Exact boundary controllability of a shallow intrinsic shell model

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Abstract

We consider a variant of a Koiter shell model based on the intrinsic geometry methods of Michael Delfour and Jean-Paul Zolésio. This model, derived in [J. Cagnol, I. Lasiecka, C. Lebiedzik, J.-P. Zolésio, Uniform stability in structural acoustic models with flexible curved walls, J. Differential Equations 186 (1) (2003) 88–121], relies heavily on the oriented distance function which describes the geometry. Here, we establish continuous observability estimates in the Dirichlet case with an explicit observability time, under an additional shallowness assumption and a checkable geometric condition. This yields (by duality) exact controllability for this class of intrinsically modelled shells.

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1. Introduction

In this work we consider the controllability properties of a new shell model. Many interesting problems today involve vibrating surfaces whose motion can be best described by a shell model. These problems arise in fields as diverse as aerospace [4], automotive [9], and medical technologies [8]. In many cases it is critical to be able to control the motion of the curved surface.

The shell model considered here is a shallow variant of the model introduced in [5]. This model is based on the intrinsic geometry methods of Michael Delfour and Jean-Paul Zolésio...
which relies heavily on the oriented, or signed, distance function to describe the geometry of the shell. In this system, differential operators are defined based on the intrinsic geometry of the shell itself. This is in contrast to standard models which use Christoffel symbols to incorporate the curvature of the shell. These models result in partial differential equations in which the variable coefficients in front of the differential operators are written explicitly in terms of the local curvilinear coordinates which describe the shell mid-surface. Hence, the application of PDE methods first introduced for constant–coefficient problems quickly runs into difficulties.

Instead, we choose to use a coordinate-free model in which the variable coefficients are handled implicitly through the definition of the tangential differential operators. The hope is that the resulting calculus is thus made much simpler and the necessary calculations are more tractable. It is unfortunately not made very easy, as the curvature is still a vital part of the system and thus the higher-order terms that cause difficulty are still very much present. However, because of the form of the intrinsic operators we are able to take advantage of the special properties of the shell geometry in analysis of these difficult terms.

1.1. Literature

The formulation behind the shell model we utilize was developed by Michael Delfour and Jean-Paul Zolésio [10–13]. In [5], the full computation of the strain energy was carried out and the resulting model was coupled to a three-dimensional wave equation. In addition, the resulting structural-acoustic system was shown to be uniformly stable. The optimal control of this structural-acoustic system was explored in [7]. In [6], existence and uniqueness of the strong form of the shell model were derived in the appropriate spaces as well as an explicit formulation for the free or Neumann-type boundary conditions for the shell. This paper is the first work to date which addresses controllability properties of the intrinsic model, under an additional shallowness assumption.

Controllability of other shell models has been studied by several other authors. Miara and Valente [21] consider the standard Koiter model and show exact boundary controllability under the additional assumption of smallness of the required terms. A related result of uniform stabilization was derived by Lasiecka, Triggiani and Valente [18], for the special case of a spherical cap. P.F. Yao [26] considers a shallow shell like that considered in our work, but his model is based on Riemannian geometry rather than the intrinsic approach used here. Boundary control of the Naghdi shell model in Riemannian geometry terms has also been considered in [25].

2. The model

In this section we will present the shell model and the set of hypotheses which will be in force for the rest of the paper. A more detailed overview of the oriented distance function and the tangential calculus, as well as definitions of the standard tangential operators, are included in Appendix A.1 for the convenience of the reader.

Consider a domain \( \mathcal{O} \subset \mathbb{R}^3 \) whose nonempty boundary \( \partial \mathcal{O} \) is a \( C^1 \) two-dimensional submanifold of \( \mathbb{R}^3 \). Define the oriented (or signed) distance function to \( \mathcal{O} \) as

\[
b(x) = d_{\mathcal{O}}(x) - d_{\mathbb{R}^3 \setminus \mathcal{O}}(x),
\]

where \( d \) is the Euclidean distance from the point \( x \) to the domain \( \mathcal{O} \). Consider a subset \( \Gamma \subseteq \partial \mathcal{O} \) which will eventually become the mid-surface of our shell. We define the projection \( p(x) \) of a
point $x$ onto $\Gamma$ as $p(x) = x - b(x)\nabla b(x)$. We define a shell $S_h$ of thickness $h_{SH}$ as

$$S_h(\Gamma) \equiv \{x \in \mathbb{R}^3 : p(x) \in \Gamma, \quad |b(x)| < \frac{h_{SH}}{2}\}.$$  \hfill (2)

When $\Gamma \neq \partial \Omega$, the shell $S_h$ has a lateral boundary

$$\Sigma_h(\Gamma) \equiv \{x \in \mathbb{R}^3 : p(x) \in \Upsilon, \quad |b(x)| < \frac{h_{SH}}{2}\}$$  \hfill (3)

where $\Upsilon \equiv \partial \Gamma$ denotes the boundary of $\Gamma$.

We shall adopt the following notation:

$$|w|_{s, \Gamma} \equiv |w|_{H^s(\Gamma)}, \quad (u, v)_\Gamma \equiv \int_{\Gamma} uv \, d\Gamma.$$  

Throughout this paper the conventions of [14] concerning tensors are used. For instance, we will make no distinction between a second-order tensor or a matrix, nor will we make a distinction between a first-order tensor and a vector. Consequently we will not distinguish simple contraction and multiplication. Finally, we use $\langle u, v \rangle$ to denote the scalar, or dot product between two vectors $u$ and $v$.

### 2.1. Model hypotheses

#### Hypothesis 1

The following assumptions are imposed on the shell $S_h$ with mid-surface $\Gamma$.

(i) The shell is assumed to be made of an isotropic and homogeneous material, so that the Lamé coefficients $\lambda > 0$ and $\mu > 0$ are constant.

(ii) The thickness $h_{SH}$ of the shell is small enough to accommodate the curvatures $H$ and $K$, i.e. the product of the thickness by the curvatures is small as compared to 1. In addition, the shell is shallow in the sense that the second fundamental form (here given in terms of the oriented distance function as $D^2 b$) and its derivative ($D^3 b$) are small. This assumption allows us to neglect certain terms in the strain energy in comparison to the model in [5,6] and yields the intrinsic version of the classical shallow shell equations. For a detailed justification of these assumptions we refer to Koiter [15,16].

(iii) (Kirchhoff hypothesis) In the classical thin plate theory named after Kirchhoff, the displacement vectors of the shell $S_h$ and of the mid-surface $\Gamma$ are related by the hypothesis that the filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular to the deformed surface, and undergo neither contraction nor extension. We generalize this hypothesis to the case of a shell using the intrinsic geometry in [5].

(iv) We will assume the boundary $\Upsilon$ consists of two open connected regions $\Upsilon_0$ and $\Upsilon_1$, with $\Upsilon = \Upsilon_0 \cup \Upsilon_1$ and $\emptyset = \Upsilon_0 \cap \Upsilon_1$. We will clamp the shell on $\Upsilon_1$, and impose a Dirichlet boundary control on $\Upsilon_0$.

#### Hypothesis 2 (Geometric condition)

We assume the existence of a vector field $h: \Gamma \to \mathbb{R}^3$ so that for all vectors $\phi$

$$\langle \phi, D_{\Gamma} h\phi \rangle = \alpha(x)|\phi|^2,$$

where $\alpha(x) \geq \alpha_0 > 0$. In addition, we assume that $\text{div}_{\Gamma} \star D_{\Gamma} \text{div}_{\Gamma} \star D_{\Gamma} e_{\Gamma} - \text{div}_{\Gamma} \text{div}_{\Gamma} \star D_{\Gamma} e_{\Gamma} = \Delta_{\Gamma}^2 e_{\Gamma}$, where $e_{\Gamma}$ is the tangential displacement of the shell.
Remark 3 (Consequences of Hypothesis 2). The existence of the vector field $h$ is essentially a hypothesis which eliminates the possibility of having closed geodesics inside the shell. This is a necessary condition for controllability [20]. This vector field also has the property that the second and third derivatives are bounded. The construction of a specific vector field satisfying condition 2 is done in [24] using Riemannian geometry techniques. We can reconstruct this proof using intrinsic notation.

The second part of our hypothesis is used in the context of uniqueness and can be shown to be true if the shell is shallow enough.

2.2. Strong form of the model

The model considered here is a special case of that derived in [5,6]. We denote by $e$ the transformation of the shell mid-surface and by $e_{Γ}$ and $e_n$ the tangential and normal components of $e$ in local coordinates. We define $w$ to be the magnitude of the normal displacement. As such, we have that

$$w = ⟨e, \nabla b⟩, \quad e_n = w \nabla b, \quad e_{Γ} = e - e_n.$$  (4)

The shallowness assumption Hypothesis 1(ii) allows the following simplification over the form presented in [5].

Proposition 4. The following strain-displacement relation holds for a shell modeled in the intrinsic geometry under Hypothesis 1(i)–(iii).

$$\varepsilon(T) = (\varepsilon_{Γ}(e_{Γ}) + wD^2b + V_{Γ}e_{Γ}) \circ p - b(S_{Γ}w + G_{Γ}w) \circ p,$$  (5)

where $\varepsilon_{Γ}$ is the tangential linear strain tensor of elasticity and $V_{Γ}, G_{Γ},$ and $S_{Γ}$ are defined by

$$V_{Γ}u = \frac{1}{2}\left((D^2bu) \otimes \nabla b + \nabla b \otimes (D^2bu)\right),$$  (6a)

$$G_{Γ}w = \frac{1}{2}\left(\nabla b \otimes \nabla_{Γ}w)D^2b + D^2b(\nabla_{Γ}w \otimes \nabla b)\right),$$  (6b)

$$S_{Γ}w = \frac{1}{2}(D^2_{Γ}w + *D^2_{Γ}w),$$  (6c)

$V_{Γ}$ is a zeroth-order operators, that in practice operates on a tangential vector $u$. $G_{Γ}$ is a first-order operator, and $S_{Γ}$ is the symmetrization of the Hessian matrix of a scalar function $w$ (the Hessian matrix is not symmetric in the tangential calculus [10]).

Proof. This is simply an intrinsic version of the model derived by Koiter (see [15, p. 27] or [2, p. 37]) which is valid when the shell is ‘shallow enough’ in the sense given by Hypothesis 1(ii). The translation from coordinate-based language to intrinsic geometry is fully explored in [10,13].

Thus, the elastic energy $E_p$ of the shell under Hypothesis 1 is given by

$$E_p = \frac{λ}{2} \int_{S_h} (\text{tr} \varepsilon(T))^2 + μ \int_{S_h} \text{tr} \varepsilon(T)^2$$
and the kinetic energy of the system $E_k$ is given by

$$E_k = \frac{\rho h}{2} \int_{\Gamma} |\bar{\partial}_t e\Gamma| \, d\Gamma + \frac{\rho h}{2} \int_{\Gamma} |\partial_tw| \, d\Gamma.$$  

Here we let $\gamma = \frac{h^2}{12}$. 

The weak form of the shell model is given by

$$\int_{0}^{T} \left[ -m(\partial_t e, \partial_t \hat{e}) + a(e, \hat{e}) \right] \, dt = 0$$  

for all test functions $\hat{e}$, with $m(e, \hat{e})$ and $a(e, \hat{e})$ given by

$$-m(e, \hat{e}) = -\rho \left[ 2(e\Gamma, \hat{e}\Gamma) + 2(w, \hat{w}) + 2\gamma(\nabla\Gamma w, \nabla\Gamma \hat{w}) \right]$$  

and

$$a(e, \hat{e}) = (2\lambda + 4\mu)\gamma(\Delta\Gamma w, \Delta\Gamma \hat{w}) - 8\mu\gamma(D^2b\nabla\Gamma w, D^2b\nabla\Gamma \hat{w}) + 2\gamma(\nabla\Gamma w, \nabla\Gamma \hat{w}) + 2\gamma(\nabla\Gamma w, \nabla\Gamma \hat{w})$$  

where $k$ is defined by

$$k = 4H^2\lambda + (8H^2 - 4K)\mu > 0.$$  

Proposition 5. The bilinear form $a(e, \hat{e})$ defined in (11) is elliptic on $V$ where the space $V$ is defined by

$$V = \left\{ e \in \left[ H^1(\Gamma) \right]^2 \times H^2(\Gamma) \left| \begin{array}{l} \frac{\partial}{\partial \nu} w = 0 \text{ on } \Gamma_1 \end{array} \right. \right\}.$$  

That is, there exists constant $c > 0$ such that

$$c|e|_V^2 \leq a(e, e) \quad \forall e \in V.$$  

Proof. This inequality is established by Bernadou and Oden [3] (see also [2]) provided the shell is shallow enough (as in Hypothesis 1(ii)). Though this model does not exactly satisfy the lemma of rigid body motion, the proof relies on the fact that the shallowness assumption allows us to consider the equations above as some perturbation of the special case of the plate. 

