G-designs and related designs

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Abstract


This is a survey on the existence of G-designs, bipartite G-designs and multipartite G-designs.

1. Introduction

G-designs and related designs have been investigated to solve construction problems occurring in graph theory and many fields of research-combinatorial mathematics, mathematical statistics, database systems, etc.

In combinatorial mathematics, there are famous problems such as the ‘fifteen school-girls problem’, the ‘nine school-boys problem’ and the ‘nine prisoners problem’. The ‘nine prisoners problem’ (Hell and Rosa [11]) is the following: In a jail there are 9 prisoners of a particularly dangerous character. Each morning they are allowed to walk handcuffed in the prison yard. Here is how they walked on Monday: (1)–(2)–(3) (4)–(5)–(6) (7)–(8)–(9). Can they be arranged for Tuesday–Saturday so that no pair of prisoners is handcuffed together twice? The solutions of the three problems given above can be obtained by constructing resolvable (15, 3, 1) K3-designs, resolvable (9, 3, 1) K3-designs and resolvable (9, 3, 1) P3-designs, respectively.

In design of experiments, there are fundamental construction problems about (v, k, λ) balanced incomplete block (BIB) designs, resolvable (v, k, λ) BIB designs and (v, k, λ) transversal designs. Such designs can be obtained by constructing (v, k, λ) Kk-designs, resolvable (v, k, λ) Kk-designs and (v, k, λ) Ck-designs, respectively.

In database systems consisting of formatted data, there are combinatorial index-file organization scheme problems about BFS2, HUBFS2, BMFS2 and HUBMFS2. They can be solved by constructing (v, k, 1) Kk-designs, (v, k, 1) Sk-designs, (m, n, k, 1) multipartite Kk-designs and (m, n, k, 1) multipartite Sk-designs, respectively.

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In this paper, $G$-designs and related designs – bipartite $G$-designs and multipartite $G$-designs – will be discussed.

2. Definitions of $G$-designs and related designs

$G$-designs were introduced by Hell and Rosa [11]. Many types of $G$-designs can be found in Bermond and Sotteau [2]. A combinatorial definition of $G$-designs can be given in terms of $G$-blocks. In a block $G$, considered as a set of elements, any two distinct elements are either linked or unlinked. The adjacency matrix $M(G) = (m_{ij})$ of a block $G$, with $k$ elements, is a symmetric matrix of order $k$ with zero diagonal, where $m_{ij} = 1$ if the elements $i$ and $j$ are linked in $G$ and 0 otherwise. A block $B$ is said to be a $G$-block if its adjacency matrix is equivalent to that of $G$.

**Definition 2.1.** A $(v, k, \lambda)$ $G$-design is an arrangement of $v$ elements into $b$ $G$-blocks such that every $G$-block contains $k$ elements and any two distinct elements are linked in exactly $\lambda$ $G$-blocks. Furthermore if each element belongs to exactly $r$ $G$-blocks then the design is said to be balanced.

**Definition 2.2.** A parallel class of $G$-blocks consists of a set of disjoint $G$-blocks such that every element occurs in one $G$-block of the class. A $(v, k, \lambda)$ $G$-design is said to be resolvable if the $b$ $G$-blocks can be partitioned into $r$ parallel classes.

Another definition of $G$-designs can be given by using the terminology of graph theory. Let $G$ be a graph with $k$ vertices and let $\lambda K_v$ be the complete multigraph with $v$ vertices in which any two distinct vertices are joined by exactly $\lambda$ edges.

**Definition 2.3.** A $(v, k, \lambda)$ $G$-design is an edge-disjoint decomposition of $\lambda K_v$ into $b$ subgraphs isomorphic to $G$. Furthermore if each vertex belongs to exactly $r$ subgraphs then the $G$-design is said to be balanced.

**Definition 2.4.** A $G$-factor is a spanning subgraph of $\lambda K_v$ such that each component of the $G$-factor is isomorphic to $G$. A $(v, k, \lambda)$ $G$-design is said to be resolvable if $\lambda K_v$ can be factorized into $r$ $G$-factors.

**Remark 2.1.** If $G$ is regular, it is easy to see that the $G$-design is balanced. In the particular case when $G$ is the complete graph $K_k$, a balanced $(v, k, \lambda)$ $K_k$-design and a resolvable $(v, k, \lambda)$ $K_k$-design are nothing but a $(v, k, \lambda)$ BIB design and a resolvable $(v, k, \lambda)$ BIB design, respectively. The $G$-design is in a sense a generalization of BIB designs.

