R(X) \). Since this contradicts the fact that \( (e_1, p_1) \subset O(X) \), the second part of 4) is true.

Now we claim that:

5) \( D_{i_1} \subset A_n \), for every \( n \in \mathbb{N} \).

To show 4) let \( n \in \mathbb{N} \). Then, by 2), \( A_n = \{ e_1, p_n \} \). Since both \( A_n \) and \( \{ e_1, q_1 \} \) are arcs with one common end point, \( A_n \cap \{ e_1, q_1 \} \) is a subarc of \( \{ e_1, q_1 \} = D_{i_1} \). If such subarc is proper then, since \( A_n \subset D_{i_1} \), there is \( i \in \{ 1, 2, \ldots, l \} \) \( - \{ i_1 \} \) such
that \((D_1 - \{q_1\}) \cap D_1 \neq \emptyset\). This contradicts the second part of 4), so \(A_n \cap \{e_1, q_1\} = \{e_1, q_1\}\), and then \(D_{i_1} = \{e_1, q_1\} \subset A_n\). This shows 5).

Now we claim that:

6) \(D_1 \cup D_2 \subset C(X) \cap \mathcal{D}\).

By 3), it follows that \(D_1 \cup D_2 \subset C(X)\). Since \(q_1 \in O(X) \cap P_X\), we have \(q_1 \in E(D_2)\). By Theorem 3.10, \(q_1 \notin E_a(D_2)\), so \(q_1 \in E(D_2) - E_a(D_2)\) and, since \(D_1\) is an arc that intersects \(D_2\) only at \(q_1\), \(D_1 \cup D_2\) is homeomorphic to \(D_1\). Hence, \(D_1 \cup D_2 \subset \mathcal{D}\). This shows 6).

If \(A_n \subset D_1 \cup D_2\) for infinitely many indices \(n\) then, by 1) and 6), \(A\) satisfies either (c) or (d). Assume then that \(A_n \subset D_1 \cup D_2\) only for finitely many indices \(n\). Taking a subsequence of \(\{A_n\}_{n \in \mathbb{N}}\), if necessary, we can consider that \(A_n \subset D_1 \cup D_2\), for all \(n \in \mathbb{N}\). Since \(A_1 \subset D_1 \cup D_2\), there exist \(i_3 \in \{1, 2, \ldots, l\} - \{i_1, i_2\}\) and \(q_2 \in D_{i_3} \cap D_{i_1}\) such that \(A_1 = \{e_1, q_1\} \cup \{q_1, q_2\} \cap \{q_2, p_1\} \cap (D_{i_3} - \{q_2\}) \neq \emptyset\). Note that \(q_2 \neq q_1\). Proceeding as in the proof of 2)–6) we infer that \(D_{i_2} = [q_1, q_2] \subset D_{i_3} \cap (D_{i_1} \cup D_{i_2}) = [q_2]\), \(D_{i_4} \cup D_{i_2} \cup D_{i_3} \subset C(X) \cap \mathcal{D}\) and \(D_{i_1} \cup D_{i_3} \subset A_n\), for every \(n \in \mathbb{N}\). We also infer that there is no \(i \in \{1, 2, \ldots, l\} - \{i_1, i_2, i_3\}\) such that \(q_2 \in D_{i_2} \cap D_{i_3} \cap D_i\) and that there is no \(i \in \{1, 2, \ldots, l\} - \{i_2\}\) such that \((D_{i_2} - \{q_1, q_2\}) \cap D_i \neq \emptyset\).

If we continue the above argument, we conclude that there exists \(Y \subset C(X) \cap \mathcal{D}\) such that \(A_n \subset Y\) for infinitely many indices \(n\). By 1), this implies that \(A\) satisfies either (c) or (d). This shows that if \(A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)\), then \(A\) satisfies either (c) or (d).