Define the following operator $C$ acting on a matrix $A$:

$$C(A) = \lambda \text{tr}(A)I + 2\mu A.$$  

(15)
Using the principle of virtual work we thus obtain the following strong form for the displacement of the shell \( e \):

\[
\rho \partial_{tt} w - \rho \gamma \Delta \Gamma \partial_{tt} w + \beta \Delta^2 \Gamma w + P_1(w, e \Gamma) = 0 \quad \text{in } (0, \infty) \times \Gamma,
\]

\[
\rho \partial_{tt} e \Gamma - \text{div} \Gamma \mathcal{C}(\varepsilon \Gamma(e \Gamma)) - \mu (D^2 b)^2 e \Gamma + P_2(w, e \Gamma) = 0 \quad \text{in } (0, \infty) \times \Gamma,
\]

\[
w = \frac{\partial}{\partial \nu} w = 0 \quad \text{in } (0, \infty) \times \Gamma,
\]

\[
e \Gamma = 0 \quad \text{in } (0, \infty) \times \Gamma,
\]

\[
e(t = 0) = e_0, \quad \partial_t e(t = 0) = e_1,
\]

where the constant \( \beta = \gamma (\lambda + 2 \mu) \); and \( P_1(w, e \Gamma), \ P_2(w, e \Gamma) \) are lower-order and coupling terms in the equations for the normal and tangential components, respectively, given by

\[
P_1(w, e \Gamma) = \lambda H \text{div} \Gamma e \Gamma + 2 \mu \text{tr}(\varepsilon \Gamma(e \Gamma) D^2 b) + kw,
\]

\[
P_2(w, e \Gamma) = 2[\mu \text{div} \Gamma(w D^2 b) - \lambda \nabla \Gamma(H w)].
\]

As a result of Proposition 5 we have that the total energy of the system (20) at time \( t \) is given by

\[
E(t) \simeq \rho l \left( |\partial_t w|^2 + |\partial_t e \Gamma|^2 + \gamma |\nabla \Gamma \partial_t w|^2 \right) + \lambda |\text{div} \Gamma e \Gamma|^2 + \beta |\Delta \Gamma w|^2
\]

\[+ 2 \mu \int_{\Gamma} \text{tr}[(\varepsilon_P \Gamma(e \Gamma))^2],\]

where

\[\varepsilon_P \Gamma(e \Gamma) \equiv \varepsilon \Gamma(e \Gamma) + V \Gamma(e \Gamma).\]

3. Statement of main result—Dirichlet control

We consider the Dirichlet mixed problem in the unknown \( \eta = (\eta \Gamma, \xi) \)

\[
\rho \partial_{tt} \xi - \rho \gamma \Delta \Gamma \partial_{tt} \xi + \beta \Delta^2 \Gamma \xi + P_1(\xi, \eta \Gamma) = 0 \quad \text{in } (0, \infty) \times \Gamma,
\]

\[
\rho \partial_{tt} \eta \Gamma - \text{div} \Gamma \mathcal{C}(\varepsilon \Gamma(\eta \Gamma)) - \mu (D^2 b)^2 \eta \Gamma + P_2(\xi, \eta \Gamma) = 0 \quad \text{in } (0, \infty) \times \Gamma,
\]

\[
\xi = \frac{\partial}{\partial \nu} \xi = 0, \quad \eta \Gamma = 0 \quad \text{in } (0, \infty) \times \gamma_1,
\]

\[
\xi = u, \quad \frac{\partial}{\partial \nu} \xi = v, \quad \eta \Gamma = U \quad \text{in } (0, \infty) \times \gamma_0,
\]

\[
\eta(t = 0) = \eta_0, \quad \partial_t \eta(t = 0) = \eta_1,
\]

with control functions \( u, v \), and \( U \in L_2(0, T; L_2(\gamma_0)) \) on the boundary \( \gamma_0 \). The dual version of (19) in the variable \( e = (e \Gamma, w) \) is the following system of equations—i.e. Eqs. (16) with appropriate final condition at time \( T \):

\[
\rho \partial_{tt} w - \rho \gamma \Delta \Gamma \partial_{tt} w + \beta \Delta^2 \Gamma w + P_1(w, e \Gamma) = 0 \quad \text{in } (0, T) \times \Gamma,
\]

\[
\rho \partial_{tt} e \Gamma - \text{div} \Gamma \mathcal{C}(\varepsilon \Gamma(e \Gamma)) - \mu (D^2 b)^2 e \Gamma + P_2(w, e \Gamma) = 0 \quad \text{in } (0, T) \times \Gamma,
\]

\[
w = \frac{\partial}{\partial \nu} w = 0 \quad \text{in } (0, T) \times \gamma,
\]

\[
e \Gamma = 0 \quad \text{in } (0, T) \times \gamma,
\]

\[
e(t = T) = e_0, \quad \partial_t e(t = T) = e_1.
\]
Theorem 6. Let the shell $S_l$ satisfy the standing Hypotheses 1 and 2. Then, for any $T > T_0$, there is a $c_b > 0$ (depending on the geometry and thus on the oriented distance function $b$) such that given initial data $(e_0, e_1) \in ([L^2(\Gamma)]^2 \times H^1_0(\Gamma)) \times ([H^{-1}(\Gamma)]^2 \times L^2(\Gamma))$, the solution $e = (e_\Gamma, w)$ of (20) satisfies the continuous observability inequality
\[ c_b E(0) \leq \int_0^T \left( (D^P_\Gamma e_\Gamma v, \tau)^2 + \langle D^P_\Gamma (e_\Gamma) v, v \rangle + |\Delta_\Gamma w|^2 + \left( \frac{\partial}{\partial \nu} \Delta_\Gamma w \right)^2 \right) \, dt \, d\mathcal{Y}. \] (21)
And thus, by duality, the nonhomogeneous problem (19) is exactly controllable by control functions $u, v, U \in L^2(0, T; L^2(\mathcal{Y}))$ at any time $T > T_0 = \frac{2C}{a_0 - 2\epsilon}$.

4. Proof of Theorem 6

The proof of Theorem 6 follows through several steps via the use of the following tangential differential multipliers:
\[ (h, \nabla_\Gamma w), \quad D^P_\Gamma e_\Gamma h = (D_\Gamma e_\Gamma + \nabla b \otimes D^2 b e_\Gamma) h \] (22)
as well as $m_1 w, m_2 e_\Gamma$, with $m_1, m_2$ functions independent of time. Here $h$ is the vector field discussed in Hypothesis 2. The multipliers introduced in Eq. (22) are simply tangential versions of the classical multipliers used for the wave equation [19,23] and Kirchhoff and Riessner–Midlin plates [17]. The curvature of the shell comes into play in the definition of the operators $\nabla_\Gamma$ and $D^P_\Gamma$, and we note that the expression $D^P_\Gamma u$ is equivalent to the classical covariant derivative of a vector $u$ [13, p. 60, Eq. 3.59]. We will see that the application of the same techniques as in the plate case is possible even though the tangential Hessian is not symmetric, largely due to the relationships between intrinsic objects that are clearly represented in the tangential calculus.

4.1. Various useful identities

In this section we present some definitions and identities that will be continuously referenced in the following calculations. Here, we use the variables $u, v, h$ to denote vector fields; $a, b$ to denote scalars; and $A, B$ to denote matrices. First, we have the standard Green’s formula in the tangential calculus [13]:
\[ \int_\Gamma a \text{div}_\Gamma v \, d\Gamma + \int_\Gamma \langle \nabla_\Gamma a, v \rangle \, d\Gamma = \int_\Gamma \langle av, \nu \rangle \, d\mathcal{Y} + \int_\Gamma 2H a \langle v, \nabla b \rangle \, d\Gamma \] (23)
where $\nu$ is the outward unit normal to the curve $\mathcal{Y}$. Also from [13] we have
\[ \langle \nabla_\Gamma w, \nabla b \rangle = 0, \quad D_\Gamma u \nabla b = 0 \] (24)
by definition for any scalar $w$ and vector $u$. In addition, if we consider a purely tangent vector $u = u_\Gamma$, i.e. $\langle u_\Gamma, \nabla b \rangle = 0$, we can take the tangential gradient of both sides of this expression and derive the following useful formula:
\[ D^2 b u_\Gamma + * D_\Gamma u_\Gamma \nabla b = 0. \] (25)

A recurring issue will be the fact that the matrix $D^2_\Gamma w$ is not symmetric and is not equivalent to the restriction of the Hessian matrix of the canonical extension $w \circ p$. In fact the two terms differ by a first-derivative correction, as is seen in the following identity [10]:
\[ D^2_\Gamma w - \left( D^2 b \nabla_\Gamma w \right) \otimes \nabla b = D^2 (w \circ p)|_\Gamma. \] (26)
(The Hessian matrix on the right-hand side of (26) is of course symmetric.) Next, we have a formula similar to (23) for the tangential divergence of a matrix $A$ (this follows exactly from (23) and the definition of $\text{div}_\Gamma A$):

$$\int_{\Gamma} \langle v, \text{div}_\Gamma A \rangle \, d\Gamma + \int_{\Gamma} \text{tr}(D_\Gamma v A) \, d\Gamma = \int_{\gamma} \langle v, A v \rangle \, d\gamma + \int_{\Gamma} 2H(v, A \nabla b) \, d\Gamma. \quad (27)$$

Define the operators $\ast$ and $\triangle$ on a third-order tensor $T$ as

$$\{ \ast T \}_{ijk} = T_{kji}, \quad (28)$$

$$\{ T^{\circ} \}_{ijk} = T_{kij}. \quad (29)$$

Since integration by parts gives

$$\text{div}_\Gamma (a h) = \langle h, \nabla_\Gamma a \rangle + a \text{div}_\Gamma h \quad (30)$$

application of (23) gives

$$\int_{\Gamma} a \langle h, \nabla_\Gamma a \rangle \, d\Gamma = \frac{1}{2} \int_{\Gamma} a^2 \langle h, v \rangle \, d\gamma - \frac{1}{2} \int_{\Gamma} a^2 \text{div}_\Gamma h \, d\Gamma + \int_{\Gamma} Ha^2 \langle h, \nabla b \rangle \, d\Gamma. \quad (31)$$

Using the fact that $\text{div}_\Gamma (h \otimes v) = \text{div}_\Gamma hv + D_\Gamma vh$ gives the corresponding equation for a vector $v$:

$$\int_{\Gamma} \langle v, D_\Gamma vh \rangle \, d\Gamma = \frac{1}{2} \int_{\gamma} \langle v, (h \otimes v) \rangle \, d\gamma - \frac{1}{2} \int_{\Gamma} |v|^2 \text{div}_\Gamma h \, d\Gamma$$

$$+ \int_{\Gamma} H\langle v, (h \otimes v) \nabla b \rangle \, d\Gamma. \quad (32)$$

**Lemma 7.** Under the assumption of Hypothesis 2, we have that given any ‘tangential’ matrix $M$ (that is, a matrix such that $M \nabla b = 0$), the vector field $h$ satisfies the following relationship:

$$\text{tr}(M) \text{tr}(MD_\Gamma h) = \alpha(x) \text{tr}(M)^2. \quad (33)$$

In particular, we have that

$$\text{div}_\Gamma h = \text{tr}(D_\Gamma h) = 2\alpha(x). \quad (34)$$

**Proof.** Firstly, assume that $M$ is a symmetric matrix. We need only to rewrite the matrix $M$ as an appropriate expansion of outer products of vectors $\phi_i$. There exist coefficients $a_i$ such that

$$M = \sum_{i=1}^{4} a_i \phi_i \otimes \phi_i.$$

Then

$$\text{tr}(MD_\Gamma h) = \sum_{i=1}^{4} \text{tr}(a_i \phi_i \otimes \phi_i D_\Gamma h) = \sum_{i=1}^{4} a_i \langle \phi_i, D_\Gamma h \phi_i \rangle = \sum_{i=1}^{4} a_i \alpha(x) |\phi_i|^2$$

$$= \alpha(x) \text{tr}(M).$$

Multiplying both sides by an additional $\text{tr}(M)$ gives the result for symmetric $M$. If $M$ is not symmetric, the fact that $D_\Gamma h$ is symmetric gives us the result, since

$$\text{tr}(MD_\Gamma h) = \text{tr}(D_\Gamma h \ast M) = \text{tr}(D_\Gamma h \ast M)$$
and
\[ \text{tr} \left( \frac{1}{2} (M + *M) D^\Gamma h \right) = \frac{1}{2} (\text{tr}(MD^\Gamma h) + \text{tr}(*MD^\Gamma h)) = \frac{1}{2} (\text{tr}(MD^\Gamma h) + \text{tr}(D^\Gamma h *M)) = \text{tr}(MD^\Gamma h). \]

Equation (34) is obtained by taking the matrix \( M = I^\Gamma = I - \nabla b \otimes \nabla b. \)

4.2. Boundary issues

We here prove some relationships which hold on the boundary \( \Gamma \) due to the application of Dirichlet boundary conditions (20c) and (20d). We adopt the notation \( \nu = \) outward unit normal to the boundary \( \Gamma \) and \( \tau = \) the corresponding tangent unit vector on the boundary \( \Gamma \).