By use of the terminology of graph theory, definitions of bipartite $G$-designs and multipartite $G$-designs can also be given. For $G$ being a graph with $k$ vertices, let $\lambda K_{m,n}$
be the complete bipartite multigraph with two partite sets of \( m \) and \( n \) vertices each in which any two distinct vertices in different sets are joined by exactly \( \lambda \) edges. Further let \( \lambda K_m^n \) be the complete multipartite multigraph with \( m \) partite sets of \( n \) vertices each in which any two distinct vertices in different sets are joined by exactly \( \lambda \) edges.

**Definition 2.5.** An \( (m, n, k, \lambda) \) bipartite \( G \)-design is an edge-disjoint decomposition of \( \lambda K_{m,n} \) into \( b \) subgraphs isomorphic to \( G \). Furthermore if each vertex belongs to exactly \( r \) subgraphs then the bipartite \( G \)-design is said to be balanced. An \( (m, n, k, \lambda) \) bipartite \( G \)-design is said to be resolvable if \( \lambda K_{m,n} \) can be factorized into \( r \) \( G \)-factors of \( \lambda K_{m,n} \).

**Definition 2.6.** An \( (m, n, k, \lambda) \) multipartite \( G \)-design is an edge-disjoint decomposition of \( \lambda K_n^m \) into \( b \) subgraphs isomorphic to \( G \). Furthermore if each vertex belongs to exactly \( r \) subgraphs then the multipartite \( G \)-design is said to be balanced. An \( (m, n, k, \lambda) \) multipartite \( G \)-design is said to be resolvable if \( \lambda K_n^m \) can be factorized into \( r \) \( G \)-factors of \( \lambda K_n^m \).

**Remark 2.2.** If \( G \) is regular, the multipartite \( G \)-design is balanced and the bipartite \( G \)-design with \( m = n \) is also balanced. In the particular case when \( G \) is the complete graph \( K_k \), a balanced \( (m, n, k, \lambda) \) multipartite \( K_k \)-design and a resolvable \( (m, n, k, \lambda) \) multipartite \( K_k \)-design are respectively particular types of partially balanced incomplete block (PBIB) designs and resolvable PBIB designs. In a sense, the multipartite \( G \)-design is a generalization of PBIB designs.

**Remark 2.3.** In any case of \( G \)-designs, bipartite \( G \)-designs and multipartite \( G \)-designs, if the design is resolvable then it is also balanced.

### 3. Necessary conditions for the existence of \( G \)-designs and related designs

In this section we give the necessary conditions for the existence of \( G \)-designs, bipartite \( G \)-designs and multipartite \( G \)-designs. Let \( e \) be the number of edges of \( G \) and let \( d \) be the g.c.d. of the degrees of vertices of \( G \). Then we have the following:

**Theorem 3.1.** If there exists a \( (v, k, \lambda) \) \( G \)-design, then

(i) \( \lambda v (v-1) \equiv 0 \) (mod \( 2e \)),

(ii) \( \lambda (v-1) \equiv 0 \) (mod \( d \)).

Moreover, if the \( G \)-design is balanced, then

(iii) \( \lambda k (v-1) \equiv 0 \) (mod \( 2e \)),

and if the \( G \)-design is resolvable, then

(iv) \( v \equiv 0 \) (mod \( k \)).
Proof. Let $b$ be the number of subgraphs (blocks) of the $G$-design. Then $b = \lambda v(v-1)/2e$, which yields (i). Let $d_i$ $(i=1, 2, \ldots, k)$ be the degrees of the vertices of $G$. Suppose that a vertex $x$ of $\lambda K_v$ appears with degree $d_i$ in $b_i$ subgraphs. Then $\lambda(v-1) = \sum b_i d_i$, which implies (ii). Next, suppose that each vertex belongs to exactly $r$ subgraphs in the balanced $G$-design. The $r = \lambda k(v-1)/2e$, which yields (iii). When each $G$-factor has $t$ components in the resolvable $G$-design, we get $t = v/k$, which implies (iv).

Remark 3.1. When $G$ is regular, (iii) is equivalent to (ii).

