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G-designs and related designs

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Abstract

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This is a survey on the existence of G-designs, bipartite G-designs and multipartite G-designs.

1. Introduction

G-designs and related designs have been investigated to solve construction problems occurring in graph theory and many fields of research-combinatorial mathematics, mathematical statistics, database systems, etc.

In combinatorial mathematics, there are famous problems such as the 'fifteen school-girls problem', the 'nine school-boys problem' and the 'nine prisoners problem'. The 'nine prisoners problem' (Hell and Rosa [11]) is the following: In a jail there are 9 prisoners of a particularly dangerous character. Each morning they are allowed to walk handcuffed in the prison yard. Here is how they walked on Monday: (1)–(2)–(3) (4)–(5)–(6) (7)–(8)–(9). Can they be arranged for Tuesday–Saturday so that no pair of prisoners is handcuffed together twice? The solutions of the three problems given above can be obtained by constructing resolvable $(15,3,1)$ K_3-designs, resolvable $(9,3,1)$ K_3-designs and resolvable $(9,3,1)$ P_3-designs, respectively.

In design of experiments, there are fundamental construction problems about (v,k,λ) balanced incomplete block (BIB) designs, resolvable (v,k,λ) BIB designs and (v,k,λ) transversal designs. Such designs can be obtained by constructing (v,k,λ) K_k-designs, resolvable (v,k,λ) K_k-designs and (v,k,λ) C_k-designs, respectively.

In database systems consisting of formatted data, there are combinatorial index-file organization scheme problems about BFS_2, HUBFS_2, BMFS_2 and HUBMFS_2. They can be solved by constructing $(v,k,1)$ K_k-designs, $(v,k,1)$ S_k-designs, $(m,n,k,1)$ multipartite K_k-designs and $(m,n,k,1)$ multipartite S_k-designs, respectively.

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In this paper, G-designs and related designs – bipartite G-designs and multipartite G-designs – will be discussed.

2. Definitions of G-designs and related designs

G-designs were introduced by Hell and Rosa [11]. Many types of G-designs can be found in Bermond and Sotteau [2]. A combinatorial definition of G-designs can be given in terms of G-blocks. In a block G, considered as a set of elements, any two distinct elements are either linked or unlinked. The adjacency matrix $M(G)=(m_{ij})$ of a block G, with k elements, is a symmetric matrix of order k with zero diagonal, where $m_{ij}=1$ if the elements i and j are linked in G and 0 otherwise. A block B is said to be a *G-block* if its adjacency matrix is equivalent to that of G.

Definition 2.1. A $(v, k\,\lambda)$ *G-design* is an arrangement of v elements into b G-blocks such that every G-block contains k elements and any two distinct elements are linked in exactly λ G-blocks. Furthermore if each element belongs to exactly r G-blocks then the design is said to be *balanced.*

Definition 2.2. A *parallel class* of G-blocks consists of a set of disjoint G-blocks such that every element occurs in one G-block of the class. A (v, k, λ) G-design is said to be *resolvable* if the b G-blocks can be partitioned into r parallel classes.

Another definition of G-designs can be given by using the terminology of graph theory. Let G be a graph with k vertices and let λK_v be the complete multigraph with v vertices in which any two distinct vertices are joined by exactly λ edges.

Definition 2.3. A (v, k, λ) *G-design* is an edge-disjoint decomposition of λK_v into b subgraphs isomorphic to G. Furthermore if each vertex belongs to exactly r subgraphs then the G-design is said to be *balanced.*

Definition 2.4. A *G-factor* is a spanning subgraph of λK_v such that each component of the G-factor is isomorphic to G. A (v, k, λ) G-design is said to be *resolvable* if λK_v can be factorized into r G-factors.

Remark 2.1. If G is regular, it is easy to see that the G-design is balanced. In the particular case when G is the complete graph K_k, a balanced (v, k, λ) K_k-design and a resolvable (v, k, λ) K_k-design are nothing but a (v, k, λ) BIB design and a resolvable (v, k, λ) BIB design, respectively. The G-design is in a sense a generalization of BIB designs.

By use of the terminology of graph theory, definitions of *bipartite G-designs* and *multipartite G-designs* can also be given. For G being a graph with k vertices, let $\lambda K_{m,n}$

be the complete bipartite multigraph with two partite sets of m and n vertices each in which any two distinct vertices in different sets are joined by exactly λ edges. Further let λK_m^n be the complete multipartite multigraph with m partite sets of n vertices each in which any two distinct vertices in different sets are joined by exactly λ edges.

Definition 2.5. An (m, n, k, λ) *bipartite G-design* is an edge-disjoint decomposition of $\lambda K_{m,n}$ into b subgraphs isomorphic to G. Furthermore if each vertex belongs to exactly r subgraphs then the bipartite G-design is said to be *balanced*. An (m, n, k, λ) bipartite G-design is said to be *resolvable* if $\lambda K_{m,n}$ can be factorized into r G-factors of $\lambda K_{m,n}$.