**Proposition 8.** Let \( \partial \Omega \) be a \( C^1 \) two-dimensional submanifold of \( \mathbb{R}^3 \) as previously described, and \( \Gamma \subset \partial \Omega \) with boundary \( \Gamma = \partial \Omega \). Let \( w \in C^1(\overline{\Gamma}) \). If \( w = 0 \) on the boundary \( \Gamma \), then \( \frac{\partial}{\partial \tau} w \equiv \langle \nabla^\Gamma w, \tau \rangle = 0 \) on \( \Gamma \).

**Proof.** We consider the unit outward normal vector \( \nu(\zeta) \) and the corresponding unit tangent vector \( \tau(\zeta) \) of a point \( \zeta \in \Gamma \). In local coordinates \((X,z)\) as described in Appendix A.1, \( \nu(\zeta) = \left[ \nu_1(\zeta), \nu_2(\zeta), 0 \right] \) and \( \tau(\zeta) = [ -\nu_2(\zeta), \nu_1(\zeta), 0 ] \). We have by definition that
\[ \langle \nabla^\Gamma w, \tau \rangle = \left\langle \nabla (w \circ p) \big|_{\Gamma}, [ -\nu_2, \nu_1, 0 ] \right\rangle = -\frac{\partial}{\partial x} (w \circ p) \bigg|_{\Gamma} \nu_2 + \frac{\partial}{\partial y} (w \circ p) \bigg|_{\Gamma} \nu_1. \quad (35) \]

Now, if \( \Gamma \) is parameterized by arc length we have that \( \nu_1 = -\frac{\partial y}{\partial s} \) and \( \nu_2 = \frac{\partial x}{\partial s} \) for \((x,y) = (x(s), y(s)) \in \Gamma \) so that
\[ \langle \nabla^\Gamma w, \tau \rangle = -\frac{\partial}{\partial x} (w \circ p) \bigg|_{\Gamma} \frac{\partial x}{\partial s} - \frac{\partial}{\partial y} (w \circ p) \bigg|_{\Gamma} \frac{\partial y}{\partial s} = \frac{\partial}{\partial s} (w \circ p) \bigg|_{\Gamma}. \quad (36) \]

And as \( w = 0 \) on \( \Gamma \), so does the expression (36) and \( \langle \nabla^\Gamma w, \tau \rangle = 0 \) on \( \Gamma \).

**Corollary 9.** If \( u \) is a vector and \( u = 0 \) on the boundary \( \Gamma \), then \( D^\Gamma u \tau = 0 \) on \( \Gamma \).

**Proof.** Follows directly from Proposition 8 and definition of \( D^\Gamma u \). 

**Proposition 10.** Let \( \partial \Omega \) be a \( C^1 \) two-dimensional submanifold of \( \mathbb{R}^3 \) as previously described, and \( \Gamma \subset \partial \Omega \) with boundary \( \Gamma = \partial \Omega \). Let \( w \in C^2(\overline{\Gamma}) \). The tangential Laplacian on the boundary \( \Gamma \) can be written as
\[ \Delta^\Gamma w \big|_{\Gamma} = \frac{\partial^2}{\partial \nu^2} w + \frac{\partial^2}{\partial \tau^2} w + \left( \frac{\partial}{\partial \nu} w \right) \text{div}^\Gamma \nu. \quad (37) \]

**Proof.** The proof follows exactly as that of Proposition 8, if one expands the left-hand side of (37) using the definitions of \( \frac{\partial}{\partial \nu} \) and \( \frac{\partial}{\partial \tau}. \)
Remark 11. We note that $D^P_{\Gamma} u \equiv D_{\Gamma} u + \nabla b \otimes D^2 bu$, so that in the case of Dirichlet boundary conditions on $\gamma$, $D^P_{\Gamma} u \equiv D_{\Gamma} u$ on $\gamma$. Naturally in this case we also have $\varepsilon^P_{\Gamma}(u) \equiv \varepsilon_{\Gamma}(u)$ on $\gamma$.

4.3. Multipliers on the $e_{\Gamma}$ equation—first step of proof

Lemma 12. Let $Q = \Gamma \times [0, T]$. With respect to the equation for the tangential components $e_{\Gamma}$ (Eq. (20b)), the following equalities hold:

\[
\int_{\Gamma} \rho \left( \frac{\partial_t e_{\Gamma}}{m_2} \right)^2 \left[ \frac{1}{2} \right] \left( \int_0^T \int_{\gamma} \left( (\lambda + 2\mu)(D^P_{\Gamma} e_{\Gamma} v, v)^2 + 2\mu(D^P_{\Gamma} e_{\Gamma} \tau, \tau)^2 \right) (h, v) d\gamma dt \right. \\
+ \left. \int_{\gamma} \left( 2\mu(\text{div}_{\Gamma} (w D^2 b), D^P_{\Gamma} e_{\Gamma} h) - \lambda(\nabla_{\Gamma} (Hw), D^P_{\Gamma} e_{\Gamma} h) \right) dQ \right. \\
+ \left. \frac{1}{2} \int_{\gamma} \rho |\partial_t e_{\Gamma}|^2 \text{div}_{\Gamma} h dQ + \text{lot}(w, e_{\Gamma}) = 0, \right) \tag{38}
\]

\[
\int_{\Gamma} \rho \langle \partial_t e_{\Gamma}, e_{\Gamma} \rangle m_2 d\Gamma \left. \right|_0^T - \int_Q \left( \rho |\partial_t e_{\Gamma}|^2 - \lambda |\text{div}_{\Gamma} e_{\Gamma}|^2 - 2\mu \text{tr}(\varepsilon^P_{\Gamma}(e_{\Gamma}))^2 \right) m_2 dQ \\
+ \int_Q \left[ \text{div}_{\Gamma} e_{\Gamma} \langle e_{\Gamma}, \nabla_{\Gamma} m_2 \rangle + \text{tr}(e_{\Gamma} \otimes \nabla_{\Gamma} m_2) \varepsilon_{\Gamma}(e_{\Gamma}) \right] dQ \\
+ \mu \int_Q m_2 K \langle e_{\Gamma}, D^2 b e_{\Gamma} \rangle dQ \\
+ \int_Q (2\mu \text{div}_{\Gamma} (w D^2 b) e_{\Gamma} - \lambda(\nabla_{\Gamma} (Hw), e_{\Gamma})) m_2 dQ = 0, \tag{39}
\]

where $h$ is the vector field given by Hypothesis 2 and $m_2 = m_2(x)$ is any function. Here the lower-order terms

\[
\text{lot}(w, e_{\Gamma}) \leq C_b \int_0^T \left( |w|^2_{2-\delta, \Gamma} + |e_{\Gamma}|^2_{1-\delta, \Gamma} \right) dt, \quad \delta > 0. \tag{40}
\]

Proof. The proof of Eq. (38) follows by application of the multiplier $D^P_{\Gamma} e_{\Gamma} h$ and that of Eq. (39) by the multiplier $m_2 e_{\Gamma}$. We multiply Eq. (20b) by $D^P_{\Gamma} e_{\Gamma} h$ and integrate over time and space:

\[
\int_{Q} \left( \rho \partial_t e_{\Gamma} - \text{div}_{\Gamma} C(\varepsilon_{\Gamma}(e_{\Gamma})) - \mu(D^2 b)^2 e_{\Gamma} + P_2(w, e_{\Gamma}), D^P_{\Gamma} e_{\Gamma} h \right) dQ = 0. \tag{41}
\]

We consider each term separately, and the bulk of the work comes in dealing with the term involving $C(\varepsilon_{\Gamma}(e_{\Gamma}))$. In analogy with the case of the linear system of elasticity, we seek to show that the above-energy level expression which results from multiplying by $D^P_{\Gamma} e_{\Gamma} h$ is in fact the divergence of energy-level terms, and thus we can apply the divergence theorem to convert these terms on $\Gamma$ to terms on $\gamma$. Unfortunately, since we cannot exchange the order of derivatives at
will, this calculation is not straightforward at all and a number of extra energy-level terms appear. However, by appropriately combining these terms we find that we can apply the formula (25) to reduce the number of derivatives by one and show that the extra terms are in fact all lower-order.

Lemma 13. The following equality holds for $e = (e_{\Gamma}, w)$ satisfying the system of equations (20):

$$
\int_{\Gamma} (\lambda|\nabla_{\Gamma} e_{\Gamma}|^2 + 2\mu(\nabla_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h)) \, d\Gamma
= \frac{1}{2} \int_{\Gamma} (\lambda|\nabla_{\Gamma} e_{\Gamma}|^2 + 2\mu(\nabla_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h)) \, d\Gamma
- \frac{1}{2} \int_{\Gamma} (\lambda|\nabla_{\Gamma} e_{\Gamma}|^2 + 2\mu(\nabla_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h)) \, d\Gamma
$$

Proof. For simplicity will ignore the integration over the time variable in this calculation. Integrating by parts the left-hand side of Eq. (42) via formulas (23) and (27) gives

$$
\int_{\Gamma} (\lambda\nabla_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h) \, d\Gamma
= \int_{\Gamma} \nabla_{\Gamma} e_{\Gamma} (D^p_{\Gamma} e_{\Gamma} h, \nabla b) \, d\Gamma + 2H \int_{\Gamma} \nabla_{\Gamma} e_{\Gamma} (D^p_{\Gamma} e_{\Gamma} h, \nabla b) \, d\Gamma
- \int_{\Gamma} \nabla_{\Gamma} e_{\Gamma} (D^p_{\Gamma} e_{\Gamma} h, \nabla b) \, d\Gamma,
$$

(43a)

$$
\int_{\Gamma} (\varepsilon_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h) \, d\Gamma
= \int_{\Gamma} (\varepsilon_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h) \, d\Gamma + \int_{\Gamma} (\varepsilon_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h) \, d\Gamma
- \int_{\Gamma} (\varepsilon_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h) \, d\Gamma.
$$

(43b)

We first note that since by construction $h(x) = h_{\Gamma}(x) \in T_\gamma(\Gamma)$, so that $D^p_{\Gamma} e_{\Gamma} h \in T_\gamma(\Gamma)$ and we have, after applying (25),

$$
\int_{\Gamma} \nabla_{\Gamma} e_{\Gamma} (D^p_{\Gamma} e_{\Gamma} h, \nabla b) \, d\Gamma = 0,
$$

$$
\int_{\Gamma} (D^p_{\Gamma} e_{\Gamma} h, \varepsilon_{\Gamma} e_{\Gamma}, D^p_{\Gamma} e_{\Gamma} h) \, d\Gamma
= -\frac{1}{2} \int_{\Gamma} (D^p_{\Gamma} e_{\Gamma} h, D^2 b e_{\Gamma}) \, d\Gamma = \text{lot}(e_{\Gamma}),
$$

(44)

in what follows we will consider the last terms of Eqs. (43a) and (43b), respectively; and then come back and consider the integrals over the boundary $\gamma$. 