The following two theorems can easily be proved using an argument similar to the proof of Theorem 3.1.

Theorem 3.2. If there exists an $(m, n, k, \lambda)$ bipartite $G$-design, then

(i) $\lambda mn \equiv 0 \pmod{e}$,
(ii) $\lambda m \equiv 0 \pmod{d}$,
(iii) $\lambda n \equiv 0 \pmod{d}$.

Moreover, if the bipartite $G$-design is balanced, then

(iv) $\lambda mnk \equiv 0 \pmod{(m+n)e}$,
and if the bipartite $G$-design is resolvable, then

(v) $m + n \equiv 0 \pmod{k}$.

Theorem 3.3. If there exists an $(m, n, k, \lambda)$ multipartite $G$-design, then

(i) $\lambda (m-1)n^2 \equiv 0 \pmod{2e}$,
(ii) $\lambda (m-1)n \equiv 0 \pmod{d}$.

Moreover, if the multipartite $G$-design is balanced, then

(iii) $\lambda (m-1)nk \equiv 0 \pmod{2e}$,
and if the multipartite $G$-design is resolvable, then

(iv) $mn \equiv 0 \pmod{k}$.

The following existence theorem is given by Wilson [37].

Theorem 3.4. For a given graph (block) $G$ and a given $\lambda$, the necessary conditions (i) and (ii) for the existence of $G$-designs given in Theorem 3.1 are sufficient for all sufficiently large integers $v$.

4. Survey of the results concerning the existence of $G$-designs and related designs

We consider the following four classes of $G$ such that $G = K_k, C_k, P_k$ and $S_k$, where $k \geq 3$ and

- $K_k$ is a complete graph with $k$ vertices and $k(k-1)/2$ edges,
- $C_k$ is a cycle with $k$ vertices and $k$ edges,
- $P_k$ is a path with $k$ vertices and $k-1$ edges,
- $S_k$ is a star with $k$ vertices and $k-1$ edges.
Remark 4.1. Since $K_k$ and $C_k$ are regular, the following designs are automatically balanced: $K_k$-designs, bipartite $K_k$-designs with $m=n$, multipartite $K_k$-designs, $C_k$-designs, bipartite $C_k$-designs with $m=n$ and multipartite $C_k$-designs.

Remark 4.2. Since $\lambda K_{m,n}$ has no subgraph $K_k$, bipartite $K_k$-designs do not exist.

4.1. $K_k$-designs and multipartite $K_k$-designs

A balanced $(m, n, k, \lambda)$ multipartite $K_k$-design and a resolvable $(m, n, k, \lambda)$ multipartite $K_k$-design are a PBIB design with $\lambda_1 = \lambda$ and $\lambda_2 = 0$ and a resolvable PBIB design with $\lambda_1 = \lambda$ and $\lambda_2 = 0$, respectively.

Theorem 4.1 [10, Theorem 5.1–5.4]. For $k$ in $3 \leq k \leq 5$ and $k = 6$ with $\lambda > 1$, there exists a balanced $(v, k, \lambda)$ $K_k$-design if and only if

(i) $\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}$,
(ii) $\lambda(v - 1) \equiv 0 \pmod{k - 1}$,

with two exceptions $(v = 15, k = 5, \lambda = 2)$ and $(v = 21, k = 6, \lambda = 2)$.

Sketch of proof. The necessity follows from Theorem 3.1. To prove the sufficiency, we note that $\lambda$ determines the values of $v$ satisfying the conditions of the theorem. It suffices to consider only those $\lambda$'s which are factors of $6$ (when $k = 3$, $4$), $20$ (when $k = 5$) and $30$ (when $k = 6$). Then we obtain the following: When $k = 3$, for $\lambda = 1$, $v \equiv 1$ or $3 \pmod{6}$, for $\lambda = 2$, $v \equiv 0$ or $1 \pmod{6}$, for $\lambda = 3$, $v \equiv 1 \pmod{2}$, for $\lambda = 6$, every $v$. When $k = 4$, for $\lambda = 1$, $v \equiv 1$ or $4 \pmod{12}$, for $\lambda = 2$, $v \equiv 1 \pmod{4}$, for $\lambda = 3$, $v \equiv 0$ or $1 \pmod{4}$, for $\lambda = 6$, every $v$. When $k = 5$, for $\lambda = 1$, $v \equiv 1$ or $5 \pmod{20}$, for $\lambda = 2$, $v \equiv 1$ or $5 \pmod{10}$, for $\lambda = 4$, $v \equiv 0$ or $1 \pmod{5}$, for $\lambda = 5$, $v \equiv 1 \pmod{4}$, for $\lambda = 10$, $v \equiv 1 \pmod{2}$, for $\lambda = 20$, every $v$. And when $k = 6$ with $\lambda > 1$, for $\lambda = 2$, $v \equiv 1$ or $6 \pmod{15}$, for $\lambda = 3$, $v \equiv 1 \pmod{5}$, for $\lambda = 5$, $v \equiv 0$ or $1 \pmod{3}$, for $\lambda = 15$, every $v$.