Now assume that \(A \in \text{Cl}\) satisfies either (c) or (d). Then \(A\) does not satisfy (a) nor (b) so, by the first part of the theorem, \(A \notin \Omega(X)\). If \(A\) satisfies (c), then \(A = [r]\), for some \(r \in R(X) \cup E_a(X)\). By Theorem 3.17, there exists a sequence \(\{p_n\}_{n \in \mathbb{N}}\) in \(O(X)\) that converges to \(r\). Then \(A = [r]\), in the Hausdorff metric, of the sequence \(\{\{p_n\}_{n \in \mathbb{N}}\}\) and, by the first part of the theorem, \(\{p_n\}_{n \in \mathbb{N}}\) in \(\Omega(X)\) for every \(n \in \mathbb{N}\). Thus, \(A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)\).

If \(A\) satisfies (d), then \(A = [e, r]\) for some \(e \in E(X) - E_a(X)\) and \(r \in R(X)\) such that \((e, r) \subset O(X)\). Let \(\{p_n\}_{n \in \mathbb{N}}\) be a sequence in \(E(X)\) that converges to \(r\). Given \(n \in \mathbb{N}\) the subarc \([e, p_n]\) of \([e, r]\) satisfies (b) so, by the first part of the theorem, \([e, p_n]\) \subset \Omega(X). By property (S) described on p. 2072, the sequence of arcs \([\{e, p_n\}_{n \in \mathbb{N}}\) converges, in the Hausdorff metric, to \([e, r]\). Thus, \(A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)\).

Corollary 4.9. If \(X \in \mathcal{L}\), then \(F_1(X) \subset \text{Cl}_{C(X)}(\Omega(X))\).

The following result is another important part of the proof of the main theorem of this paper. By Theorem 4.8, \(\text{Cl}_{C(X)}(\Omega(X))\) is the union of \(F_1(X)\) with all the sets of the form \(C([e, r], [e, r])\), where \([e, r]\) satisfies (d) of such theorem. Every such set \(C([e, r], [e, r])\) is an arc in \(C(X)\), with end points \([e, r]\) that intersects \(F_1(X)\) only at \([e, r]\). If \([e_1, r_1]\) and \([e_2, r_2]\) are distinct elements of \(C(X)\) that satisfy (d) then \(C([e_1, r_1]) \cap C([e_2, r_2]) = \emptyset\). Let \(\{e_n\}_{n \in \mathbb{N}}\) be a sequence of elements in \(C(X)\) that satisfy (d) and \(e \in X\) is the limit of the sequence \(\{e_n\}_{n \in \mathbb{N}}\), then \(C([e_n, [e_n, r_n]])_{n \in \mathbb{N}}\) converges, in the Hausdorff metric, to \([e]\). Though this is the case, we will show that \(\text{Cl}_{C(X)}(\Omega(X))\) is homeomorphic to \(X\), using a different argument.

Theorem 4.10. If \(X \in \mathcal{L}\) is different from an arc, then \(\text{Cl}_{C(X)}(\Omega(X))\) is homeomorphic to \(X\).

Proof. Since the result is true when \(X\) is a simple closed curve, assume that \(X\) is not a simple closed curve. We will first define a one-to-one function \(g : \Omega(X) \rightarrow X\) which we will extend to a homeomorphism \(G : \text{Cl}_{C(X)}(\Omega(X)) \rightarrow X\). To define \(g\) let us consider the following class of arcs in \(X\):

\[
\mathcal{E} = \{[e, r] \mid e \in E(X) - E_a(X), r \in R(X) \text{ and } (e, r) \subset O(X)\}.
\]

Since \(X\) is different from an arc and a simple closed curve, for every \(e \in E(X) - E_a(X)\) there is a unique \(r_e \in R(X)\) such that \([e, r_e] \in \mathcal{E}\). For every such \([e, r_e]\), fix a point \(s_e \in (e, r_e)\) as well as two homeomorphisms \(f_1^{s_e} : [e, r_e] \rightarrow [s_e, r_e]\) and \(f_2^{s_e} : [e, r_e] \rightarrow [e, s_e]\) so that

\[
f_1^{s_e}(e) = s_e, \quad f_1^{s_e}(r_e) = r_e, \quad f_2^{s_e}(e) = s_e \quad \text{and} \quad f_2^{s_e}(r_e) = e.
\]