Proposition 14. The following equalities hold for \( e_\Gamma \) satisfying the system of equations (20):

\[
\int_\Gamma \langle \nabla_\Gamma \text{div}_\Gamma e_\Gamma, D^P_\Gamma e_\Gamma h \rangle \, d\Gamma \\
= \int_\Gamma \text{div}_\Gamma e_\Gamma \langle D^P_\Gamma e_\Gamma h, v \rangle \, d\Gamma - \int_\Gamma \alpha(x)|\text{div}_\Gamma e_\Gamma|^2 \, d\Gamma + \frac{1}{2} \int_\Gamma \text{div}_\Gamma h(\text{div}_\Gamma e_\Gamma)^2 \, d\Gamma \\
- \frac{1}{2} \int_\Gamma |\text{div}_\Gamma e_\Gamma|^2 \langle h, v \rangle \, d\Gamma + \text{lot}(e_\Gamma), \tag{45a}
\]

\[
\int_\Gamma \langle \varepsilon_\Gamma(e_\Gamma), D^P_\Gamma e_\Gamma h \rangle \, d\Gamma \\
= \int_\Gamma \langle \varepsilon_\Gamma(e_\Gamma) v, D^P_\Gamma e_\Gamma h \rangle \, d\Gamma - \int_\Gamma \text{tr}(\varepsilon_\Gamma(e_\Gamma)^2) \, d\Gamma + \text{lot}(e_\Gamma) \\
+ \frac{1}{2} \int_\Gamma \text{div}_\Gamma h \text{tr}(\varepsilon_\Gamma(e_\Gamma)^2) \, d\Gamma - \frac{1}{2} \int_\Gamma \text{tr}(\varepsilon_\Gamma(e_\Gamma)^2) \langle h, v \rangle \, d\Gamma. \tag{45b}
\]

**Proof.** We first derive a useful identity.

Proposition 15. For any \( u : \Gamma \to \mathbb{R}^3 \) and \( h : \Gamma \to \mathbb{R}^3 \) we have that

\[
D_\Gamma((D_\Gamma u)h) = D_\Gamma u D_\Gamma h + \ast(\ast h D^2_\Gamma u), \tag{46}
\]

so that

\[
\text{div}_\Gamma(D_\Gamma uh) = \text{tr}(D_\Gamma u D_\Gamma h) + \text{tr}(\ast h D^2_\Gamma u) \tag{47}
\]

and

\[
\varepsilon_\Gamma(D_\Gamma uh) = \frac{1}{2}[D_\Gamma u D_\Gamma h + \ast D_\Gamma h \ast D_\Gamma u + \ast h D^2_\Gamma u + \ast(D^2_\Gamma u)h], \tag{48}
\]

where \( D^2_\Gamma u \equiv (D^2_\Gamma u)^\ast \), and the operators \( \ast \) and \( \circ \) on a third-order tensor are defined in Eqs. (28) and (29), respectively.

**Proof.** We work first with the standard Jacobian \( D \) of vector functions \( \tilde{u} : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( \tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3 \).

\[
\{D((D\tilde{u})\tilde{h})\}_{ik} = \partial_k(\tilde{h}_j \partial_j \tilde{u}_i) = \partial_k\tilde{h}_j \partial_j \tilde{u}_i + \tilde{h}_j \partial_k \partial_j \tilde{u}_i \\
= (D\tilde{h})_{jk}(D\tilde{u})_{ij} + \tilde{h}_j(D(D\tilde{u}))_{ijk} \\
= (D\tilde{u} D\tilde{h})_{ik} + (\ast h(D(D\tilde{u}))^{\ast})_{ki}
\]

so that

\[
D((D\tilde{u})\tilde{h}) = D\tilde{u} D\tilde{h} + \ast(\ast h(D(D\tilde{u}))^{\ast}).
\]

Using this, we have
\[ D_{\Gamma}((D_{\Gamma}u)h) \equiv D((D_{\Gamma}uh) \circ p)|_{\Gamma} \]
\[ = D((D_{\Gamma}u \circ p)(h \circ p))|_{\Gamma} \]
\[ = (D_{\Gamma}u \circ p)D(h \circ p) + *(h \circ p)(D(D_{\Gamma}u \circ p))|_{\Gamma} \]
\[ = (D_{\Gamma}u \circ p)|_{\Gamma}D(h \circ p) + *(h \circ p)|_{\Gamma}D(D_{\Gamma}u \circ p)|_{\Gamma} \]
\[ = D_{\Gamma}uD_{\Gamma}h + *(hD_{\Gamma}^2u). \]

Equations (47) and (48) then follow from the definitions of \( \text{div}_{\Gamma} \) and \( \varepsilon_{\Gamma}. \)

Next, we use the definition of \( D_{\Gamma}^P e_{\Gamma} \) to simplify
\[ \text{div}_{\Gamma}(D_{\Gamma}^P e_{\Gamma}h) = \text{div}_{\Gamma}(D_{\Gamma}e_{\Gamma}h) + \text{div}_{\Gamma}\left(\nabla b\{D^2be_{\Gamma}, h\}\right) \]
\[ = \text{div}_{\Gamma}(D_{\Gamma}e_{\Gamma}h) + \text{tr}(D^2b)|D^2be_{\Gamma}, h| + \{\nabla b, \nabla\{D^2be_{\Gamma}, h\}\} \]
\[ = \text{div}_{\Gamma}(D_{\Gamma}e_{\Gamma}h) + H|D^2be_{\Gamma}, h|. \tag{49} \]

The last term of Eq. (43a) thus is given by
\[ \int_{\Gamma} \text{div}_{\Gamma} e_{\Gamma} \text{div}_{\Gamma}(D_{\Gamma}^P e_{\Gamma}h) d\Gamma = \int_{\Gamma} \text{div}_{\Gamma} e_{\Gamma} \text{div}_{\Gamma}(D_{\Gamma}e_{\Gamma}h) d\Gamma \]
\[ + \int_{\Gamma} H \text{div}_{\Gamma} e_{\Gamma}\{D^2be_{\Gamma}, h\} d\Gamma \]
\[ = \int_{\Gamma} \text{div}_{\Gamma} e_{\Gamma} \text{div}_{\Gamma}(D_{\Gamma}e_{\Gamma}h) d\Gamma + \text{lot}(e_{\Gamma}). \tag{50} \]

We focus now on showing that how the integrand on the right-hand side of Eq. (50) is related to the tangential divergence of a desired energy term. Firstly we have that
\[ \text{div}_{\Gamma} e_{\Gamma} \text{div}_{\Gamma}(D_{\Gamma}e_{\Gamma}h) = \text{div}_{\Gamma} e_{\Gamma} \text{tr}(D_{\Gamma}e_{\Gamma}D_{\Gamma}h) + \text{div}_{\Gamma} e_{\Gamma} \text{tr}(hD_{\Gamma}^2e_{\Gamma}) \tag{51} \]
using (47). Next, we consider the expression
\[ \frac{1}{2} \text{div}_{\Gamma} [h(\text{div}_{\Gamma} e_{\Gamma})^2] = \frac{1}{2}(\text{div}_{\Gamma} e_{\Gamma})^2 \text{div}_{\Gamma} h + \frac{1}{2}\{h, \nabla\{\text{div}_{\Gamma} e_{\Gamma}\}\} \]
\[ = \frac{1}{2} \text{div}_{\Gamma} h(\text{div}_{\Gamma} e_{\Gamma})^2 + \text{div}_{\Gamma} e_{\Gamma} \text{tr}(h^*D_{\Gamma}^2e_{\Gamma}) \tag{52} \]
using (30). If the last term of (51) and (52) are the same, we could easily combine two equations. They are clearly not equal, but we can use (26), from which it follows that
\[ *D_{\Gamma}^2e_{\Gamma} = D_{\Gamma}^2e_{\Gamma} - (\nabla b \otimes D_{\Gamma}e_{\Gamma}D^2b)^\circ + *(\nabla b \otimes D_{\Gamma}e_{\Gamma}D^2b) \tag{53} \]
so that
\[ \text{tr}(h^*D_{\Gamma}^2e_{\Gamma}) - \text{tr}(hD_{\Gamma}^2e_{\Gamma}) = -\text{tr}(h(\nabla b \otimes D_{\Gamma}e_{\Gamma}D^2b)^\circ) \]
\[ + \text{tr}(h^*(\nabla b \otimes D_{\Gamma}e_{\Gamma}D^2b)) \tag{54} \]
and we have eliminated one derivative on \( e_{\Gamma}. \) Next we have immediately that the first term of (54)
\[ \text{tr}(h(\nabla b \otimes D_{\Gamma}e_{\Gamma}D^2b)^\circ) = \text{tr}(\nabla b \otimes (hD_{\Gamma}e_{\Gamma}D^2b)) = hD_{\Gamma}e_{\Gamma}D^2b\nabla b = 0 \tag{55} \]
since $D^2 b \nabla b = 0$. In addition, we have that for matrix $M$ and vectors $u, v$ that $\text{tr}(u \otimes M v) = \langle u M, v \rangle$ so that the second term of (54) becomes

$$
\text{tr}(^* h (\nabla b \otimes D\Gamma e\Gamma D^2 b)) = \text{tr}(\nabla b \otimes D\Gamma e\Gamma D^2 b, h) = \langle \nabla b D\Gamma e\Gamma D^2 b, h \rangle = -\langle ^*(D^2 b e\Gamma), D^2 b h \rangle
$$

(56)

upon application of formula (25). Thus we have eliminated another derivative, and we can combine Eqs. (54)–(56) to give that

$$
\text{div}\ e\Gamma \text{tr}(^* h ^* (\nabla b \otimes D\Gamma e\Gamma D^2 b)) = \text{div}\ e\Gamma \text{tr}(^* h (D^2 e\Gamma \cdot D^2 b h))
$$

(57)

Thus, we can combine (51) and (52) by means of (54) to give

$$
\text{div}\ e\Gamma \text{div} (D\Gamma e\Gamma h) = \text{div}\ e\Gamma \text{tr}(D\Gamma e\Gamma D\Gamma h) + \frac{1}{2} \text{div} (h(\text{div} e\Gamma)^2)
$$

$$
- \frac{1}{2} \text{div} h(\text{div} e\Gamma)^2 + \text{lot}(e\Gamma).
$$

(58)

Substituting back into Eq. (50) and applying the divergence theorem, as well as the fact that $\langle h, \nabla b \rangle = 0$ gives that

$$
\int_{\Gamma} \text{div}\ e\Gamma \text{div} (D\Gamma e\Gamma h) = \frac{1}{2} \int_{\Gamma} |\text{div} e\Gamma|^2 \langle h, v \rangle \, d\Gamma + \int_{\Gamma} \text{div}\ e\Gamma \text{tr}(D\Gamma e\Gamma D\Gamma h) \, d\Gamma
$$

$$
- \frac{1}{2} \int_{\Gamma} \text{div} h(\text{div} e\Gamma)^2 \, d\Gamma + \text{lot}(e\Gamma).
$$

(59)

Application of the property of $D\Gamma h$ given in Lemma 7 (Eq. (33)) gives

$$
\int_{\Gamma} \text{div}\ e\Gamma \text{div} (D\Gamma e\Gamma h) = \frac{1}{2} \int_{\Gamma} |\text{div} e\Gamma|^2 \langle h, v \rangle \, d\Gamma + \int_{\Gamma} \alpha(x)|\text{div} e\Gamma|^2 \, d\Gamma
$$

$$
- \frac{1}{2} \int_{\Gamma} \alpha(x) \text{div} h(\text{div} e\Gamma)^2 \, d\Gamma + \text{lot}(e\Gamma)
$$

(60)

and noting that $\text{div} h = 2\alpha(x)$ (as in (34)) allows us to cancel the terms containing $\alpha$.

Next, we wish to derive an analogous expression involving the strain tensor. We again consider

$$
\text{tr}(\varepsilon\Gamma(\varepsilon\Gamma) (D\Gamma e\Gamma h)) = \text{tr}(\varepsilon\Gamma(\varepsilon\Gamma) (D\Gamma e\Gamma h)) + \text{tr}(\varepsilon\Gamma(\varepsilon\Gamma) (\nabla b[D^2 e\Gamma, h])).
$$

(61)

Letting $\langle D^2 b e\Gamma, h \rangle = f$ and expanding gives that

$$
\varepsilon\Gamma(f \nabla b) = \frac{1}{2} (\nabla f \nabla b + ^* \nabla f \nabla b) + 2(D\Gamma(f \nabla b)) = D^2 b f + \frac{1}{2} (\nabla b \otimes \nabla f + \nabla f \otimes \nabla b)
$$