Definition 2.6. An (m, n, k, λ) *multipartite G-design* is an edge-disjoint decomposition of λK_m^n into b subgraphs isomorphic to G. Furthermore if each vertex belongs to exactly r subgraphs then the multipartite G-design is said to be *balanced*. An $(m. n, k, \lambda)$ multipartite G-design is said to be *resolvable* if λK_m^n can be factorized into r G-factors of λK_m^n.

Remark 2.2. If G is regular, the multipartite G-design is balanced and the bipartite G-design with $m = n$ is also balanced. In the particular case when G is the complete graph K_k, a balanced (m, n, k, λ) multipartite K_k-design and a resolvable (m, n, k, λ) multipartite K_k-design are respectively particular types of partially balanced incomplete block (PBIB) designs and resolvable PBIB designs. In a sense, the multipartite G-design is a generalization of PBIB designs.

Remark 2.3. In any case of G-designs, bipartite G-designs and multipartite G-designs, if the design is resolvable then it is also balanced.

3. Necessary conditions for the existence of G-designs and related designs

In this section we give the necessary conditions for the existence of G-designs, bipartite G-designs and multipartite G-designs. Let e be the number of edges of G and let d be the g.c.d. of the degrees of vertices of G. Then we have the following:

Theorem 3.1. *If there exists a* (v, k, λ) *G-design, then*

(i) $\lambda v(v-1) \equiv 0 \pmod{2e}$,

(ii) $\lambda(v-1) \equiv 0 \pmod{d}$.

Moreover, if the G-design is balanced, then

(iii) $\lambda k(v-1) \equiv 0 \pmod{2e}$,

and if the G-design is resolvable, then

(iv) $v \equiv 0 \pmod{k}$.

Proof. Let b be the number of subgraphs (blocks) of the G-design. Then $b=\lambda v(v-1)/2e$, which yields (i). Let d_i $(i=1,2,\ldots,k)$ be the degrees of the vertices of G. Suppose that a vertex x of λK_v appears with degree d_i in b_i subgraphs. Then $\lambda(v-1)=\sum b_i d_i$, which implies (ii). Next, suppose that each vertex belongs to exactly r subgraphs in the balanced G-design. The $r=\lambda k(v-1)/2e$, which yields (iii). When each G-factor has t components in the resolvable G-design, we get $t=v/k$, which implies (iv). $\square$

Remark 3.1. When G is regular, (iii) is equivalent to (ii).

The following two theorems can easily be proved using an argument similar to the proof of Theorem 3.1.

Theorem 3.2. *If there exists an* (m,n,k,λ) *bipartite G-design, then*
(i) $\lambda mn\equiv 0 \pmod e$,
(ii) $\lambda m\equiv 0 \pmod d$,
(iii) $\lambda n\equiv 0 \pmod d$.
Moreover, if the bipartite G-design is balanced, then
(iv) $\lambda mnk\equiv 0 \pmod{(m+n)e}$,
and if the bipartite G-design is resolvable, then
(v) $m+n\equiv 0 \pmod k$.

Theorem 3.3. *If there exists an* (m,n,k,λ) *multipartite G-design, then*
(i) $\lambda m(m-1)n^2\equiv 0 \pmod{2e}$,
(ii) $\lambda(m-1)n\equiv 0 \pmod d$.
Moreover, if the multipartite G-design is balanced, then
(iii) $\lambda(m-1)nk\equiv 0 \pmod{2e}$,
and if the multipartite G-design is resolvable, then
(iv) $mn\equiv 0 \pmod k$.

The following existence theorem is given by Wilson [37].

Theorem 3.4. *For a given graph* (*block*) *G and a given* λ, *the necessary conditions* (i) *and* (ii) *for the existence of G-designs given in Theorem* 3.1 *are sufficient for all sufficiently large integers* v.

4. Survey of the results concerning the existence of G-designs and related designs

We consider the following four classes of G such that $G=K_k$, C_k, P_k and S_k, where $k\geqslant 3$ and
K_k is a complete graph with k vertices and $k(k-1)/2$ edges,
C_k is a cycle with k vertices and k edges,
P_k is a path with k vertices and $k-1$ edges,
S_k is a star with k vertices and $k-1$ edges.

Remark 4.1. Since K_k and C_k are regular, the following designs are automatically balanced: K_k-designs, bipartite K_k-designs with $m=n$, multipartite K_k-designs, C_k-designs, bipartite C_k-designs with $m=n$ and multipartite C_k-designs.

Remark 4.2. Since $\lambda K_{m,n}$ has no subgraph K_k, bipartite K_k-designs do not exist.