Note that the sets \((e, s_e)\) and \([s_e, r_e]\) are disjoint. We are ready to define \(g\). Let \(A \in \Omega(X)\). By Theorem 4.8, \(A\) satisfies either (a) or (b) of such theorem. Assume first that \(A = [a]\), for some \(a \in O(X)\) and that, for every \([e, r] \in \mathcal{E}\), we have \(a \notin [e, r]\). Define

\[
g(A) = g([a]) = a.
\]
Let us assume now that $A = \{q\}$, for some $q \in (E(X) - E_d(X)) \cup O(X)$ so that $q \in [e, r_e]$, for some $[e, r_e] \in \mathcal{E}$. Define

$$g(A) = g(\{q\}) = f_{1g}^e(q).$$

Note that $g(A) \in [s_e, r_e] \subset [e, r_e]$, so $g(A)$ is an element of some member of $\mathcal{E}$. Note also that we have defined $g$ for every member of $\Omega(X)$ that satisfies (a) of Theorem 4.8. Now assume that $A$ satisfies (b) of Theorem 4.8. Then there exists $[e, r_e] \in \mathcal{E}$ and $p \in (e, r_e)$ such that $A = [e, p] \subset [e, r_e]$. Define

$$g(A) = g([e, p]) = f_{2g}^e(p).$$

Note that $g(A) \in (e, s_e) \subset [e, r_e]$, so $g(A)$ is an element of some member of $\mathcal{E}$. Note also that $g$ is a well-defined function so that $g(A) \in O(X)$, for every $A \in \Omega(X)$. We claim that:

1) $g$ is a one-to-one function.

To show 1) let $A, D \in \Omega(X)$ be such that $g(A) = g(D)$. Assume first that $A = [a]$, for some $a \in O(X)$ and that, for every $[e, r] \in \mathcal{E}$, we have $a \not\in [e, r]$. Then $g(A) = a$, so $g(D) = a$. This implies that $D$ is an element of $\Omega(X)$ so that $g(D)$ does not belong to any member of $\mathcal{E}$. Then, by the way $g$ is defined, it follows that $D = [a] = A$.

Now assume that $A = \{q\}$, for some $q \in (E(X) - E_d(X)) \cup O(X)$ so that $q \in [e, r_e]$ and $[e, r_e] \in \mathcal{E}$. Then $g(A) = f_{1g}^e(q)$, so $g(D) = f_{1g}^e(q) \in [s_e, r_e] \subset [e, r_e]$. This implies that $D$ is not of the form of $A$ described in the previous paragraph and also that $D \not\subset [e, r_e]$. If $D = [e, p]$ for some $p \in (e, r_e)$, then $g(D) = f_{2g}^e(p) \in (e, s_e)$. This contradicts the fact that $g(D) \in [s_e, r_e]$. Then $D = [r]$, for some $r \in (E(X) - E_d(X)) \cup O(X)$ so that $r \in [e, r_e]$. Hence, $g(D) = f_{1g}^e(r)$, so $f_{1g}^e(q) = f_{1g}^e(r)$ and, since $f_{1g}^e$ is a one-to-one function, $q = r$. Thus, $D = A$.

Now assume that there exists $[e, r_e] \in \mathcal{E}$ and $p \in (e, r_e)$ such that $A = [e, p] \subset [e, r_e]$. Then $g(A) = f_{2g}^e(p)$, so $g(D) = f_{2g}^e(p) \in (e, s_e) \subset [e, r_e]$. This implies that $D$ is not of the form of $A$ described in the first paragraph of the proof of 1), and also that $D \subset [e, r_e]$. If $D = [q]$, for some $q \in (E(X) - E_d(X)) \cup O(X)$ so that $q \in [e, r_e]$, then $g(D) = f_{1g}^e(q) \in [s_e, r_e]$. This contradicts the fact that $g(D) \in (e, s_e)$, so $D = [e, r]$ for some $r \in (e, r_e)$. Then $g(D) = f_{2g}^e(r)$, so $f_{2g}^e(p) = f_{2g}^e(r)$ and, since $f_{2g}^e$ is a one-to-one function, we have $p = r$. Thus, $D = A$. This completes the proof of 1).