(62)

and

$$
\frac{1}{2} \text{tr}(\varepsilon\Gamma(\varepsilon\Gamma)(\nabla b \otimes \nabla f + \nabla f \otimes \nabla b))
$$

$$
= \frac{1}{4} \text{tr}(D\Gamma e\Gamma \nabla b \otimes \nabla f + ^* D\Gamma e\Gamma \nabla f \otimes \nabla b + D\Gamma e\Gamma \nabla f \otimes \nabla b + ^* D\Gamma e\Gamma \nabla f \otimes \nabla b)
$$

$$
+ ^* D\Gamma e\Gamma \nabla f \otimes \nabla b)
$$
\[
\begin{align*}
&= \frac{1}{4} \left[ \langle \nabla b D \Gamma e \Gamma, \nabla \Gamma f \rangle + \langle \ast D \Gamma e \Gamma, \nabla \Gamma f \rangle + \langle D \Gamma e \Gamma \nabla b, \nabla \Gamma f \rangle \right] \\
&= -\frac{1}{2} \langle D^2 b e \Gamma, \nabla \Gamma \{ D^2 b e \Gamma, h \} \rangle,
\end{align*}
\]
where we have used the fact that \( D \Gamma e \Gamma \nabla b = 0 \) (Eq. (24)) as well as Eq. (25) in the last step. This then yields that
\[
\int_{\Gamma} \text{tr}(\epsilon \Gamma (e \Gamma) \epsilon \Gamma (D \Gamma e \Gamma h)) \, d\Gamma = \int_{\Gamma} \text{tr}(\epsilon \Gamma (e \Gamma) \epsilon \Gamma (D \Gamma e \Gamma h)) \, d\Gamma + \text{lot}(e \Gamma).
\]
Similarly to the previous case we have, upon applying (48), that
\[
\text{tr}(\epsilon \Gamma (e \Gamma) \epsilon \Gamma (D \Gamma e \Gamma h)) = \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) (D \Gamma e \Gamma h + \ast D \Gamma h \ast D \Gamma e \Gamma)] + \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) (\ast h D^2 \Gamma e \Gamma + \ast (D^2 \Gamma e \Gamma) h)].
\]
Next, we can see that
\[
\frac{1}{2} \text{div} \Gamma (h \text{tr}(\epsilon \Gamma (e \Gamma)^2)) = \frac{1}{2} \text{div} \Gamma h \text{tr}(\epsilon \Gamma (e \Gamma)^2) + \frac{1}{2} [h, \nabla \Gamma \text{tr}(\epsilon \Gamma (e \Gamma)^2)]
\]
\[
= \frac{1}{2} \text{div} \Gamma h \text{tr}(\epsilon \Gamma (e \Gamma)^2)
\]
\[
+ \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) (\ast h (D^2 \Gamma e \Gamma) + D^2 \Gamma e \Gamma h)]
\]
by simply expanding the tangential gradient of \( \text{tr}(\epsilon \Gamma (e \Gamma)^2) \). Again, we seek to define the relationship between the last two terms of (65) and (66). Application of (26) gives that
\[
D^2 \Gamma e \Gamma = \ast D^2 \Gamma e \Gamma - \ast (\nabla b \otimes D \Gamma e \Gamma D^2 b) + (\nabla b \otimes D \Gamma e \Gamma D^2 b)^\ast
\]
so that
\[
\frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) \ast h D^2 \Gamma e \Gamma] = \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) \ast h D^2 \Gamma e \Gamma] - \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) \ast h (\nabla b \otimes D \Gamma e \Gamma D^2 b)]
\]
\[
+ \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) \ast h (\nabla b \otimes D \Gamma e \Gamma D^2 b)^\ast]
\]
(68a)
and
\[
\frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma)^\ast (D^2 \Gamma e \Gamma) h] = \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) D^2 \Gamma e \Gamma h] - \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma) (\nabla b \otimes D \Gamma e \Gamma D^2 b) h]
\]
\[
+ \frac{1}{2} \text{tr}[\epsilon \Gamma (e \Gamma)^\ast ((\nabla b \otimes D \Gamma e \Gamma D^2 b)^\ast) h].
\]
(68b)
Let us consider separately the last 4 terms of Eq. (68). We will use the definitions (28) and (29) as well as the symmetry of \( \epsilon \Gamma (e \Gamma) \) and \( D^2 b \).
\[
\text{tr}[\epsilon \Gamma (e \Gamma) \ast h (\nabla b \otimes D \Gamma e \Gamma D^2 b)]
\]
\[
= \frac{1}{2} \text{tr}[ (\nabla b \otimes D \Gamma e \Gamma D^2 b) h (D \Gamma e \Gamma + \ast D \Gamma e \Gamma)]
\]
\[
= \frac{1}{2} \langle D \Gamma e \Gamma \nabla b, D \Gamma e \Gamma D^2 bh \rangle + \frac{1}{2} \langle \ast D \Gamma e \Gamma \nabla b, D \Gamma e \Gamma D^2 bh \rangle
\]
\[
= -\frac{1}{2} \langle D^2 b e \Gamma, D \Gamma e \Gamma D^2 bh \rangle = \text{lot}(e \Gamma).
\]
(69)
In the last step we have applied formulas (25) and (24). Similarly,
\[
\text{tr}\left[\varepsilon_\Gamma (e_\Gamma )^* h (\nabla b \otimes \Delta e_\Gamma D^2 b)^2\right]
= \frac{1}{2} \text{tr}\left[\left(D e_\Gamma + * D e_\Gamma \right)\left(* h D e_\Gamma D^2 b \otimes \nabla b\right)\right]
= \frac{1}{2} \left\{* h D e_\Gamma D^2 b, D e_\Gamma \nabla b\right\} + \frac{1}{2} \left\{* h D e_\Gamma D^2 b, * D e_\Gamma \nabla b\right\}
= -\frac{1}{2} \left\{D^2 b e_\Gamma, * h D e_\Gamma D^2 b\right\}
= \text{lot}(e_\Gamma), \quad (70)
\]

\[
\text{tr}\left[\varepsilon_\Gamma (e_\Gamma )\left(\nabla b \otimes D e_\Gamma D^2 b h\right)\right]
= \frac{1}{2} \text{tr}\left[\left(D e_\Gamma + * D e_\Gamma \right)\left(\nabla b \otimes D e_\Gamma D^2 b h\right)\right]
= \frac{1}{2} \left\{D e_\Gamma \nabla b, D e_\Gamma D^2 b h\right\} + \frac{1}{2} \left\{* D e_\Gamma \nabla b, D e_\Gamma D^2 b h\right\}
= -\frac{1}{2} \left\{D^2 b e_\Gamma, D e_\Gamma D^2 b h\right\}
= \text{lot}(e_\Gamma) \quad (71)
\]

and finally
\[
\text{tr}\left[\varepsilon_\Gamma (e_\Gamma )^* \left((\nabla b \otimes D e_\Gamma D^2 b)^2\right) h\right]
= \frac{1}{2} \text{tr}\left[\left(D e_\Gamma + * D e_\Gamma \right)\left(\nabla b \otimes D^2 b D e_\Gamma h\right)\right]
= \frac{1}{2} \left\{D e_\Gamma \nabla b, D^2 b D e_\Gamma h\right\} + \frac{1}{2} \left\{* D e_\Gamma \nabla b, D^2 b D e_\Gamma h\right\}
= -\frac{1}{2} \left\{D^2 b e_\Gamma, D^2 b D e_\Gamma h\right\}
= \text{lot}(e_\Gamma). \quad (72)
\]

Combining Eqs. (65) and (66) with the aid of the relationships (69)–(72) gives
\[
\text{tr}(\varepsilon_\Gamma (e_\Gamma) \varepsilon_\Gamma (D e_\Gamma h))
= \frac{1}{2} \text{tr}\left[\varepsilon_\Gamma (e_\Gamma) \left(D e_\Gamma D h + * D h * D e_\Gamma\right)\right] + \frac{1}{2} \text{tr}\left[\varepsilon_\Gamma (e_\Gamma)\left(* h D^2 e_\Gamma + D^2 e_\Gamma h\right)\right]
= \frac{1}{2} \text{tr}\left[\varepsilon_\Gamma (e_\Gamma) \left(D e_\Gamma D h + * D h * D e_\Gamma\right)\right] + \frac{1}{2} \text{div}_\Gamma (h \text{tr}(\varepsilon_\Gamma (e_\Gamma)^2)) - \frac{1}{2} \text{div}_\Gamma h \text{tr}(\varepsilon_\Gamma (e_\Gamma)^2) + \text{lot}(e_\Gamma). \quad (73)
\]

And thus, as before, the extra terms are transformed to lower-order terms by the application of Eq. (25). Substituting (73) into (64) and applying the divergence theorem and the geometric lemma (Eqs. (33) and (34)) as before gives
\[
\int_\Gamma \text{tr}(\varepsilon_\Gamma (e_\Gamma) \varepsilon_\Gamma (D^2 e_\Gamma h)) d\Gamma = \frac{1}{2} \int_\gamma \text{tr}(\varepsilon_\Gamma (e_\Gamma)^2) (h, v) d\gamma + \text{lot}(e_\Gamma). \quad (74)
\]

Combining (60) with (43a) and (74) with (43b) gives the conclusion, Eqs. (45). □
The next step in the proof of Lemma 13 is to explicitly calculate the boundary integrals appearing in (45a) and (45b) to show that they are, in fact, of the desired form.

**Proposition 16.** The following equalities hold for \( e_\Gamma \) satisfying the system of equations (20):

\[
\int_\Gamma \text{div}_\Gamma e_\Gamma \langle D_\Gamma^P e_\Gamma h, v \rangle \, d\gamma = \int_\Gamma \langle \text{div}_\Gamma e_\Gamma \rangle^2 \langle h, v \rangle \, d\gamma = \int_\Gamma \langle D_\Gamma^P e_\Gamma v, v \rangle \langle h, v \rangle, \tag{75a}
\]

\[
\int_\Gamma \langle \varepsilon_\Gamma (e_\Gamma) v, D_\Gamma^P e_\Gamma h \rangle \, d\gamma - \frac{1}{2} \int_\Gamma \text{tr}(\varepsilon_\Gamma (e_\Gamma)^2) \langle h, v \rangle \, d\gamma
\]

\[
= \frac{1}{2} \int_\Gamma \left( \langle D_\Gamma^P e_\Gamma v, \tau \rangle^2 + \langle D_\Gamma^P (e_\Gamma) v, v \rangle^2 \right) \langle h, v \rangle \, d\gamma. \tag{75b}
\]

**Proof.** We use the fact that \((v, \tau, \nabla b)\) forms a basis for \(\mathbb{R}^3\), and thus the collection \(M_i\) is a basis for \(M_3(\mathbb{R})\), where

\[
M_1 = v \otimes v, \quad M_2 = v \otimes \tau, \quad M_3 = \tau \otimes v,
\]

\[
M_4 = \tau \otimes \tau, \quad M_5 = \tau \otimes \nabla b, \quad M_6 = v \otimes \nabla b,
\]

\[
M_7 = \nabla b \otimes \tau, \quad M_8 = \nabla b \otimes v, \quad M_9 = \nabla b \otimes \nabla b \tag{76}
\]

so that matrix \(A = (A .. M_i) M_i\). Orthogonality gives

\[
\text{tr}(A) = A .. I = A .. (\nabla b \otimes \nabla b) + A .. (v \otimes v) + A .. (\tau \otimes \tau). \tag{77}
\]

In addition, we will use (without pointing out each time) Proposition 8, which allows us to conclude that with the Dirichlet boundary condition (20d) imposed on \( e_\Gamma \), the expression \( D_\Gamma^P e_\Gamma \tau = D_\Gamma e_\Gamma \tau = 0 \) on \( \gamma \).

We consider the left-hand side of Eq. (75a). Noting that \( \langle D_\Gamma^P e_\Gamma h, v \rangle = \langle D_\Gamma^P (e_\Gamma) v, h \rangle \), we expand \( D_\Gamma^P (e_\Gamma) v \) on the basis defined above and use that by definition \( D_\Gamma^P e_\Gamma \nabla b = \nabla b \cdot D_\Gamma^P e_\Gamma = 0 \) as well as orthogonality. Thus

\[
D_\Gamma^P (e_\Gamma) v = D_\Gamma^P (e_\Gamma) (v \otimes v) + D_\Gamma^P (e_\Gamma) (\tau \otimes v) \tau = \langle v, D_\Gamma^P e_\Gamma v \rangle v + \langle v, D_\Gamma^P e_\Gamma \tau \rangle \tau
\]

and, after applying Proposition 8 and Eq. (77) we have

\[
\langle D_\Gamma e_\Gamma h, v \rangle = \langle v, D_\Gamma e_\Gamma v \rangle \langle h, v \rangle = \langle v, D_\Gamma^P e_\Gamma \rangle \langle h, v \rangle = \text{div}_\Gamma e_\Gamma \langle h, v \rangle \tag{78}
\]

which gives (75a). We proceed similarly in the next case

\[
\varepsilon_\Gamma^P (e_\Gamma) v = \langle v, \varepsilon_\Gamma^P (e_\Gamma) v \rangle v + \langle v, \varepsilon_\Gamma^P (e_\Gamma) \tau \rangle \tau = \langle v, \varepsilon_\Gamma^P (e_\Gamma) v \rangle v + \frac{1}{2} \langle v, D_\Gamma^P e_\Gamma \tau \rangle \tau, \tag{79}
\]

using Proposition 8. So
\begin{align*}
\langle \varepsilon^p_F(e_F)v, D^p_F e_F h \rangle &= \langle v, \varepsilon^p_F(e_F)v \rangle \langle D^p_F e_F h, v \rangle + \frac{1}{2} \langle v, \ast D^p_F e_F \tau \rangle \langle \tau, D^p_F e_F h \rangle. \tag{80}
\end{align*}