In all these cases—with two exceptions $(v = 15, k = 5, \lambda = 2)$ and $(v = 21, k = 6, \lambda = 2)$—the existence of the relevant BIB design has been proved. It is known that the two exceptional BIB designs do not exist. 

Example 4.1. A balanced $(7, 3, 1) K_3$-design is given by

$B_1 = (1, 2, 4) = \{(1, 2), (1, 4), (2, 4)\}$,
$B_2 = (2, 3, 5) = \{(2, 3), (2, 5), (3, 5)\}$,
$B_3 = (3, 4, 6) = \{(3, 4), (3, 6), (4, 6)\}$,
$B_4 = (4, 5, 7) = \{(4, 5), (4, 7), (5, 7)\}$,
$B_5 = (5, 6, 1) = \{(5, 6), (5, 1), (6, 1)\}$,
$B_6 = (6, 7, 2) = \{(6, 7), (6, 2), (7, 2)\}$,
$B_7 = (7, 1, 3) = \{(7, 1), (7, 3), (1, 3)\}$,
Example 4.2. A resolvable \((9, 3, 1)\) \(K_3\)-design is given by
\[
\begin{align*}
B_1 &= (1, 6, 7) = \{(1, 6), (6, 7), (7, 1)\}, \\
B_2 &= (2, 3, 5) = \{(2, 3), (3, 5), (5, 2)\}, \\
B_3 &= (8, 4, 9) = \{(8, 4), (4, 9), (9, 8)\}, \\
B_4 &= (2, 7, 8) = \{(2, 7), (7, 8), (8, 2)\}, \\
B_5 &= (3, 4, 6) = \{(3, 4), (4, 6), (6, 3)\}, \\
B_6 &= (1, 5, 9) = \{(1, 5), (5, 9), (9, 1)\}, \\
B_7 &= (3, 8, 1) = \{(3, 8), (8, 1), (1, 3)\}, \\
B_8 &= (4, 5, 7) = \{(4, 5), (5, 7), (7, 4)\}, \\
B_9 &= (2, 6, 9) = \{(2, 6), (6, 9), (9, 2)\}, \\
B_{10} &= (4, 1, 2) = \{(4, 1), (1, 2), (2, 4)\}, \\
B_{11} &= (5, 6, 8) = \{(5, 6), (6, 8), (8, 5)\}, \\
B_{12} &= (3, 7, 9) = \{(3, 7), (7, 9), (9, 3)\}.
\end{align*}
\]

Let \(F_i = B_{3i-2} \cup B_{3i-1} \cup B_{3i} (1 \leq i \leq 4)\). Then these \(K_3\)-factors \(F_i\) comprise a \(K_3\)-factorization of \(K_9\). This example is known as 'a solution of the 9 school-boys problem'.

4.2. \(C_k\)-designs, bipartite \(C_k\)-designs and multipartite \(C_k\)-designs

Balanced \(C_k\)-designs have been studied by Kotzig [19], Rosa [23, 24], Rosa and Huang [25], Bermond, Huang and Sotteau [3], and Bermond and Thomassen [4]. In [3], it is conjectured that the necessary conditions (i) and (ii) in Theorem 3.1 are sufficient for balanced \((v, k, \lambda)\) \(C_k\)-designs. This is true for small \(k\) as the following demonstrates (e.g., [3, 25]).

**Theorem 4.2.** For \(k\) in \(4 \leq k \leq 8\), there exists a balanced \((v, k, \lambda)\) \(C_k\)-design if and only if
\[
\begin{align*}
(i) \quad \lambda v(v-1) &\equiv 0 \pmod{2k}, \\
(ii) \quad \lambda(v-1) &\equiv 0 \pmod{2}.
\end{align*}
\]

For bipartite \(C_k\)-designs, we have the following (see [26]).