We now define $G$. Let $A \in \text{Cl}_{C(X)}(\Omega(X))$. If $A \in \Omega(X)$, then $G(A) = g(A)$. Assume now that $A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$. Then $A$ satisfies either (c) or (d) of Theorem 4.8. If $A$ satisfies (c), then $A = \{s\}$, for some $s \in R(X) \cup E_d(X)$. Define

$$G(A) = G(\{s\}) = s.$$ 

If $A$ satisfies (d), then $A = [e, r_e]$, for some $[e, r_e] \in \mathcal{E}$. Define

$$G(A) = G([e, r_e]) = e.$$ 

The function $G$ is well defined and $G(A) \in X - O(X)$, for every $A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$. Note that:

2) if $A \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$, then $G(A) \in R(X) \cup E_d(X)$ if $A$ satisfies (c) of Theorem 4.8, while $G(A) \in E(X) - E_d(X)$, if $A$ satisfies (d) of Theorem 4.8.

We claim that:

3) $G$ is a one-to-one function.

To show 3) note that since $G$ extends $g$, by 1), $G$ is one-to-one in $\Omega(X)$. Note also that $G$ extends $g$, $g(A) \in O(X)$ for each $A \in \Omega(X)$ and $G(B) \in X - O(X)$, for every $B \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$. According to this and 2), to complete the proof of 3), it is enough to show that if $A, B \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$ are such that $G(A) = G(B)$ and either both $A$ and $B$ satisfy (c) of Theorem 4.8, or both $A$ and $B$ satisfy (d) of Theorem 4.8, then $A = B$. So let $A$ and $B$ be as assumed. If both $A$ and $B$ satisfy (c), then $A = \{s\}$ and $B = \{t\}$ for some $s, t \in R(X) \cup E_d(X)$. Thus, $s = G(A) = G(B) = t$, so $A = B$. If both $A$ and $B$ satisfy (d), then $A = [e_1, r_1]$ and $B = [e_2, r_2]$, for some $[e_1, r_1], [e_2, r_2] \in \mathcal{E}$. Then $e_1 = G(A) = G(B) = e_2$ and, since for every $e \in (E(X) - E_d(X)$ there is a unique $r_e \in R(X)$ such that $[e, r_e] \in \mathcal{E}$, we have $r_1 = r_2$. Then $A = B$ and the proof of 3) is complete.

Now we claim that:

4) $G$ is an onto function.

To show 4) let $p \in X$. Then

$$p \in (E(X) - E_d(X)) \cup E_d(X) \cup R(X) \cup O(X).$$ 

Assume first $p \in (E(X) - E_d(X))$. Let $r_p \in R(X)$ be such that $[p, r_p] \in \mathcal{E}$. Then $[p, r_p] \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$ and $G([p, r_p]) = p$. Assume now that $p \in R(X) \cup E_d(X)$. Then $[p] \in \text{Cl}_{C(X)}(\Omega(X)) - \Omega(X)$ and $G([p]) = p$. Assume now that $p \in O(X)$ and
that, for every \([e, r] \in \mathcal{E}\), we have \(p \notin [e, r]\). Then \([p] \in \Omega(X)\) and \(G([p]) = g([p]) = p\). Finally assume that \(p \in O(X)\) and that \(p \notin [e, r]\), for some \([e, r] \in \mathcal{E}\). Then, by the definition of \(f^{se}_1\), we have \(t = (f^{se}_1)^{-1}(p) \in [e, r]\). Thus, \(A = [t] \in \Omega(X)\) and \(G(A) = g(A) = f^{se}_1(t) = p\). If \(p \in (e, s_e)\) and we let \(q = (f^{se}_2)^{-1}(p)\), then \(q \in (e, r_e)\), \([q, e] \in \Omega(X)\) and \(G([q, e]) = g([q, e]) = f^{se}_2(q) = p\). This shows that \(G\) is onto.

Now we claim that

5) \(G\) is a continuous function in \(\Omega(X)\).