The first term of Eq. (80) is
\begin{align*}
\langle v, \varepsilon^p_F(e_F)v \rangle \langle D^p_F e_F h, v \rangle &= \frac{1}{2} \left( \langle v, D^p_F e_F v \rangle + \langle v, \ast D^p_F e_F v \rangle \right) \langle v, D^p_F e_F v \rangle \langle h, v \rangle \\
&= \langle (v, D^p_F e_F v) \rangle^2 \langle h, v \rangle \tag{81}
\end{align*}
and, writing \( h = \langle h, v \rangle v + \langle h, \tau \rangle \tau \) gives the second term as
\begin{align*}
\frac{1}{2} \langle v, \ast D^p_F e_F \tau \rangle \langle \tau, D^p_F e_F h \rangle &= \frac{1}{2} \langle v, \ast D^p_F e_F \tau \rangle \langle \ast D^p_F e_F \tau, \langle h, v \rangle v + \langle h, \tau \rangle \tau \rangle \\
&= \frac{1}{2} \langle \langle \tau, D^p_F e_F v \rangle \rangle^2 \langle h, v \rangle. \tag{82}
\end{align*}

Finally, we note that
\begin{align*}
\text{tr} \left( \varepsilon^p_F(e_F) \right)^2 &= \varepsilon^p_F(e_F) \ast \varepsilon^p_F(e_F) = \left( \varepsilon^p_F(e_F) \ast M_i \right) \left( \varepsilon^p_F(e_F) \ast M_j \right) (M_i \ast M_j) \\
&= \left( \varepsilon^p_F(e_F) \ast (v \otimes v) \right)^2 + \left( \varepsilon^p_F(e_F) \ast (\tau \otimes \tau) \right)^2 \\
&= \langle (v, D^p_F e_F v) \rangle^2 \tag{83}
\end{align*}
since \( \langle \tau, \varepsilon^p_F(e_F) \rangle = 0 \) by application of Proposition 8. Combining equations (80)–(83) results in the conclusion Eq. (75b). \( \square \)

Combining the results of Propositions 14 and 16 results in the equality presented in Lemma 13. We note that we have used the equality \( \text{div}_F e_F = \langle v, D^p_F e_F v \rangle \) proved in Eq. (78). \( \square \)

We proceed with the other terms in Eq. (41).
\begin{align*}
\int_{Q} \langle (D^2 b)^2 e_F, D^p_F e_F h \rangle dQ &= \text{lot}(e_F) \tag{84}
\end{align*}
and the two terms that arise from \( F_2(e_F, w) \) stay as they are for the purposes of Lemma 12, Eq. (38).

Finally, the time-derivative term gives:
\begin{align*}
\int_{Q} \langle \partial_t e_F, D^p_F e_F h \rangle dQ \\
&= \int_{\Gamma} \langle \partial_t e_F, D^p_F e_F h \rangle d\Gamma \bigg|_0^T - \int_{Q} \langle \partial_t e_F, D_F \partial_t e_F h \rangle + \langle \partial_t e_F, \nabla b \rangle \langle D^2 b \partial_t e_F, h \rangle \rangle dQ \tag{85}
\end{align*}
\[
\begin{align*}
&= \int_{\Gamma} \langle \partial_t e_\Gamma, D_P e_\Gamma h \rangle \mid_{t=0}^{T} d\Gamma - \frac{1}{2} \int_0^T \int_{\Gamma} \langle \partial_t e_\Gamma, h \rangle \langle \partial_t e_\Gamma, v \rangle \mid_{t=0}^{T} d\Gamma dt + \frac{1}{2} \int_{Q} |\partial_t e_\Gamma|^2 \text{div}_\Gamma h dQ \\
&= \int_{\Gamma} \langle \partial_t e_\Gamma, D_P e_\Gamma h \rangle \mid_{t=0}^{T} d\Gamma + \frac{1}{2} \int_{Q} |\partial_t e_\Gamma|^2 \text{div}_\Gamma h dQ
\end{align*}
\]

(86)

after an application of boundary conditions. Note that the last term of (85) is zero because \(\partial_t e_\Gamma\) is in the tangent plane.

Combining Lemma 13 (Eq. (42)) and equations (84) and (86) results, in the conclusion Eq. (38).

For the second equation of Lemma 12, we multiply Eq. (20b) by \(m_2(x)e_\Gamma\):

\[
\int_{Q} \langle \rho \partial_{tt} e_\Gamma - \text{div}_\Gamma C(e_\Gamma \langle e_\Gamma \rangle) - \mu(D^2b)^2 e_\Gamma + P_2(w, e_\Gamma), m_2 e_\Gamma \rangle dQ = 0.
\]

(87)

Applying the Green’s formulas (23) and (27), the formula (25), and using the boundary condition (20d):

\[
\int_{Q} \langle \partial_{tt} e_\Gamma, m_2 e_\Gamma \rangle dQ = \int_{\Gamma} \langle \partial_t e_\Gamma, m_1 e_\Gamma \rangle \mid_{t=0}^{T} - \int_{Q} m_2 |\partial_t e_\Gamma|^2 dQ,
\]

(88)

\[
\int_{Q} \langle \text{div}_\Gamma \text{div}_\Gamma e_\Gamma, m_2 e_\Gamma \rangle dQ = -\int_{Q} m_2 |\text{div}_\Gamma e_\Gamma|^2 dQ - \int_{Q} \text{div}_\Gamma e_\Gamma \langle e_\Gamma, \text{div}_\Gamma m_2 \rangle dQ,
\]

(89)

\[
\int_{Q} \langle \text{div}_\Gamma \varepsilon_\Gamma (e_\Gamma), m_2 e_\Gamma \rangle dQ = -\int_{Q} H m_2 \langle e_\Gamma, D^2b e_\Gamma \rangle dQ - \int_{Q} (m_2 \text{tr}(D_{\Gamma} e_\Gamma \varepsilon_\Gamma (e_\Gamma)))
\]

\[
+ \text{tr}(e_\Gamma \otimes \nabla_{\Gamma} m_2 \varepsilon_\Gamma (e_\Gamma)) dQ.
\]

(90)

Combining equations (88)–(90) with the other terms, and using the relation \((D^2b)^2 - 2H D^2b + K = 0\) gives Eq. (39). \(\square\)

4.4. Multipliers on the \(w\) equation—second step of proof

**Lemma 17.** Let \(Q = \Gamma \times [0, T]\). With respect to the equation for the normal component \(w\) (Eq. (20a)), the following equalities hold:

\[
\begin{align*}
\rho \int_{\Gamma} \langle \partial_t w(h, \nabla_\Gamma w) + \gamma(\nabla_\Gamma \partial_t w, \nabla_\Gamma w) + \gamma(\nabla_\Gamma \partial_t w, * h D^2 \partial w) \rangle d\Gamma \mid_{t=0}^{T} \\
- \frac{1}{2} \int_0^T \int_{\Gamma} \beta |\Delta_\Gamma w|^2 \langle h, v \rangle d\Gamma dt + \int_0^T \int_{\Gamma} \beta \frac{\partial}{\partial v}(\Delta_\Gamma w) \langle h, \nabla_\Gamma w \rangle d\Gamma dt \\
+ \int_{Q} \langle \lambda H \text{div}_\Gamma e_\Gamma \langle h, \nabla_\Gamma w \rangle + 2\mu \text{tr}(\varepsilon_\Gamma (e_\Gamma D^2b) \langle h, \nabla_\Gamma w \rangle \rangle dQ
\end{align*}
\]
\[
+ \beta \left[ \int_Q \alpha |\Delta_{\Gamma} w|^2 \, dQ + \int_Q \Delta_{\Gamma} w \tr(\nabla_{\Gamma} w D_{\Gamma}^2 h) \, dQ \right] + \frac{1}{2} \int_Q \rho |\partial_t w|^2 \div_{\Gamma} h \, dQ \\
+ \text{lot}(w, e_{\Gamma}) = 0,
\]
(91a)

\[
\rho \int_{\Gamma} (\partial_t w w + \gamma \langle \nabla_{\Gamma} w, \nabla_{\Gamma} w \rangle) m_1 \, d\Gamma \bigg|_0^T \\
+ \int_Q (\beta |\Delta_{\Gamma} w|^2 - \rho |\partial_t w|^2 - \rho \gamma |\nabla_{\Gamma} \partial_t w|^2) m_1 \, dQ - \rho \gamma \int_{\Gamma} \partial_t w \langle \nabla_{\Gamma} \partial_t w, \nabla_{\Gamma} m_1 \rangle \, d\Gamma \\
+ \beta \int_Q (w \Delta_{\Gamma} w \Delta_{\Gamma} m_1 + 2\Delta_{\Gamma} w \langle \nabla_{\Gamma} w, \nabla_{\Gamma} m_1 \rangle) \, dQ \\
+ \int_Q (\lambda H \div_{\Gamma} e_{\Gamma} w + 2\mu \tr(\epsilon_{\Gamma}(e_{\Gamma}) D_{\Gamma}^2 b) w + k w^2) m_1 \, dQ = 0,
\]
(91b)

where \( h \) is the vector field given by Hypothesis 2, \( m_1 = m_1(x) \) is any function, and the lower-order terms \( \text{lot}(w, e_{\Gamma}) \) are defined in (40).

**Proof.** The proof of Eq. (91a) follows from the application of the multiplier \( \langle h, \nabla_{\Gamma} w \rangle \) and that of (91b) from the multiplier \( m_1 w \). Multiplying equation (20a) by \( \langle h, \nabla_{\Gamma} w \rangle \) and integrating over time and space gives
\[
\int_Q (\rho \partial_{tt} w - \rho \gamma \Delta_{\Gamma} \partial_t w + \beta \Delta_{\Gamma}^2 w + P_1(w, e_{\Gamma})) \langle h, \nabla_{\Gamma} w \rangle \, dQ = 0.
\]
(92)

Again, we consider each term separately. The first time-derivative term gives
\[
\int_Q \partial_{tt} w \langle h, \nabla_{\Gamma} w \rangle \, dQ = \int_{\Gamma} \partial_t w \langle h, \nabla_{\Gamma} w \rangle \, d\Gamma \bigg|_0^T - \int_Q \partial_t w \langle h, \nabla_{\Gamma} \partial_t w \rangle \, dQ \\
= \int_{\Gamma} \partial_t w \langle h, \nabla_{\Gamma} w \rangle \, d\Gamma \bigg|_0^T + \frac{1}{2} \int_Q |\partial_t w|^2 \div_{\Gamma} h \, dQ
\]
(93)
after application of the formula (31), boundary conditions (20c), and properties of \( h \). The next time-derivative term involves the Laplacian:
\[
- \int_Q \Delta_{\Gamma} \partial_t w \langle h, \nabla_{\Gamma} w \rangle \, dQ \\
= - \int_0^T \int_{\Gamma} \langle \nabla_{\Gamma} \partial_t w, v \rangle \langle h, \nabla_{\Gamma} w \rangle \, d\gamma \, dt + \int_Q 2H \langle h, \nabla_{\Gamma} w \rangle \nabla_{\Gamma} \partial_t w \nabla b \, dQ \\
+ \int_Q \langle \nabla_{\Gamma} \partial_t w, \nabla_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle \rangle \, dQ \\
= \int_{\Gamma} \langle \nabla_{\Gamma} \partial_t w, \nabla_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle \rangle \, d\Gamma \bigg|_0^T - \int_Q \langle \nabla_{\Gamma} \partial_t w, \nabla_{\Gamma} \langle h, \nabla_{\Gamma} \partial_t w \rangle \rangle \, dQ
\]
(94)
where we have used the boundary conditions to eliminate the term on \( \mathcal{Y} \) and (24) gives

\[
\langle \nabla \partial_t w, \nabla w \rangle = 0.
\]

Expanding gives

\[
-\int_Q \frac{\Delta \partial_t w}{\Delta} \langle h, \nabla \partial_t w \rangle \, dQ = \int_\Gamma \left( \left. \langle \nabla \partial_t w, * D \nabla w \rangle \right|_0^T \right.
\]

\[
-\int_Q |\nabla \partial_t w|^2 \, \text{div} \, h \, dQ - \int_Q \langle \nabla \partial_t w, * D^2 (\partial_t w) h \rangle \, dQ.
\]

(95)

We proceed in the same way as in the calculation of the divergence terms in the \( e_\Gamma \) equation:

\[
\langle \nabla \partial_t w, * D^2 \partial_t w h \rangle = \langle D^2 \partial_t w \nabla \partial_t w, h \rangle
\]

and we note that

\[
\frac{1}{2} \text{div} \, (|\nabla \partial_t w|^2 h) = \frac{1}{2} \text{div} \, h |\nabla \partial_t w|^2 + \langle * D^2 \partial_t w \nabla \partial_t w, h \rangle.
\]

(96)