4.1. K_k-designs and multipartite K_k-designs

A balanced (m,n,k,λ) multipartite K_k-design and a resolvable (m,n,k,λ) multipartite K_k-design are a PBIB design with $\lambda_1=\lambda$ and $\lambda_2=0$ and a resolvable PBIB design with $\lambda_1=\lambda$ and $\lambda_2=0$, respectively.

Theorem 4.1 [10, Theorem 5.1–5.4]. *For k in $3\leqslant k\leqslant 5$ and $k=6$ with $\lambda>1$, there exists a balanced (v, k, λ) K_k-design if and only if*
(i) $\lambda v(v-1)\equiv 0 \pmod{k(k-1)}$,
(ii) $\lambda(v-1)\equiv 0 \pmod{k-1}$,
with two exceptions $(v=15, k=5, \lambda=2)$ and $(v=21, k=6, \lambda=2)$.

Sketch of proof. The necessity follows from Theorem 3.1. To prove the sufficiency, we note that λ determines the values of v satisfying the conditions of the theorem. It suffices to consider only those λ's which are factors of 6 (when $k=3, 4$), 20 (when $k=5$) and 30 (when $k=6$). Then we obtain the following: When $k=3$, for $\lambda=1$, $v\equiv 1$ or $3 \pmod 6$, for $\lambda=2$, $v\equiv 0$ or $1 \pmod 3$, for $\lambda=3$, $v\equiv 1 \pmod 2$, for $\lambda=6$, every v. When $k=4$, for $\lambda=1$, $v\equiv 1$ or $4 \pmod{12}$, for $\lambda=2$, $v\equiv 1 \pmod 3$, for $\lambda=3$, $v\equiv 0$ or $1 \pmod 4$, for $\lambda=6$, every v. When $k=5$, for $\lambda=1$, $v\equiv 1$ or $5 \pmod{20}$, for $\lambda=2$, $v\equiv 1$ or $5 \pmod{10}$, for $\lambda=4$, $v\equiv 0$ or $1 \pmod 5$, for $\lambda=5$, $v\equiv 1 \pmod 4$, for $\lambda=10$, $v\equiv 1 \pmod 2$, for $\lambda=20$, every v. And when $k=6$ with $\lambda>1$, for $\lambda=2$, $v\equiv 1$ or $6 \pmod{15}$, for $\lambda=3$, $v\equiv 1 \pmod 5$, for $\lambda=5$, $v\equiv 0$ or $1 \pmod 3$, for $\lambda=15$, every v.

In all these cases—with two exceptions $(v=15, k=5, \lambda=2)$ and $(v=21, k=6, \lambda=2)$—the existence of the relevant BIB design has been proved. It is known that the two exceptional BIB designs do not exist. $\square$

Example 4.1. A balanced $(7,3,1)$ K_3-design is given by

$$B_1=(1,2,4)=\{(1,2),(1,4),(2,4)\},$$
$$B_2=(2,3,5)=\{(2,3),(2,5),(3,5)\},$$
$$B_3=(3,4,6)=\{(3,4),(3,6),(4,6)\},$$
$$B_4=(4,5,7)=\{(4,5),(4,7),(5,7)\},$$
$$B_5=(5,6,1)=\{(5,6),(5,1),(6,1)\},$$
$$B_6=(6,7,2)=\{(6,7),(6,2),(7,2)\},$$
$$B_7=(7,1,3)=\{(7,1),(7,3),(1,3)\},$$

Example 4.2. A resolvable $(9,3,1)$ K_3-design is given by

$$B_1=(1,6,7)=\{(1,6),(6,7),(7,1)\},$$
$$B_2=(2,3,5)=\{(2,3),(3,5),(5,2)\},$$
$$B_3=(8,4,9)=\{(8,4),(4,9),(9,8)\},$$
$$B_4=(2,7,8)=\{(2,7),(7,8),(8,2)\},$$
$$B_5=(3,4,6)=\{(3,4),(4,6),(6,3)\},$$
$$B_6=(1,5,9)=\{(1,5),(5,9),(9,1)\},$$
$$B_7=(3,8,1)=\{(3,8),(8,1),(1,3)\},$$
$$B_8=(4,5,7)=\{(4,5),(5,7),(7,4)\},$$
$$B_9=(2,6,9)=\{(2,6),(6,9),(9,2)\},$$
$$B_{10}=(4,1,2)=\{(4,1),(1,2),(2,4)\},$$
$$B_{11}=(5,6,8)=\{(5,6),(6,8),(8,5)\},$$
$$B_{12}=(3,7,9)=\{(3,7),(7,9),(9,3)\}.$$

Let $F_i=B_{3i-2}\cup B_{3i-1}\cup B_{3i}(1\leqslant i\leqslant 4)$. Then these K_3-factors F_i comprise a K_3-factorization of K_9. This example is known as 'a solution of the 9 school-boys problem'.