To show 5) let \(A \in \Omega(X)\) and \(\varepsilon > 0\). We need to find \(\delta > 0\) such that

\[
G(\overline{B}_X(A, \delta) \cap \text{Cl}(\Omega(X))) \subset B_X(G(A), \varepsilon). \tag{4.1}
\]

Assume first that \(A = \{a\}\), for some \(a \in O(X)\) and that, for every \([e, r] \in \mathcal{E}\), we have \(a \notin [e, r]\). Then \(G(A) = g(A) = a\). Since \(X\) is locally connected and \(O(X)\) is open in \(X\) (Corollary 3.13), there is an open and connected subset \(U\) of \(X\) such that \(a \in U \subset O(X)\). If \(U\) intersects some element \([e, r] \in \mathcal{E}\), then \(r \in U\), and this contradicts the fact that \(U \subset O(X)\). Hence, \(U\) does not intersect any element of \(\mathcal{E}\). Let \(\delta > 0\) be such that \(\delta < \varepsilon\) and \(B_X(a, \delta) \subset U\). If \(B \in \overline{B}_X(A, \delta) \cap \text{Cl}(\Omega(X))\), then \(B \subset \{a\}\) and \(B \in B_X(a, \delta) \subset U \subset O(X)\) so, by Theorem 4.8, \(B = [b]\) for some \(b \in O(X)\) and, for every \([e, r] \in \mathcal{E}\), we have \(b \notin [e, r]\). Then \(B \in \Omega(X)\), so \(G(B) = g(B) = f^{se}_1(b) \in B_X(f^{se}_1(q), \varepsilon) = B_X(G(A), \varepsilon)\), and (4.1) is satisfied.

Now assume that \(A = \{q\}\), for some \(q \in O(X)\) so that \(q \in (e, r_e)\) and \([e, r_e] \in \mathcal{E}\). Then \(G(A) = g(A) = f^{se}_2(q)\). Since \([e, r_e]\) is a free arc in \(X\), there is \(\delta_1 > 0\) such that \(B_X(q, \delta_1) \subset (e, r_e)\). Since the function \(f^{se}_1\) is continuous, there exists \(\delta > 0\) such that \(\delta < \delta_1\) and

\[
f^{se}_1(B_X(q, \delta)) \subset B_X(f^{se}_1(q), \varepsilon).
\]

If \(B \in B_X(A, \delta) \cap \text{Cl}(\Omega(X))\), then \(B \subset N(\delta, A) = B_X(q, \delta) \subset B_X(q, \delta_1) \subset (e, r_e)\), so, by Theorem 4.8, \(B = [b]\), for some \(b \in B_X(q, \delta)\). Then \(B \in \Omega(X)\), so \(G(B) = g(B) = f^{se}_1(b) \in B_X(f^{se}_1(q), \varepsilon) = B_X(G(A), \varepsilon)\), and (4.1) is satisfied.

Now assume that \(A = \{e\}\), for some \(e \in E(X) - E_0(X)\). Let \(r_e \in r_X\) be so that \([e, r_e] \in \mathcal{E}\). Then \(G(A) = g([e]) = f^{se}_2(e) = s_e\). Since \([e, r_e]\) is a free arc in \(X\), there is \(\delta_1 > 0\) such that \(B_X(e, \delta_1) \subset (e, r_e)\). Since the functions \(f^{se}_1\) and \(f^{se}_2\) are continuous, there exists \(\delta > 0\) such that \(\delta < \delta_1\), and

\[
f^{se}_1(B_X(e, \delta)) \subset B_X(f^{se}_1(e), \varepsilon) = B_X(s_e, \varepsilon).
\]

and

\[
f^{se}_2(B_X(e, \delta)) \subset B_X(f^{se}_2(e), \varepsilon) = B_X(s_e, \varepsilon).
\]

If \(B \in B_X(A, \delta) \cap \text{Cl}(\Omega(X))\), then \(B \subset N(\delta, A) = B_X(e, \delta) \subset B_X(e, \delta_1) \subset [e, r_e]\). Thus, by Theorem 4.8, either \(B = [b]\) for some \(b \in B_X(e, \delta)\) or \(B = [e, p]\) for some \(p \in B_X(e, \delta)\). Then \(B \in \Omega(X)\). Moreover, in the first case, \(G(B) = g([b]) = f^{se}_1(b) \in B_X(s_e, \varepsilon)\) and, in the second case, \(G(B) = g([e, p]) = f^{se}_2(p) \in B_X(s_e, \varepsilon)\). Thus, \(G(B) \in B_X(G(A), \varepsilon)\), so (4.1) is satisfied.