**Theorem 4.3.** There exists an \((m, n, k, 1)\) bipartite \(C_k\)-design if and only if
\[
\begin{align*}
(i) \quad mn &\equiv 0 \pmod{2k}, \\
(ii) \quad m, n &\equiv 0 \pmod{2}, \\
(iii) \quad m, n &> k.
\end{align*}
\]

The following theorem has been established by Enomoto, Miyamoto and Ushio [9].
Theorem 4.4. There exists a resolvable \((m, n, k, 1)\) bipartite \(C_k\)-design if and only if

(i) \(m = n \equiv 0 \pmod{2}\),
(ii) \(k \equiv 0 \pmod{2}, k \geq 4\),
(iii) \(2n \equiv 0 \pmod{k}\)

with precisely one exception, namely \(m = n = k = 6\).

Sketch of proof. The necessity follows from Theorem 3.2. To prove the sufficiency, it suffices to consider the following facts: If \(K_{m,n}\) has a \(C_k\)-factorization, then \(K_{m,n}\) also has a \(C_k\)-factorization. If \(k \equiv 0 \pmod{4}\) and \(m = n \equiv 0 \pmod{k/2}\), then \(K_{m,n}\) has a \(C_k\)-factorization. \(K_6,6\) does not have a \(C_6\)-factorization. When \(s\) is odd and \(s \geq 5\), \(K_{2s,2s}\) has a \(C_{2s}\)-factorization. When \(s \geq 2\), \(K_{6s,6s}\) has a \(C_6\)-factorization.  

Multipartite \(C_k\)-designs have been studied by Cockayne and Hartnell [8], and Laskar and Auerbach [21].

Example 4.3. A balanced \((9,4,1)\) \(C_4\)-design is given by

\[
\begin{align*}
B_1 &= \{(1, 2), (2, 6), (6, 3), (3, 1)\}, \\
B_2 &= \{(2, 3), (3, 7), (7, 4), (4, 2)\}, \\
B_3 &= \{(3, 4), (4, 8), (8, 5), (5, 3)\}, \\
B_4 &= \{(4, 5), (5, 9), (9, 6), (6, 4)\}, \\
B_5 &= \{(5, 6), (6, 1), (1, 7), (7, 5)\}, \\
B_6 &= \{(6, 7), (7, 2), (2, 8), (8, 6)\}, \\
B_7 &= \{(7, 8), (8, 3), (3, 9), (9, 7)\}, \\
B_8 &= \{(8, 9), (9, 4), (4, 1), (1, 8)\}, \\
B_9 &= \{(9, 1), (1, 5), (5, 2), (2, 9)\}.
\end{align*}
\]

Example 4.4. A resolvable \((12,12,6,1)\) bipartite \(C_6\)-design is given below: Let partite sets of \(K_{12,12}\) be \(\{i|1 \leq i \leq 12\}\) and \(\{\bar{i}|1 \leq i \leq 12\}\). And let