4.2. *C_k-designs, bipartite C_k-designs and multipartite C_k-designs*

Balanced C_k-designs have been studied by Kotzig [19], Rosa [23, 24], Rosa and Huang [25], Bermond, Huang and Sotteau [3], and Bermond and Thomassen [4]. In [3], it is conjectured that the necessary conditions (i) and (ii) in Theorem 3.1 are sufficient for balanced (v, k, λ) C_k-designs. This is true for small k as the following demonstrates (e.g., [3, 25]).

Theorem 4.2. *For k in $4\leqslant k\leqslant 8$, there exists a balanced (v, k, λ) C_k-design if and only if*
(i) $\lambda v(v-1)\equiv 0 \pmod{2k}$,
(ii) $\lambda(v-1)\equiv 0 \pmod 2$.

For bipartite C_k-designs, we have the following (see [26]).

Theorem 4.3. *There exists an $(m,n,k,1)$ bipartite C_k-design if and only if*
(i) $mn\equiv 0 \pmod{2k}$,
(ii) $m, n\equiv 0 \pmod 2$,
(iii) $m,n\geqslant k$.

The following theorem has been established by Enomoto, Miyamoto and Ushio [9].

Theorem 4.4. *There exists a resolvable* $(m, n, k, 1)$ *bipartite* C_k*-design if and only if*

(i) $m = n \equiv 0 \pmod 2$,

(ii) $k \equiv 0 \pmod 2$, $k \geqslant 4$,

(iii) $2n \equiv 0 \pmod k$

with precisely one exception, namely $m = n = k = 6$.

Sketch of proof. The necessity follows from Theorem 3.2. To prove the sufficiency, it suffices to consider the following facts: If $K_{n,n}$ has a C_k-factorization, then $K_{sn,sn}$ also has a C_k-factorization. If $k \equiv 0 \pmod 4$ and $m = n \equiv 0 \pmod{k/2}$, then $K_{m,n}$ has a C_k-factorization. $K_{6,6}$ does not have a C_6-factorization. When s is odd and $s \geqslant 5$, $K_{2s,2s}$ has a C_{2s}-factorization. When $s \geqslant 2$, $K_{6s,6s}$ has a C_6-factorization. □

Multipartite C_k-designs have been studied by Cockayne and Hartnell [8], and Laskar and Auerbach [21].

Example 4.3. A balanced $(9, 4, 1)$ C_4-design is given by

$$B_1 = \{(1,2), (2,6), (6,3), (3,1)\},$$

$$B_2 = \{(2,3), (3,7), (7,4), (4,2)\},$$

$$B_3 = \{(3,4), (4,8), (8,5), (5,3)\},$$

$$B_4 = \{(4,5), (5,9), (9,6), (6,4)\},$$

$$B_5 = \{(5,6), (6,1), (1,7), (7,5)\},$$

$$B_6 = \{(6,7), (7,2), (2,8), (8,6)\},$$

$$B_7 = \{(7,8), (8,3), (3,9), (9,7)\},$$

$$B_8 = \{(8,9), (9,4), (4,1), (1,8)\},$$

$$B_9 = \{(9,1), (1,5), (5,2), (2,9)\}.$$

Example 4.4. A resolvable $(12, 12, 6, 1)$ bipartite C_6-design is given below: Let partite sets of $K_{12,12}$ be $\{i \mid 1 \leqslant i \leqslant 12\}$ and $\{\bar{i} \mid 1 \leqslant i \leqslant 12\}$. And let

$$B_1 = \{(1,\bar{1}), (\bar{1},5), (5,\bar{5}), (\bar{5},9), (9,\bar{9}), (\bar{9},1)\},$$

$$B_2 = \{(2,\bar{2}), (\bar{2},6), (6,\bar{6}), (\bar{6},10), (10,\overline{10}), (\overline{10},2)\},$$

$$B_3 = \{(3,\bar{3}), (\bar{3},7), (7,\bar{7}), (\bar{7},11), (11,\overline{11}), (\overline{11},3)\},$$

$$B_4 = \{(4,\bar{4}), (\bar{4},8), (8,\bar{8}), (\bar{8},12), (12,\overline{12}), (\overline{12},4)\},$$

$$B_5 = \{(1,\bar{4}), (\bar{4},11), (11,\bar{2}), (\bar{2},9), (9,\overline{12}), (\overline{12},1)\},$$

$$B_6 = \{(2,\bar{5}), (\bar{5},12), (12,\bar{3}), (\bar{3},4), (4,\bar{7}), (\bar{7},2)\},$$

$$B_7=\{(3,\bar{6}),(\bar{6},7),(7,\overline{10}),(\overline{10},5),(5,\bar{8}),(\bar{8},3)\},$$

$$B_8=\{(6,\bar{9}),(\bar{9},10),(10,\bar{1}),(\bar{1},8),(8,\overline{11}),(\overline{11},6)\},$$

$$B_9=\{(1,\bar{2}),(\bar{2},3),(3,\bar{1}),(\bar{1},2),(2,\bar{3}),(\bar{3},1)\},$$