Now assume that there exists \([e, r_e] \in \mathcal{E}\) and \(p \in (e, r_e)\) such that \(A = [e, p] \subset [e, r_e]\). Then \(G(A) = g(A) = f^{se}_2(p)\). Let \(\delta_1 > 0\) be so that \(N(\delta_1, A) \subset [e, r_e]\), \(B_X(e, \delta_1) \cap B_X(p, \delta_1) = \emptyset\) and \(B_X(p, \delta_1) \subset [e, r_e]\). Since the function \(f^{se}_2\) is continuous, there exists \(\delta > 0\) such that \(\delta < \delta_1\) and

\[
f^{se}_2(B_X(p, \delta)) \subset B_X(f^{se}_2(p), \varepsilon).
\]

If \(B \in B_X(A, \delta) \cap \text{Cl}(\Omega(X))\), then \(B \subset N(\delta, A) \subset N(\delta_1, A) \subset [e, r_e]\), \(B_X(e, \delta) \cap B \neq \emptyset\) and \(B_X(p, \delta) \cap B \neq \emptyset\). Hence, by Theorem 4.8, \(B = [r] \cap \forall\epsilon \in \mathcal{E}\). Then \(B \in \Omega(X)\), so \(G(B) = g(B) = f^{se}_2(r)\) and, since \(r \in B_X(p, \delta)\), we have \(G(B) = f^{se}_2(r) \in B_X(f^{se}_2(p), \varepsilon) = B_X(G(A), \varepsilon)\), so (4.1) is satisfied. This completes the proof of 5).

Now we claim that:

6) \(G\) is a continuous function in \(\mathcal{E}\).

To show 6) let \(A \in \mathcal{E}\) and \(\varepsilon > 0\). We need to find \(\delta > 0\) such that (4.1) holds. Let \([e, r_e] \in \mathcal{E}\) be such that \(A = [e, r_e]\). Then \(G(A) = e\). Let \(\delta_1 > 0\) be such that \(B_X(e, \delta_1) \subset (e, r_e)\) and \(B_X(e, \delta_1) \cap B_X(r_e, \delta_1) = \emptyset\). Since the function \(f^{se}_2\) is continuous, there exists \(\delta > 0\) such that \(\delta < \delta_1\) and

\[
f^{se}_2(B_X(r_e, \delta)) \subset B_X(f^{se}_2(r_e), \varepsilon) = B_X(e, \varepsilon).
\]

If \(B \in B_X(A, \delta) \cap \text{Cl}(\Omega(X))\), then \(e, r_e \in A \subset N(\delta, B) \subset N(\delta_1, B)\) so \(B \cap B_X(e, \delta_1) \neq \emptyset\) and \(B \cap B_X(r_e, \delta_1) \neq \emptyset\). This implies that \(B\) is a nondegenerate element of \(\text{Cl}(\mathcal{E}(X))\) that intersects \([e, r_e]\). Thus, by Theorem 4.8, \(B \subset [e, r_e]\) and there exists \(p \in B_X(r_e, \delta)\) such that \(B = [e, p] \subset [e, r_e]\). Then either \(G(B) = e\) if \(p = r_e\) or \(G(B) = g(B) = f^{se}_2(p)\) if \(p \neq r_e\). In both cases we have \(G(B) \in B_X(G(A), \varepsilon)\), so (4.1) is satisfied. This completes the proof of 6).
Now we are going to prove that:

7) $G$ is a continuous function in $F_1(R(X))$.

To show 7) let $A \in F_1(R(X))$ and assume that $A$ is the limit, in the Hausdorff metric, of the sequence $\{A_n\}_{n \in \mathbb{N}}$ in $\text{Cl}(X)(\Omega(X))$. Let us consider that, for infinitely many elements $n$, $A_n$ satisfies either (b) or (d) of Theorem 4.8 or $A_n \in E_1(\Omega(X))$. Then, for such indices $n$, we have $A_n \cap E(X) \neq \emptyset$ so, by Theorem 3.11, $A \cap E(X) \neq \emptyset$. Since this contradicts the fact that $A \in F_1(R(X))$ we can assume that, for every $n \in \mathbb{N}$, $A_n$ satisfies either (a) or (c) and even more, that $A_n \in F_1(R(X) \cup O(X))$. Since by Theorem 3.12 every limit of ramification points of $X$ is an end point of $X$, we can assume that $A_n \in E(\Omega(X))$ for each $n \in \mathbb{N}$. Let $A = \{p\}$ and, for $n \in \mathbb{N}$, let $A_n = \{p_n\}$. Note that $G(A) = p$ and that $p$ is the limit of the sequence $\{p_n\}_{n \in \mathbb{N}}$. If for infinitely many indices $n$, $p_n$ is not an element of any member of $\mathcal{E}$ then, for such indices $n$, we have $G(A_n) = g(A_n) = p_n$. This implies that the sequence $\{G(A_n)\}_n$ of such indices $n$, converges to $G(A)$.