\[
\begin{align*}
B_1 &= \{(1, \bar{1}), (\bar{1}, 5), (5, \bar{5}), (\bar{5}, 9), (9, 9), (\bar{9}, 1)\}, \\
B_2 &= \{(2, \bar{2}), (\bar{2}, 6), (6, \bar{6}), (\bar{6}, 10), (10, \bar{10}), (\bar{10}, 2)\}, \\
B_3 &= \{(3, \bar{3}), (\bar{3}, 7), (7, \bar{7}), (\bar{7}, 11), (11, \bar{11}), (\bar{11}, 3)\}, \\
B_4 &= \{(4, \bar{4}), (\bar{4}, 8), (8, \bar{8}), (\bar{8}, 12), (12, \bar{12}), (\bar{12}, 4)\}, \\
B_5 &= \{(1, \bar{4}), (\bar{4}, 11), (11, \bar{11}), (\bar{11}, 2), (\bar{2}, 9), (9, \bar{12}), (\bar{12}, 1)\}, \\
B_6 &= \{(2, \bar{5}), (\bar{5}, 12), (12, \bar{5}), (\bar{3}, 4), (4, \bar{7}), (\bar{7}, 2)\}.
\end{align*}
\]
\[ B_7 = \{ (3, 6), (6, 7), (7, 10), (10, 5), (5, 8), (8, 3) \}, \]
\[ B_8 = \{ (6, 9), (9, 10), (10, 1), (1, 8), (8, 11), (11, 6) \}, \]
\[ B_9 = \{ (1, 2), (2, 3), (3, 1), (1, 2), (2, 3), (3, 1) \}, \]
\[ B_{10} = \{ (4, 5), (5, 6), (6, 4), (4, 5), (5, 6), (6, 4) \}, \]
\[ B_{11} = \{ (7, 8), (8, 9), (9, 7), (7, 8), (8, 9), (9, 7) \}, \]
\[ B_{12} = \{ (10, 11), (11, 12), (12, 10), (10, 11), (11, 12), (12, 10) \}, \]
\[ B_{13} = \{ (1, 5), (5, 3), (3, 4), (4, 2), (2, 6), (6, 1) \}, \]
\[ B_{14} = \{ (4, 8), (8, 6), (6, 7), (7, 8), (5, 9), (9, 4) \}, \]
\[ B_{15} = \{ (7, 11), (11, 9), (9, 10), (10, 8), (8, 12), (12, 7) \}, \]
\[ B_{16} = \{ (10, 2), (2, 12), (12, 1), (1, 11), (11, 3), (3, 10) \}, \]
\[ B_{17} = \{ (1, 7), (7, 3), (3, 5), (5, 2), (2, 8), (8, 1) \}, \]
\[ B_{18} = \{ (4, 10), (10, 6), (6, 12), (12, 5), (5, 11), (11, 4) \}, \]
\[ B_{19} = \{ (7, 1), (1, 9), (9, 3), (3, 8), (8, 2), (2, 7) \}, \]
\[ B_{20} = \{ (10, 4), (4, 12), (12, 6), (6, 11), (11, 5), (5, 10) \}, \]
\[ B_{21} = \{ (1, 10), (10, 3), (3, 12), (12, 2), (2, 11), (11, 1) \}, \]
\[ B_{22} = \{ (4, 1), (1, 6), (6, 3), (3, 5), (5, 2), (2, 4) \}, \]
\[ B_{23} = \{ (7, 4), (4, 9), (9, 6), (6, 8), (8, 5), (5, 7) \}, \]
\[ B_{24} = \{ (10, 7), (7, 12), (12, 9), (9, 11), (11, 8), (8, 10) \}. \]

Let \( F_i = B_{4i-3} \cup B_{4i-2} \cup B_{4i-1} \cup B_{4i} (1 \leq i \leq 6) \). Then these \( C_v \)-factors \( F_i \) comprise a \( C_v \)-factorization of \( K_{12,12} \).

4.3. \( P_k \)-designs, bipartite \( P_k \)-designs and multipartite \( P_k \)-designs

The following is due to Tarsi [28].

**Theorem 4.5.** There exists a \((v, k, \lambda)\) \( P_k \)-design if and only if \( \lambda v(v-1) \equiv 0 \pmod{2(k-1)} \) and \( v \geq k \).

Balanced \( P_k \)-designs have been studied under the name of 'Handcuffed Designs' by Hell and Rosa [11], Lawless [20], and Hung and Mendelsohn [18].

**Theorem 4.6** [18]. There exists a balanced \((v, k, \lambda)\) \( P_k \)-design if and only if \( \lambda v(v-1) \equiv 0 \pmod{2(k-1)} \) and \( \lambda k(v-1) \equiv 0 \pmod{2(k-1)} \).
Resolvable $P_k$-designs have been investigated by Horton [13], and Bermond, Heinrich and Yu [5].

**Theorem 4.7** [5]. There exists a resolvable $(v, k, \lambda) P_k$-design if and only if $v \equiv 0 \pmod{k}$ and $\lambda k (v-1) \equiv 0 \pmod{2(k-1)}$.

Bipartite and resolvable bipartite $P_k$-designs have been, respectively, studied by Truszczynski [32] and Ushio [36].

**Theorem 4.8** [36]. There exists a resolvable $(m, n, 3, 1)$ bipartite $P_k$-design if and only if

(i) $m+n \equiv 0 \pmod{3}$,
(ii) $m \leq 2n$,
(iii) $n \leq 2m$,
(iv) $3mn/2(n+m)$ is an integer.