$$B_{10}=\{(4,\bar{5}),(\bar{5},6),(6,\bar{4}),(\bar{4},5),(5,\bar{6}),(\bar{6},4)\},$$

$$B_{11}=\{(7,\bar{8}),(\bar{8},9),(9,\bar{7}),(\bar{7},8),(8,\bar{9}),(\bar{9},7)\},$$

$$B_{12}=\{(10,\overline{11}),(\overline{11},12),(12,\overline{10}),(\overline{10},11),(11,\overline{12}),(\overline{12},10)\},$$

$$B_{13}=\{(1,\bar{5}),(\bar{5},3),(3,\bar{4}),(\bar{4},2),(2,\bar{6}),(\bar{6},1)\},$$

$$B_{14}=\{(4,\bar{8}),(\bar{8},6),(6,\bar{7}),(\bar{7},5),(5,\bar{9}),(\bar{9},4)\},$$

$$B_{15}=\{(7,\overline{11}),(\overline{11},9),(9,\overline{10}),(\overline{10},8),(8,\overline{12}),(\overline{12},7)\},$$

$$B_{16}=\{(10,\bar{2}),(\bar{2},12),(12,\bar{1}),(\bar{1},11),(11,\bar{3}),(\bar{3},10)\},$$

$$B_{17}=\{(1,\bar{7}),(\bar{7},3),(3,\bar{9}),(\bar{9},2),(2,\bar{8}),(\bar{8},1)\},$$

$$B_{18}=\{(4,\overline{10}),(\overline{10},6),(6,\overline{12}),(\overline{12},5),(5,\overline{11}),(\overline{11},4)\},$$

$$B_{19}=\{(7,\bar{1}),(\bar{1},9),(9,\bar{3}),(\bar{3},8),(8,\bar{2}),(\bar{2},7)\},$$

$$B_{20}=\{(10,\bar{4}),(\bar{4},12),(12,\bar{6}),(\bar{6},11),(11,\bar{5}),(\bar{5},10)\},$$

$$B_{21}=\{(1,\overline{10}),(\overline{10},3),(3,\overline{12}),(\overline{12},2),(2,\overline{11}),(\overline{11},1)\},$$

$$B_{22}=\{(4,\bar{1}),(\bar{1},6),(6,\bar{3}),(\bar{3},5),(5,\bar{2}),(\bar{2},4)\},$$

$$B_{23}=\{(7,\bar{4}),(\bar{4},9),(9,\bar{6}),(\bar{6},8),(8,\bar{5}),(\bar{5},7)\},$$

$$B_{24}=\{(10,\bar{7}),(\bar{7},12),(12,\bar{9}),(\bar{9},11),(11,\bar{8}),(\bar{8},10)\}.$$

Let $F_i=B_{4i-3}\cup B_{4i-2}\cup B_{4i-1}\cup B_{4i}(1\leqslant i\leqslant 6)$. Then these C_6-factors F_i comprise a C_6-factorization of $K_{12,12}$

4.3. P_k-designs, bipartite P_k-designs and multipartite P_k-designs

The following is due to Tarsi [28].

Theorem 4.5. *There exists a (v,k,λ) P_k-design if and only if $\lambda v(v-1)\equiv 0 \pmod{2(k-1)}$ and $v\geqslant k$.*

Balanced P_k-designs have been studied under the name of 'Handcuffed Desings' by Hell and Rosa [11], Lawless [20], and Hung and Mendelsohn [18].

Theorem 4.6 [18]. *There exists a balanced (v,k,λ) P_k-design if and only if $\lambda v(v-1)\equiv 0$ $\pmod{2(k-1)}$ and $\lambda k(v-1)\equiv 0 \pmod{2(k-1)}$.*

Resolvable P_k-designs have been investigated by Horton [13], and Bermond, Heinrich and Yu [5].

Theorem 4.7 [5]. *There exists a resolvable* (v, k, λ) P_k*-design if and only if* $v \equiv 0 \pmod{k}$ *and* $\lambda k(v-1) \equiv 0 \pmod{2(k-1)}$.

Bipartite and resolvable bipartite P_k-designs have been, respectively, studied by Truszczynski [32] and Ushio [36].

Theorem 4.8 [36]. *There exists a resolvable* $(m, n, 3, 1)$ *bipartite* P_k*-design if and only if*

(i) $m+n \equiv 0 \pmod{3}$,

(ii) $m \leqslant 2n$,

(iii) $n \leqslant 2m$,

(iv) $3mn/2(m+n)$ *is an integer.*

The following theorem was obtained by Ushio.

Theorem 4.9. *When* k *is even, there exists a resolvable* $(m, n, k, 1)$ *bipartite* P_k*-design if and only if* $m=n\equiv 0 \pmod{k(k-1)/2}$.