Now assume that $p$ is so that $[e, p] \in \mathcal{E}$ for some $e \in E(X) \setminus E_0(X)$. If for infinitely many indices $n$, we have $p_n \in [e, p]$ then, for such indices $n$, $G(A_n) = g(A_n) = f^{x_n}(p_n)$. By the continuity of $f^{x_n}$, the sequence $(f^{x_n}(p_n))_n$ of such indices $n$ converges to $f^{x_n}(p) = p = G(A)$. From this, and the fact that the order of $p$ in $X$ is finite, it follows that $G(A)$ is the limit of the sequence $\{G(A_n)\}_{n \in \mathbb{N}}$. This shows 7).

Now we claim that:

8) $G$ is a continuous function in $F_1(E_0(X))$.

To show 8) let $A \in F_1(E_0(X))$ and assume that $A$ is the limit, in the Hausdorff metric, of the sequence $\{A_n\}_{n \in \mathbb{N}}$ in $\text{Cl}(X)(\Omega(X))$. Let $p \in E_0(X)$ be such that $A = \{p\}$. Then $G(A) = p$. By Theorem 3.7, $X = D_1 \cup D_2 \cup \cdots \cup D_l$, where each $D_i \in \mathcal{D}$ and $|D_i \cap D_j| < \infty$ for every $i, j \in \{1, 2, \ldots , l\}$ with $i \neq j$. Since $p \in E(X) \setminus X - P_X$, there exists $i \in \{1, 2, \ldots , l\}$ such that $p \in D_i - P_X$. By Theorem 3.8, $D_i - P_X$ is open in $X$, so $\mathcal{U} = \{B \in C(X): B \subset D_i - P_X\}$ is an open subset of $C(X)$ that contains $A$. Hence, we can assume that $A_n \subset D_i - P_X$ for every $n \in \mathbb{N}$. Let $[n_k]_{k \in \mathbb{N}}$ be a sequence in $\mathbb{N}$. Let us assume that, for every $k \in \mathbb{N}$, we have $A_{n_k} = \{p_k\}$ and that either $p_k \in E(X) \cup E_0(X)$ or $p_k \in O(X)$ and $p_k$ is not an element of any member of $\mathcal{E}$. Then $p$ is the limit of the sequence $\{p_k\}_{k \in \mathbb{N}}$ and $G(A_{n_k}) = p_k$, for every $k \in \mathbb{N}$. This implies that $G(A)$ is the limit of the sequence $\{G(A_{n_k})\}_{k \in \mathbb{N}}$.

Now assume that $A_{n_k}$ is contained in some member $[e_k, r_k]$ of $\mathcal{E}$, for every $k \in \mathbb{N}$. Then, $G(A_{n_k}) \in [e_k, r_k]$, for every $k \in \mathbb{N}$. Note that each $[e_k, r_k]$ is a connected subset of $X$ that intersects $D_i$. Since $P_X$ is finite and every point of $X$ is of finite order in $X$, it follows that only for finitely many indices $k$, we have $[e_k, r_k] \subset D_i$. Then, we can consider that $[e_k, r_k] \subset D_i$, for each $k \in \mathbb{N}$. We can also consider that the sequence $[e_k]_{k \in \mathbb{N}}$ converges to $e \in D_i$. Given $k \in \mathbb{N}$ let us consider the arc $[e_k, e_k]$ in $D_i$. Since $e_k \in E(X)$ and $(e_k, r_k) \subset O(X)$, we have $r_k \in [e, e_k]$. Then, $A_{n_k} \cup G(A_{n_k}) \subset [e_k, r_k] \subset [e, e_k]$. By property (S) described on p. 2072, the sequence of arcs $[e, e_k]_{k \in \mathbb{N}}$ converges in the Hausdorff metric, to $[e]$. This implies that $e = p$ and that $G(A) = p$ is the limit of the sequence $\{G(A_{n_k})\}_{k \in \mathbb{N}}$. This shows 8).