The following theorem was obtained by Ushio.

**Theorem 4.9.** When $k$ is even, there exists a resolvable $(m, n, k, 1)$ bipartite $P_k$-design if and only if $m=n \equiv 0 \pmod{k(k-1)/2}$.

**Sketch of proof.** The necessity follows from Theorem 3.2. To prove the sufficiency, it suffices to consider the following facts: If $K_{m,n}$ has a $P_k$-factorization, then $K_{m,n}$ also has a $P_k$-factorization. Let $k$ be even. When $n=k(k-1)/2$, $K_{m,n}$ has a $P_k$-factorization. □

**Example 4.5.** A $(5,3,1)$ $P_{3}$-design is given by

$B_1 = \{(1,2), (1,3)\}$,
$B_2 = \{(1,4), (1,5)\}$,
$B_3 = \{(2,3), (2,4)\}$,
$B_4 = \{(3,4), (3,5)\}$,
$B_5 = \{(2,5), (4,5)\}$.

**Example 4.6.** A resolvable $(9,3,1)$ $P_{3}$-design is displayed below

$B_1 = \{(1,2), (2,3)\}$, $B_2 = \{(4,5), (5,6)\}$, $B_3 = \{(7,8), (8,9)\}$,
$B_4 = \{(3,7), (7,5)\}$, $B_5 = \{(9,4), (4,2)\}$,
$B_6 = \{(5,9), (9,1)\}$, $B_7 = \{(8,3), (3,4)\}$, $B_8 = \{(2,6), (6,7)\}$,
$B_9 = \{(7,2), (2,5)\}$, $B_{10} = \{(3,1), (1,4)\}$, $B_{11} = \{(9,6), (6,8)\}$,
$B_{12} = \{(9,7), (7,1)\}$, $B_{13} = \{(5,3), (3,6)\}$, $B_{14} = \{(2,8), (8,4)\}$,
$B_{15} = \{(2,9), (9,3)\}$, $B_{16} = \{(1,5), (5,8)\}$, $B_{17} = \{(7,4), (4,6)\}$.
Let $F_i = B_{3i-2} \cup B_{3i-1} \cup B_{3i} (1 \leq i \leq 6)$. Then these $P_3$-factors $F_i$ comprise a $P_3$-factorization of $K_9$. This is known as ‘a solution of the 9 prisoners problem’.

**Example 4.7.** A resolvable $(3, 2, 3, 1)$ multipartite $P_3$-design is given below:

Let partite sets of $K_3^3$ be $\{1, 2\}, \{3, 4\}, \{5, 6\}$. And let

$B_1 = \{(1, 3), (1, 6)\}, \quad B_2 = \{(2, 4), (2, 5)\}, \quad B_3 = \{(3, 2), (3, 5)\},$

$B_4 = \{(4, 1), (4, 6)\}, \quad B_5 = \{(5, 1), (5, 4)\}, \quad B_6 = \{(6, 2), (6, 3)\}.$

Let $F_i = B_{3i-1} \cup B_{3i} (1 \leq i \leq 3)$. Then these $P_3$-factors $F_i$ comprise a $P_3$-factorization of $K_3^3$.

### 4.4. $S_k$-designs, bipartite $S_k$-designs and multipartite $S_k$-designs

Many researchers (e.g., [6, 7, 27, 38]) have studied $S_k$-designs.

**Theorem 4.10** [27]. There exists a $(v, k, \lambda)$ $S_k$-design if and only if $\lambda v(v-1) \equiv 0 \pmod{2(k-1)}$ and $v \geq 2(k-1)$ for $\lambda = 1$, $v \geq k$ for even $\lambda$, $v \equiv k + (k-1)/\lambda$ for odd $\lambda \geq 3$.

The $(v, k, 1)$ $S_k$-designs can be applied to combinatorial binary-valued index-file organization schemes of order two in database systems, which have been studied by Yamanoto, Ikeda, Shige-eda, Ushio and Hamada [39].

Huang and Rosa [15] and Huang [16] have presented existence conditions for balanced $S_k$-designs.

**Theorem 4.11** [16]. There exists a balanced $(v, k, \lambda)$ $S_k$-design if and only if $\lambda(v-1) \equiv 0 \pmod{2(k-1)}$.

For bipartite $S_k$-designs, Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [38] have presented the following.