Sketch of proof. The necessity follows from Theorem 3.2. To prove the sufficiency, it suffices to consider the following facts: If $K_{m,n}$ has a P_k-factorization, then $K_{sm,sn}$ also has a P_k-factorization. Let k be even. When $n=k(k-1)/2$, $K_{n,n}$ has a P_k-factorization. □

Example 4.5. A $(5, 3, 1)$ P_3-design is given by

$$B_1=\{(1,2),(1,3)\},$$
$$B_2=\{(1,4),(1,5)\},$$
$$B_3=\{(2,3),(2,4)\},$$
$$B_4=\{(3,4),(3,5)\},$$
$$B_5=\{(2,5),(4,5)\}.$$

Example 4.6. A resolvable $(9, 3, 1)$ P_3-design is displayed below

$$B_1=\{(1,2),(2,3)\},\quad B_2=\{(4,5),(5,6)\},\quad B_3=\{(7,8),(8,9)\},$$
$$B_4=\{(3,7),(7,5)\},\quad B_5=\{(6,1),(1,8)\},\quad B_6=\{(9,4),(4,2)\},$$
$$B_7=\{(5,9),(9,1)\},\quad B_8=\{(8,3),(3,4)\},\quad B_9=\{(2,6),(6,7)\},$$
$$B_{10}=\{(7,2),(2,5)\},\quad B_{11}=\{(3,1),(1,4)\},\quad B_{12}=\{(9,6),(6,8)\},$$
$$B_{13}=\{(9,7),(7,1)\},\quad B_{14}=\{(5,3),(3,6)\},\quad B_{15}=\{(2,8),(8,4)\},$$
$$B_{16}=\{(2,9),(9,3)\},\quad B_{17}=\{(1,5),(5,8)\},\quad B_{18}=\{(7,4),(4,6)\}.$$

Let $F_i = B_{3i-2} \cup B_{3i-1} \cup B_{3i} (1 \leqslant i \leqslant 6)$. Then these P_3-factors F_i comprise a P_3-factorization of K_9. This is known as 'a solution of the 9 prisoners problem'.

Example 4.7. A resolvable (3, 2, 3, 1) multipartite P_3-design is given below:
Let partite sets of K_3^2 be $\{1,2\}$, $\{3,4\}$, $\{5,6\}$. And let

$$B_1 = \{(1,3), (1,6)\}, \quad B_2 = \{(2,4), (2,5)\}, \quad B_3 = \{(3,2), (3,5)\},$$

$$B_4 = \{(4,1), (4,6)\}, \quad B_5 = \{(5,1), (5,4)\}, \quad B_6 = \{(6,2), (6,3)\}.$$

Let $F_i = B_{2i-1} \cup B_{2i} (1 \leqslant i \leqslant 3)$. Then these P_3-factors F_i comprise a P_3-factorization of K_3^2.

4.4. S_k-designs, bipartite S_k-designs and multipartite S_k-designs

Many researchers (e.g., [6, 7, 27, 38]) have studied S_k-designs.

Theorem 4.10 [27]. *There exists a (v, k, λ) S_k-design if and only if $\lambda v(v-1) \equiv 0 \pmod{2(k-1)}$ and $v \geqslant 2(k-1)$ for $\lambda = 1$, $v \geqslant k$ for even λ, $v \geqslant k + (k-1)/\lambda$ for odd $\lambda \geqslant 3$.*

The $(v, k, 1)$ S_k-designs can be applied to combinatorial binary-valued index-file organization schemes of order two in database systems, which have been studied by Yamanoto, Ikeda, Shige-eda, Ushio and Hamada [39].

Huang and Rosa [15] and Huang [16] have presented existence conditions for balanced S_k-designs.

Theorem 4.11 [16]. *There exists a balanced (v, k, λ) S_k-design if and only if $\lambda(v-1) \equiv 0 \pmod{2(k-1)}$.*

For bipartite S_k-designs, Yamamoto, Ikeda, Shige-eda, Ushio and Hamada [38] have presented the following.

Theorem 4.12. *There exists an $(m, n, k, 1)$ bipartite S_k-design if and only if*
(i) *$m \equiv 0 \pmod{k-1}$ when $n < k-1$,*
(ii) *$n \equiv 0 \pmod{k-1}$ when $m < k-1$,*
(iii) *$mn \equiv 0 \pmod{k-1}$ when $m \geqslant k-1$ and $n \geqslant k-1$.*

The following is due to Ushio, Tazawa and Yamamoto [33].

Theorem 4.13. *There exists an $(m, n, k, 1)$ multipartite S_k-design if and only if*
(i) *$m(m-1)n^2 \equiv 0 \pmod{2(k-1)}$,*
(ii) *$mn \geqslant 2(k-1)$.*

The $(m, n, k, 1)$ multipartite S_k-designs can be applied to combinatorial multiple-valued index-file organization schemes of order two in database systems, which have been studied by Yamamoto, Tazawa, Ushio and Ikeda [41].