So, we wish to combine equations (95) and (96). Again, we use the relation

\[
* D^2 w = D^2 w - D^2 b \nabla \Gamma w \otimes \nabla b + * (D^2 b \nabla \Gamma w \otimes \nabla b)
\]

(97)

but note that

\[
\langle * D^2 \partial_t w \nabla \partial_t w, h \rangle = \langle D^2 \partial_t w \nabla \partial_t w, h \rangle - \langle D^2 b \nabla \partial_t w \nabla b, \nabla \partial_t w \rangle \langle h, \nabla b \rangle.
\]

(98)

The third term of (98) is zero since \( \langle \nabla b, \nabla \partial_t w \rangle = 0 \) and the fourth terms is zero since \( \langle h, \nabla b \rangle = 0 \). Thus, we can immediately substitute (96) into (95) to give

\[
-\int Q \frac{\Delta \partial_t w}{\Delta} \langle h, \nabla \partial_t w \rangle \, dQ = \int_\Gamma \left( \left. \langle \nabla \partial_t w, \nabla \partial_t w \rangle + \langle \nabla \partial_t w, * D \nabla \partial_t w \rangle \right|_0^T \right.
\]

\[
-\frac{1}{2} \int_Q |\nabla \partial_t w|^2 \, \text{div} \, h \, dQ - \int_Q \langle \nabla \partial_t w, * D^2 (\partial_t w) h \rangle \, dQ.
\]

(99)

after cancellations. Applying the divergence theorem gives

\[
\int_Q \text{div} \, (|\nabla \partial_t w|^2 h) \, dQ = \int_Q \int_0^T |\nabla \partial_t w|^2 \langle h, v \rangle \, dT + \int_Q |\nabla \partial_t w|^2 \langle h, \nabla b \rangle = 0
\]

(100)

as the shell is clamped on the boundary (via Proposition (8)) and as usual \( \langle h, \nabla b \rangle = 0 \). Thus, combining equations (94)–(100) gives

\[
-\int Q \frac{\Delta \partial_t w}{\Delta} \langle h, \nabla \partial_t w \rangle \, dQ = \int_\Gamma \left( \left. \langle \nabla \partial_t w, \nabla \partial_t w \rangle + \langle \nabla \partial_t w, * D \nabla \partial_t w \rangle \right|_0^T \right.
\]

(101)

Next, we proceed to the third term of Eq. (92).
Proposition 18. The following equality holds for $w$ satisfying the system of equations (20) and $h$ satisfying Hypothesis 2:

$$\int_Q \Delta^2_{\Gamma} w \langle h, \nabla_{\Gamma} w \rangle \, dQ = \int_0^T \int_{\gamma} \frac{\partial}{\partial v} (\Delta_{\Gamma} w) \langle h, \nabla_{\Gamma} w \rangle \, d\gamma \, dt$$

$$+ \int_Q \alpha |\Delta_{\Gamma} w|^2 \, dQ + \int_Q \Delta_{\Gamma} w \text{tr}(\nabla_{\Gamma} w D_{\Gamma}^2 h) \, dQ$$

$$- \frac{1}{2} \int_0^T \int_{\gamma} |\Delta_{\Gamma} w|^2 \langle h, v \rangle \, d\gamma \, dt + \text{lot}(w) = 0. \quad (102)$$

Proof. As before, we suppress the time-integral for convenience. Integration by parts gives immediately that

$$\int_{\Gamma} \Delta^2_{\Gamma} w \langle h, \nabla_{\Gamma} w \rangle \, d\Gamma = \int_{\gamma} \frac{\partial}{\partial v} (\Delta_{\Gamma} w) \langle h, \nabla_{\Gamma} w \rangle \, d\gamma - \int_{\Gamma} \Delta_{\Gamma} w \langle \nabla_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle, v \rangle \, d\gamma$$

$$+ \int_{\Gamma} \Delta_{\Gamma} w \Delta_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle \, d\Gamma. \quad (103)$$

We deal first with the last term

$$\Delta_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle = \text{div}_{\Gamma} (\nabla_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle) = \text{div}_{\Gamma} (\nabla_{\Gamma} w) = 2 \text{tr}(\nabla_{\Gamma} w D_{\Gamma}^2 h) + \text{tr}(\nabla_{\Gamma} w D_{\Gamma}^2 h) + \text{tr}(\nabla_{\Gamma} w D_{\Gamma}^2 h) \quad (104)$$

so that

$$\int_{\Gamma} \Delta_{\Gamma} w \Delta_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle \, d\Gamma = 2 \int_{\Gamma} \Delta_{\Gamma} w \text{tr}(\nabla_{\Gamma} w D_{\Gamma}^2 h) \, d\Gamma + \int_{\Gamma} \Delta_{\Gamma} w \text{tr}(\nabla_{\Gamma} w D_{\Gamma}^2 h) \, d\Gamma$$

$$+ \int_{\Gamma} \Delta_{\Gamma} w \text{tr}(\nabla_{\Gamma} w D_{\Gamma}^2 h) \, d\Gamma. \quad (105)$$

We wish to treat the last term of Eq. (105) similarly as before, and turn it into something that can be nicely integrated by parts. Here we encounter again the issue of order of derivatives, and find that we can use the same method to tackle this problem. We note that expansion gives

$$\frac{1}{2} \text{div}_{\Gamma}(h |\Delta_{\Gamma} w|^2) = \frac{1}{2} \text{div}_{\Gamma} h |\Delta_{\Gamma} w|^2 + \Delta_{\Gamma} w \text{tr}(h^* D_{\Gamma}^2 (\nabla_{\Gamma} w)). \quad (106)$$

Thus, using Eqs. (26), (25) and $D^2 b \nabla b = 0$ gives again

$$\text{tr}(h^* D_{\Gamma}^2 (\nabla_{\Gamma} w)) = \text{tr}(h^* D_{\Gamma}^2 (\nabla_{\Gamma} w)) + \text{tr}(h (\nabla b \otimes D_{\Gamma} (\nabla_{\Gamma} w) D^2 b) \otimes)$$

$$- \text{tr}(h^* (\nabla b \otimes D_{\Gamma} (\nabla_{\Gamma} w) D^2 b))$$

$$= \text{tr}(h^* D_{\Gamma}^2 (\nabla_{\Gamma} w)) + \{D^2 b \nabla_{\Gamma} w, D^2 b h\}. \quad (107)$$

Note that this is identical to the steps of (55)–(56), since $\nabla_{\Gamma} w$ is always a purely tangential vector (property (24)). We can use Eq. (106) and integration by parts,
\[
\int_{\Gamma} \Delta_{\Gamma} w \, \text{tr} (\ast D_{\Gamma}^2 (\nabla_{\Gamma} w)) \, d\Gamma \\
= \frac{1}{2} \int_{\Gamma} \text{div}_{\Gamma} (h |\Delta_{\Gamma} w|^2) \, d\Gamma - \frac{1}{2} \int_{Q} \text{div} \, h |\Delta_{\Gamma} w|^2 \, d\Gamma + \int_{\Gamma} \Delta_{\Gamma} w \{ D^2 b \nabla_{\Gamma} w, D^2 b h \} \, d\Gamma \\
= \frac{1}{2} \int_{\gamma} |\Delta_{\Gamma} w|^2 \langle h, v \rangle - \frac{1}{2} \int_{\Gamma} |\Delta_{\Gamma} w|^2 \, \text{div}_{\Gamma} h \, d\Gamma + \text{lot}(w) \quad (108)
\]
and then combine Eq. (105) with (108) to give
\[
\int_{\Gamma} \Delta_{\Gamma} w \Delta_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle \, d\Gamma \\
= 2 \int_{\Gamma} \Delta_{\Gamma} w \, \text{tr} (\ast D_{\Gamma}^2 w D_{\Gamma} h) \, d\Gamma + \int_{\Gamma} \Delta_{\Gamma} w \, \text{tr} (\ast \nabla_{\Gamma} w D_{\Gamma}^2 h) \, d\Gamma \\
- \frac{1}{2} \int_{\Gamma} \text{div}_{\Gamma} h |\Delta_{\Gamma} w|^2 \, d\Gamma + \frac{1}{2} \int_{\gamma} |\Delta_{\Gamma} w|^2 \langle h, v \rangle \, d\gamma + \text{lot}(w). \quad (109)
\]
Noting that \( \Delta_{\Gamma} w = \text{tr} (\ast D_{\Gamma}^2 w) \) and applying the geometric lemma (Eqs. (33) and (34)) as before then gives that
\[
\int_{\Gamma} \Delta_{\Gamma} w \Delta_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle \, d\Gamma = \int_{\Gamma} |\Delta_{\Gamma} w|^2 \, d\Gamma + \int_{\Gamma} \Delta_{\Gamma} w \, \text{tr} (\ast \nabla_{\Gamma} w D_{\Gamma}^2 h) \, d\Gamma \\
+ \frac{1}{2} \int_{\gamma} |\Delta_{\Gamma} w|^2 \langle h, v \rangle \, d\gamma + \text{lot}(w). \quad (110)
\]
We need finally to deal with the second boundary term appearing in (103), but this is a straightforward application of Propositions 8 and 10, and previous calculations involving the boundary
\[
\langle \nabla_{\Gamma} \langle h, \nabla_{\Gamma} w \rangle, v \rangle = (\ast D_{\Gamma} h \, \nabla_{\Gamma} w + \ast D_{\Gamma}^2 w h, v) \\
= \langle \nabla_{\Gamma} w, v \rangle + (\ast D_{\Gamma}^2 w h, v) \\
= \langle h, D_{\Gamma}^2 w v \rangle \\
= D_{\Gamma}^2 w \langle v \otimes v \rangle \langle h, v \rangle + D_{\Gamma}^2 w \langle \tau \otimes v \rangle \langle h, \tau \rangle \\
= (\text{div}_{\Gamma} \nabla_{\Gamma} w) \langle h, v \rangle + \langle v D_{\Gamma} w, \tau \rangle \langle h, \tau \rangle \\
= \Delta_{\Gamma} w \langle h, v \rangle + \langle v, * D_{\Gamma}^2 w \tau \rangle \langle h, \tau \rangle. \quad (111)
\]
Now, by applying Proposition 8 one gets immediately that \( * D_{\Gamma}^2 w \tau = 0 \), but it is not obvious that this is true for \( * D_{\Gamma}^2 w \tau \). However, property (26) shows that
\[
* D_{\Gamma}^2 w \tau = D_{\Gamma}^2 w \tau - D^2 b \nabla_{\Gamma} w \langle \nabla b, \tau \rangle + \nabla b \{ D^2 b \nabla_{\Gamma} w, \tau \} \quad (112)
\]
and \( \langle \nabla b, \tau \rangle = 0 \) by definition. The last term gives
\[
\langle D^2 b \nabla_{\Gamma} w, \tau \rangle = -\ast \langle D_{\Gamma} (\nabla_{\Gamma} w) \nabla b, \tau \rangle = -\langle \nabla b, D_{\Gamma} (\nabla_{\Gamma} w) \tau \rangle = -\langle \nabla b, D_{\Gamma}^2 w \tau \rangle = 0 \quad (113)
\]
using (25) again. Thus we see that \( * D_{\Gamma}^2 w \tau = 0 \) and
\[
\{ \nabla_\Gamma \langle h, \nabla_\Gamma w \rangle, v \} = \Delta_\Gamma w \langle h, v \rangle. \tag{114}
\]
Combining (114) and (110) gives the conclusion, Eq. (102). \[\square\]

As before, terms coming from \( P_1(w, e_\Gamma) \) are left alone for the purposes of Lemma 17, Eq. (91a) (with the exception of \( kw \langle h, \nabla_\Gamma w \rangle = \text{lot}(w) \) immediately). Thus, (91a) arises from combining Eqs. (92), (93), (101), and (102).

Equation (91b) arises from multiplying the original equation (20a) by \( m_1(x)w \), integrating by parts, and applying the boundary conditions (20c), where \( m_1(x) \) is any function:

\[
\int_Q \partial_{tt} w m_1 \, dQ = \int_\Gamma \partial_t w m_1 \, d\Gamma \bigg|_0^T - \int_Q |\partial_t w|^2 m_1 \, dQ, \tag{115}
\]
\[
\int_Q \Delta_\Gamma \partial_t w m_1 \, dQ = -\int_\Gamma \Delta_\Gamma \partial_t w m_1 \, d\Gamma \bigg|_0^T - \int_Q |\nabla_\Gamma \partial_t w|^2 \, dQ - \int_Q \partial_t w \langle \nabla_\Gamma \partial_t w, \nabla_\Gamma m_1 \rangle \, dQ, \tag{116}
\]
\[
\int_Q \Delta_\Gamma^2 w m_1 \, dQ = \int_Q |\Delta_\Gamma w|^2 m_1 \, dQ + 2 \int_Q \Delta_\Gamma w \langle \nabla_\Gamma w, \nabla_\Gamma m_1 \rangle \, dQ + \int_Q w \Delta_\Gamma w \Delta_\Gamma m_1 \, dQ. \tag{117}
\]

4.5. Observability estimate for the coupled system

**Lemma 19.** The following estimate holds for \((w, e_\Gamma)\) satisfying the coupled system (20):

\[
c_b \mathcal{E}(0) \, dt \leq C \text{ lot}(w, e_\Gamma) + \int_0^T \int_\Gamma \bigg( |D_{\Gamma}^P e_\Gamma \nu, \tau|^2 + |D_{\Gamma}^P (e_\Gamma) \nu, \nu|^2 + |\Delta_\Gamma w|^2 + \left( \frac{\partial}{\partial \nu} \Delta_\Gamma w \right)^2 \bigg) \, d\Gamma \, dt,
\tag{118}
\]

where \(\mathcal{E}(t)\) is given by Eq. (18) and the dependence of \(C, C_b\) on \(h\) has not been noted.