To end the proof note that, by 3)–8), $G: \text{Cl}(X)(\Omega(X)) \rightarrow X$ is a homeomorphism. Thus, $\text{Cl}(X)(\Omega(X))$ is homeomorphic to $X$. \qed

5. Main theorem

In this section we will show that if $X \in \mathcal{L}$ is different from an arc and a simple closed curve, then $X$ has unique hyperspace $C(X)$.

**Theorem 5.1.** Let $X$ and $Y$ be two continua different from an arc, and such that $C(X)$ is homeomorphic to $C(Y)$. If $X \in \mathcal{L}$, then $X$ is homeomorphic to $Y$.

**Proof.** By Theorem 2.2, $Y$ is locally connected and, by Theorem 4.10, we have:

1) $\text{Cl}(X)(\Omega(X))$ is homeomorphic to $X$.

Since $C(X)$ is homeomorphic to $C(Y)$, by Theorem 4.1, $\Omega(X)$ is homeomorphic to $\Omega(Y)$. Thus,

2) $\text{Cl}(X)(\Omega(X))$ is homeomorphic to $\text{Cl}(Y)(\Omega(Y))$.

Now we claim that:

3) $Y$ contains at most a finite number of simple closed curves.

To show 3) assume first that $S_1$ and $S_2$ are two different simple closed curves in $Y$. By Theorem 4.5, we have $F_1(Y) \subset \text{Cl}(Y)(\Omega(Y))$. Thus, $F_1(S_1) \cup F_1(S_2) \subset \text{Cl}(Y)(\Omega(Y))$, so $F_1(S_1)$ and $F_1(S_2)$ are different subsets of $\text{Cl}(Y)(\Omega(Y))$. By 1)
and 2), \( \text{Cl}(Y(\Omega(Y))) \) is homeomorphic to \( X \). Hence, there exist \( A_1, A_2 \subset X \) such that \( F_1(S_1) \) is homeomorphic to \( A_1 \) and \( F_1(S_2) \) is homeomorphic to \( A_2 \). Since \( F_1(S_1) \neq F_1(S_2) \), we have \( A_1 \neq A_2 \). Since \( S_1 \) is homeomorphic to \( F_1(S_i) \), for each \( i = 1, 2 \), we conclude that \( A_1 \) and \( A_2 \) are two different simple closed curves in \( X \). We have shown that if \( Y \) contains two different simple closed curves, then \( X \) contains two different simple closed curves as well. Since \( X \) is a local dendrite, by Theorem 3.5, \( X \) contains at most a finite number of simple closed curves. Thus, \( Y \) contains at most a finite number of simple closed curves. This shows 3).

Since \( Y \) is a locally connected continuum with at most a finite number of simple closed curves, by Theorem 3.5, \( Y \) is a local dendrite. Hence, by Theorem 3.21, \( Y \in \mathfrak{LD} \). Applying Theorem 4.10, it follows that \( \text{Cl}(Y(\Omega(Y))) \) is homeomorphic to \( Y \). Using this, 1) and 2), we conclude that \( X \) is homeomorphic to \( Y \). ☐

**Corollary 5.2.** If \( X \in \mathfrak{LD} \) is different from an arc and a simple closed curve, then \( X \) has unique hyperspace \( C(X) \).

**Proof.** Let \( X \) be as assumed, and \( Y \) be a continuum such that \( C(X) \) is homeomorphic to \( C(Y) \). If \( Y \) is either an arc or a simple closed curve then, by [2, Lemma 11, p. 38], \( X \) is either an arc or a simple closed curve. Since this is a contradiction, \( Y \) is different from an arc and a simple closed curve. Thus, by Theorem 5.1, \( X \) is homeomorphic to \( Y \). ☐

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