Example 4.8. A $(6, 4, 1)$ S_4-design is given by

$$B_1 = \{(1, 2), (1, 3), (1, 4)\},$$
$$B_2 = \{(2, 3), (2, 4), (2, 6)\},$$
$$B_3 = \{(3, 4), (3, 5), (3, 6)\},$$
$$B_4 = \{(5, 1), (5, 2), (5, 4)\},$$
$$B_5 = \{(6, 1), (6, 4), (6, 5)\}.$$

Example 4.9. A balanced $(7, 4, 1)$ S_4-design is given by

$$B_1 = \{(1, 2), (1, 3), (1, 4)\},$$
$$B_2 = \{(2, 3), (2, 4), (2, 7)\},$$
$$B_3 = \{(3, 4), (3, 6), (3, 7)\},$$
$$B_4 = \{(4, 5), (4, 6), (4, 7)\},$$
$$B_5 = \{(5, 1), (5, 2), (5, 3)\},$$
$$B_6 = \{(6, 1), (6, 2), (6, 5)\},$$
$$B_7 = \{(7, 1), (7, 5), (7, 6)\}.$$

5. Related unsolved problems

The necessary conditions for the existence of G-designs, bipartite G-designs and multipartite G-designs given in Theorems 3.1, 3.2 and 3.3 are not always sufficient.

Problem 5.1. Find a necessary and sufficient condition for the existence of G-designs.

Problem 5.2. Find a necessary and sufficient condition for the existence of bipartite G-designs.

Problem 5.3. Find a necessary and sufficient condition for the existence of multipartite G-designs.

These problems seem to be very difficult. It may be simpler to consider the following problem.

Problem 5.4. Solve Problems 5.1 through 5.3 when $G = K_k$, $G = C_k$, $G = P_k$ and $G = S_k$.

Conjecture 5.1 [3]. There exists a balanced (v, k, λ) C_k-design if and only if
(i) $\lambda v(v-1) \equiv 0 \pmod{2k}$,
(ii) $\lambda(v-1) \equiv 0 \pmod 2$.

Conjecture 5.2 [32]. There exists an (m, n, k, λ) bipartite P_k-design $(m \geqslant n)$ if and only if
(i) $\lambda mn \equiv 0 \pmod{k-1}$,
(ii) $m \geqslant [k/2]$,
(iii) $n \geqslant [(k-1)/2]$,
except for the following parameters (m, n, k, λ):
(1) m is even, n and $k-1$ are odd, $k-1 > \lambda n$,
(2) m and $k-1$ are odd, n is even, $k-1 > \lambda m$,
(3) m and n are odd, $k-1 > \lambda n$.

Conjecture 5.3. When k is odd, there exists a resolvable $(m, n, k, 1)$ bipartite P_k-design if and only if
(i) $m+n \equiv 0 \pmod k$,
(ii) $(k-1)m \leqslant (k+1)n$,
(iii) $(k-1)n \leqslant (k+1)m$,
(iv) $kmn/(k-1)(m+n)$ is an integer.

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References

[1] J. Akiyama and M. Kano, Factors and factorizations of graphs – a survey, J. Graph Theory 9 (1985) 1–42.
[2] J.C. Bermond and D. Sotteau, Graph decompositions and G-designs, in: C.St.J.A. Nash-Williams and J. Sheehan, eds., Proc. 5th British Combinatorial Conf. (1975) 53–72.
[3] J.C. Bermond, C. Huang and D. Sotteau, Balanced cycle and circuit designs: even cases, Ars Combin. 5 (1978) 293–318.
[4] J.C. Bermond and C. Thomassen, Cycles in digraphs – a survey, J. Graph Theory 5 (1981) 1–43.
[5] J.C. Bermond, K. Heinrich and M.L. Yu, Existence of resolvable path designs, in: Y. Alavi, G. Chartrand, T. Mckee and A. Schwenk, eds., Proc. 6th Inter. Conf. The Theory and Applications of Graphs (1988).
[6] P. Cain, Decomposition of complete graphs into stars, Bull. Austral. Math. Soc. 10 (1974) 23–30.
[7] P. Cain, Decomposition of complete graphs into 6-stars and into 10-stars, in: A. Dold and B. Eckmann, eds., Combinatorial Mathematics 3, Lecture Notes in Math. 452 (Springer, Berlin, 1975) 136–142.
[8] E.J. Cockayne and B.L. Hartnell, Edge partitions of complete multipartite graphs into equal length circuits, J. Combin. Theory Ser. B 23 (1977) 174–183.
[9] H. Enomoto, T. Miyamoto and K. Ushio, C_k-factorization of complete bipartite graphs, Graphs Combin. 4 (1988) 111–113.
[10] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 255–369.