**Theorem 4.12.** There exists an $(m, n, k, 1)$ bipartite $S_k$-design if and only if

(i) $m \equiv 0 \pmod{k-1}$ when $n < k-1$,

(ii) $n \equiv 0 \pmod{k-1}$ when $m < k-1$,

(iii) $mn \equiv 0 \pmod{k-1}$ when $m \geq k-1$ and $n \geq k-1$.

The following is due to Ushio, Tazawa and Yamamoto [33].

**Theorem 4.13.** There exists an $(m, n, k, 1)$ multipartite $S_k$-design if and only if

(i) $m(m-1)n^2 \equiv 0 \pmod{2(k-1)},$

(ii) $mn \geq 2(k-1)$. 

The \((m, n, k, 1)\) multipartite \(S_k\)-designs can be applied to combinatorial multiple-valued index-file organization schemes of order two in database systems, which have been studied by Yamamoto, Tazawa, Ushio and Ikeda [41].

**Example 4.8.** A \((6, 4, 1)\) \(S_4\)-design is given by
\[
\begin{align*}
B_1 &= \{ (1,2), (1,3), (1,4) \}, \\
B_2 &= \{ (2,3), (2,4), (2,6) \}, \\
B_3 &= \{ (3,4), (3,5), (3,6) \}, \\
B_4 &= \{ (5,1), (5,2), (5,4) \}, \\
B_5 &= \{ (6,1), (6,4), (6,5) \}.
\end{align*}
\]

**Example 4.9.** A balanced \((7, 4, 1)\) \(S_4\)-design is given by
\[
\begin{align*}
B_1 &= \{ (1,2), (1,3), (1,4) \}, \\
B_2 &= \{ (2,3), (2,4), (2,7) \}, \\
B_3 &= \{ (3,4), (3,5), (3,7) \}, \\
B_4 &= \{ (4,5), (4,6), (4,7) \}, \\
B_5 &= \{ (5,1), (5,2), (5,3) \}, \\
B_6 &= \{ (6,1), (6,2), (6,5) \}, \\
B_7 &= \{ (7,1), (7,5), (7,6) \}.
\end{align*}
\]

5. Related unsolved problems

The necessary conditions for the existence of \(G\)-designs, bipartite \(G\)-designs and multipartite \(G\)-designs given in Theorems 3.1, 3.2 and 3.3 are not always sufficient.

**Problem 5.1.** Find a necessary and sufficient condition for the existence of \(G\)-designs.

**Problem 5.2.** Find a necessary and sufficient condition for the existence of bipartite \(G\)-designs.

**Problem 5.3.** Find a necessary and sufficient condition for the existence of multipartite \(G\)-designs.

These problems seem to be very difficult. It may be simpler to consider the following problem.

**Problem 5.4.** Solve Problems 5.1 through 5.3 when \(G = K_4\), \(G = C_4\), \(G = P_4\) and \(G = S_4\).
Conjecture 5.1 [3]. There exists a balanced \((v, k, \lambda)\) \(C_k\)-design if and only if
(i) \(\lambda v(v-1) \equiv 0 \pmod{2k}\),
(ii) \(\lambda(v-1) \equiv 0 \pmod{2}\).

Conjecture 5.2 [32]. There exists an \((m, n, k, \lambda)\) bipartite \(P_k\)-design \((m \geq n)\) if and only if
(i) \(\lambda mn \equiv 0 \pmod{k-1}\),
(ii) \(m \geq \lceil k/2 \rceil\),
(iii) \(n \geq \lceil (k-1)/2 \rceil\),
except for the following parameters \((m, n, k, \lambda)\):
(1) \(m\) is even, \(n\) and \(k-1\) are odd, \(k-1 > \lambda n\),
(2) \(m\) and \(k-1\) are odd, \(n\) is even, \(k-1 > \lambda m\),
(3) \(m\) and \(n\) are odd, \(k-1 > \lambda n\).

Conjecture 5.3. When \(k\) is odd, there exists a resolvable \((m, n, k, 1)\) bipartite \(P_k\)-design if and only if
(i) \(m+n \equiv 0 \pmod{k}\),
(ii) \((k-1)m \leq (k+1)n\),
(iii) \((k-1)n \leq (k+1)m\),
(iv) \(kn(k-1)(m+n)\) is an integer.

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References