[11] P. Hell and A. Rosa, Graph decompositions, handcuffed prisoners and balanced P-designs, Discrete Math. 2 (1972) 229–252.
[12] A.J.W. Hilton, Hamilton decompositions of complete graphs, J. Combin. Theory Ser. B 36 (1984) 125–134.
[13] J.D. Horton, Resolvable path designs, J. Combin. Theory Ser. A (1985) 117–131.
[14] J.D. Horton, B.K. Roy, P.J. Schellenberg and D.R. Stinson, On decomposing graphs into isomorphic uniform 2-factors, Ann. Discrete Math. 27 (1985) 297–320.
[15] C. Huang and A. Rosa, On the existence of balanced bipartite designs, Utilitas Math. 4 (1973) 55–75.
[16] C. Huang, On the existence of balanced bipartite designs II, Discrete Math. 9 (1974) 147–159.
[17] C. Huang, Resolvable balanced bipartite designs, Discrete Math. 14 (1976) 319–335.
[18] S.H.Y. Hung and N.S. Mendelsohn, Handcuffed designs, Discrete Math. 18 (1977) 23–33.
[19] A. Kotzig, On the decomposition of complete graphs into $4k$-gons, Math. Fyz. Casop. 15 (1965) 229–233.
[20] J.F. Lawless, On the construction of handcuffed designs, J. Combin. Theory Ser. A 16 (1974) 76–86.
[21] R. Laskar and B. Auerbach, On decomposition of r-partite graphs into edge-disjoint hamilton circuits, Discrete Math. 14 (1976) 265–268.
[22] D.A. Preece, Balance and designs: another terminological tangle, Utilitas Math, 21C (1982) 85–186.
[23] A. Rosa, On the cyclic decomposition of the complete graph into $(4m+2)$-gons, Math. Fyz. Casop. 16 (1966) 349–353.
[24] A. Rosa, On the cyclic decomposition of the complete graph into polygons with odd number of edges, Časopis Pěst. Mat. 91 (1966) 53–63.
[25] A. Rosa and C. Huang, Another class of balanced graph designs: balanced circuit designs, Discrete Math. 12 (1975) 269–293.
[26] D. Sotteau, Decomposition of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$, J. Combin. Theory Ser. B 30 (1981) 75–81.
[27] M. Tarsi, Decomposition of complete multigraphs into stars, Discrete Math. 26 (1979) 273–278.
[28] M. Tarsi, Decomposition of a complete multigraph into simple paths: nonbalanced handcuffed designs, J. Combin. Theory Ser. A 34 (1983) 60–70.
[29] S. Tazawa, K. Ushio and S. Yamamoto, Partite-claw-decomposition of a complete multi-partite graph, Hiroshima Math. J. 8 (1978) 195–206.
[30] S. Tazawa, Claw-decomposition and evenly-partite-claw-decomposition of complete multi-partite graphs, Hiroshima Math. J. 9 (1979) 503-531.
[31] H. Tverberg, On the decomposition of K_n into complete bipartite graphs, J. Graph Theory 6 (1982) 493–494.
[32] M. Truszczynski, Note on the decomposition of $\lambda K_{m,n}(\lambda K^*_{m,n})$ into paths, Discrete Math. 55 (1985) 89–96.
[33] K. Ushio, S. Tazawa and S. Yamamoto, On claw-decomposition of a complete multi-partite graph, Hiroshima Math. J. 8 (1978) 207–210.
[34] K. Ushio, Bipartite decomposition of complete multipartite graphs, Hiroshima Math. J. 11 (1981) 321–345.
[35] K. Ushio, On balanced claw designs of complete multi-partite graphs, Discrete Math. 38 (1982) 117–119.
[36] K. Ushio, P_3-factorization of complete bipartite graphs, Discrete Math. 72 (1988) 361–366.
[37] R.M. Wilson, Decompositions of complete graphs into subgraphs isomorphic to a given graph, in: C.St.J.A. Nash-Williams and J. Sheehan, eds., Proc. 5th British Combinatorial Conf. (1975) 647–659.
[38] S. Yamamoto, H. Ikeda, S. Shige-eda, K. Ushio and N. Hamada, On claw-decomposition of complete graphs and complete bigraphs, Hiroshima Math. J. 5 (1975) 33–42.
[39] S. Yamamoto, H. Ikeda, S. Shige-eda, K. Ushio and N. Hamada, Design of a new balanced file organization scheme with the least redundancy, Inform. and Control 28 (1975) 156–175.
[40] S. Yamamoto, S. Tazawa, K. Ushio and H. Ikeda, Design of a generalized balanced multiple-valued file organization scheme of order two, in: E. Lowenthal and N.B. Dale, eds., Proc. ACM-SIGMOD Inter. Conf. on Management of Data (1978) 47–51.
[41] S. Yamamoto, S. Tazawa, K. Ushio and H. Ikeda, Design of a balanced multiple-valued file organization scheme with the least redundancy, ACM Trans. Database Systems 4 (1979) 518–530.