**Proof.** Combining equalities (38) and (91a) of Lemmas 12 and 17, respectively, gives the following equality:

\[
\int_Q \alpha \left( \rho |\partial_t w|^2 + \rho |\partial_t e_\Gamma|^2 + \beta |\Delta_\Gamma w|^2 \right) \, dQ = \mathcal{G} - S_1|_0^T - S_2 + \text{lot}(w, e_\Gamma), \tag{119}
\]

where

\[
S_1 = \rho \int_\Gamma \langle \partial_t w \langle h, \nabla_\Gamma w \rangle + \langle \partial_t e_\Gamma, D_{\Gamma}^P e_\Gamma h \rangle + \gamma \langle \nabla_\Gamma \partial_t w, \nabla_\Gamma w \rangle + \gamma \langle \nabla_\Gamma \partial_t w, h D_{\Gamma}^2 w \rangle \rangle \, d\Gamma,
\tag{120}
\]
\[ S_2 = \int_{\Omega} \left( 2\mu [\text{div}_\Gamma (w D^2 b), D^P_{\Gamma} e_\Gamma h] - \lambda [\nabla_\Gamma (H w), D^P_{\Gamma} e_\Gamma h] + \beta \Delta_\Gamma w \text{tr}(\varepsilon_{\Gamma} (e_{\Gamma} D^2 b)) \right) dQ \]

+ \lambda H \text{div}_\Gamma e_\Gamma \langle h, \nabla_\Gamma w \rangle + 2\mu \text{tr}(\varepsilon_{\Gamma} (e_{\Gamma} D^2 b) \langle h, \nabla_\Gamma w \rangle) dQ \quad (121)\]

and the desired boundary terms \( G \) are given by

\[ G = \frac{1}{2} \int_0^T \int_{\Gamma} \left( \frac{\lambda + 2\mu}{2} [D^P_{\Gamma} (e_{\Gamma} v), v]^2 + 2\mu [D^P_{\Gamma} e_{\Gamma}, \tau]^2 + \beta |\Delta_\Gamma w|^2 \right) \langle h, v \rangle d\Gamma dt \]

\[ - \int_0^T \int_{\Gamma} \beta \frac{\partial}{\partial v} (\Delta_\Gamma w) \langle h, \nabla_\Gamma w \rangle d\Gamma dt. \quad (122)\]

We wish to reconstruct the full energy \( E(t) \) on the left side of Eq. (119). In order to do so we add, to (119), Eq. (39) with \( m_2 = \frac{1}{2} \alpha(x) \) and (91b) with \( m_1 = -\frac{1}{2} \alpha(x) \). This yields

\[ \frac{1}{2} \int_{\Omega} \left( \alpha \left( 3\rho |\partial_t w|^2 + \rho\gamma |\nabla_\Gamma \partial_t w|^2 + |\partial_t e_{\Gamma}|^2 \right) + \beta |\Delta_\Gamma w|^2 + \lambda |\text{div}_\Gamma e_{\Gamma}|^2 \right) \]

\[ + 2\mu \text{tr} \left[ (\varepsilon^P_{\Gamma} (e_{\Gamma}))^2 \right] dQ \]

\[ = G - S_2 - S_3 + \text{lot}(w, e_{\Gamma}) \]

\[ - \left( S_1 - \frac{\rho}{2} \int_{\Gamma} \alpha \left( \partial_t w + \gamma (\nabla_\Gamma \partial_t w, \nabla_\Gamma w) - (\partial_t e_{\Gamma}, e_{\Gamma}) \right) d\Gamma \right) \bigg|_0^T \quad (123) \]

with

\[ S_3 = \frac{1}{2} \int_{\Omega} \left[ \text{div}_\Gamma e_{\Gamma} \langle e_{\Gamma}, \nabla_\Gamma \alpha \rangle + \text{tr} \left( (e_{\Gamma} \otimes \nabla_\Gamma \alpha) \varepsilon_{\Gamma} (e_{\Gamma}) \right) \right. \]

\[ - \beta \left( w \Delta_\Gamma w \Delta_\Gamma \alpha + 2\Delta_\Gamma w \langle \nabla_\Gamma w, \Delta_\Gamma \alpha \rangle \right) dQ + \frac{\rho \gamma}{2} \int_{\Omega} \alpha \partial_t w \langle \nabla_\Gamma \partial_t w, \nabla_\Gamma \alpha \rangle dQ \]

\[ - \frac{1}{2} \int_{\Omega} \left( \lambda H \text{div}_\Gamma e_{\Gamma} w + 2\mu \text{tr}(\varepsilon_{\Gamma} (e_{\Gamma} D^2 b)) \right) dQ. \quad (124) \]

Now, the left side of Eq. (123)

\[ \frac{1}{2} \int_{\Omega} \left( \alpha \left( 3\rho |\partial_t w|^2 + \rho\gamma |\nabla_\Gamma \partial_t w|^2 + |\partial_t e_{\Gamma}|^2 \right) + \beta |\Delta_\Gamma w|^2 + \lambda |\text{div}_\Gamma e_{\Gamma}|^2 \right) \]

\[ + 2\mu \text{tr} \left[ (\varepsilon^P_{\Gamma} (e_{\Gamma}))^2 \right] dQ \]

\[ = \rho \int_{\Omega} \alpha |\partial_t w|^2 dQ + \frac{1}{2} \int_0^T \alpha E(t) dt \geq \frac{1}{2} \int_0^T \alpha E(t) dt \geq \frac{\alpha_0}{2} \int_0^T E(t) dt \quad (125) \]

so that we have
\[
\alpha_0 \int_0^T E(t) \, dt \leq 2G - 2S_2 - 2S_3 + 2 \text{lot}(w, e_G)
\]

\[
- 2 \left( S_1 - \frac{\rho}{2} \int \alpha (\partial_tw \, w + \gamma (\nabla_G \partial_tw, \nabla_G w) - \langle \partial_t e_G, e_G \rangle) \, d\Gamma \right) \bigg|_0^T
\]

(126)

as desired, and since the left-hand side of (126) is strictly positive, all that remains is to take absolute values and estimate the terms on the right-hand side. Firstly, because by Hypothesis 2, \((h, \nu) \geq 0\) on \(\Gamma\), we have that

\[
|G| \leq C_h \frac{1}{2} \int_0^T \int_{\Gamma} \left( (\lambda + 2\mu) \left| D^P_{\Gamma} (e_G) v, v \right|^2 + 2\mu \left| D^P_{\Gamma} e_G v, \tau \right|^2 + (\lambda + 2\mu) \gamma |\nabla_G w|^2 \right) \, d\gamma \, dt
\]

\[
+ \int_0^T \int_{\Gamma} (\lambda + 2\mu) \gamma \left( \frac{1}{2} \left( \frac{\partial}{\partial \nu} (\nabla_G w) \right)^2 + \frac{1}{2} (h, \nabla_G w)^2 \right) \, d\gamma \, dt
\]

\[
\leq C_h \int_0^T \int_{\Gamma} \left( |D^P_{\Gamma} (e_G) v, v|^2 + |D^P_{\Gamma} e_G v, \tau|^2 + |\nabla_G w|^2 + \left( \frac{\partial}{\partial \nu} (\nabla_G w) \right)^2 \right) \, d\gamma \, dt
\]

\[
+ C_h \text{lot}(w).
\]

(127)

Next, the terms of \(S_2\) and \(S_3\) all contain an energy-level term multiplied by a lower-order term so we have, for example,

\[
\int_{Q} \left| \text{div}_{\Gamma} (w D^2 b), D^P_{\Gamma} e_G h \right| \, dQ \leq C_{\epsilon, b} \int_{Q} \left| \text{div}_{\Gamma} (w D^2 b) \right|^2 \, dQ + \epsilon \int_{Q} \left| D^P_{\Gamma} e_G h \right|^2 \, dQ
\]

and

\[
\int_{Q} \left| \nabla_G w \text{tr}(\nabla_G w D^2 \nabla_G h) \right| \, dQ \leq \epsilon \int_{Q} \left| \nabla_G w \right|^2 \, dQ + C_\epsilon \int_{Q} \left| \text{tr}(\nabla_G w D^2 \nabla_G h) \right|^2 \, dQ
\]

\[
\leq \epsilon \int_{Q} \left| \nabla_G w \right|^2 \, dQ + C_\epsilon G_h^2 \int_{Q} \left| \nabla_G w \right|^2 \, dQ,
\]

where the dependence of \(C_\epsilon\) and \(\epsilon\) on \(h\) has not been noted. We note that the derivatives of \(h\) are bounded by assumption, so we can call \(G_h = \max_{\Gamma} (D^2_{\Gamma} h, \nabla_{\Gamma} (\text{div}_{\Gamma} h))\) and thus estimate all required terms. Combining all of these terms we have that

\[
|S_2 + S_3| \leq C_\epsilon \text{lot}(w, e_G) + \epsilon \int_0^T E(t) \, dt.
\]

(128)

Finally, we have similarly that

\[
2 \left| S_1 - \frac{\rho}{2} \int \alpha (\partial_tw \, w + \gamma (\nabla_G \partial_tw, \nabla_G w) - \langle \partial_t e_G, e_G \rangle) \, d\Gamma \right|_0^T \leq C \left[ E(0) + E(T) \right].
\]

(129)
Combining (126) through (129) results in the estimate

\[
(\alpha_0 - 2\epsilon) \int_0^T E(t) \, dt \leq C [E(0) + E(T)] + C e \log(w, e_\Gamma) \\
+ C_b \int_0^T \int \left( |D_\Gamma^3 e_\Gamma v, \tau|^2 + |D_\Gamma^3 (e_\Gamma) v, v|^2 + |\Delta_\Gamma w|^2 \right) \\
+ \left( \frac{\partial}{\partial \nu} \Delta_\Gamma w \right)^2 \, dY \, dt.
\]

(130)

Finally, we note that \( E(0) = E(T) \) and \( \int_0^T E(t) \, dt = T E(0) \) so that

\[
\frac{(\alpha_0 - 2\epsilon)T - 2C}{C_b} E(0) \leq C e \log(w, e_\Gamma) \\
+ \int_0^T \int \left( |D_\Gamma^3 e_\Gamma v, \tau|^2 + |D_\Gamma^3 (e_\Gamma) v, v|^2 + |\Delta_\Gamma w|^2 \right) \\
+ \left( \frac{\partial}{\partial \nu} \Delta_\Gamma w \right)^2 \, dY \, dt
\]

(131)

so that (118) holds with

\[
c_b = \frac{(\alpha_0 - 2\epsilon)T - 2C}{C_b} > 0 \quad \text{for} \quad T > \frac{2C}{\alpha_0 - 2\epsilon} \equiv T_0.
\]

(132)

**Proposition 20.** We can absorb the lower-order terms in the estimate (118) by means of a compactness/uniqueness argument.

**Proof.** The proof proceeds similarly to that in [1,26]. In terms of uniqueness, we use the method of Yao [26] and show that the required statement is essentially the Cauchy problem of three coupled fourth-order equations with the same principal part \( \Delta_\Gamma^2 \). The statement to be proved is

**Proposition 21.** Let \( \zeta \) be a complex number and \( \hat{Y} \in Y \) be relatively open. Let \( \eta = (e_\Gamma, w) \) solve problem

\[
\zeta^2 M \eta + A \eta = 0 \quad \text{in} \, \Gamma, \\
e_\Gamma = D_\Gamma e_\Gamma v = 0 \quad \text{on} \, \hat{Y}, \\
w = \frac{\partial}{\partial \nu} w = \Delta_\Gamma w = \frac{\partial}{\partial \nu} \Delta_\Gamma w = 0 \quad \text{on} \, \hat{Y},
\]

(133)

then,

\[
e_\Gamma = w = 0 \quad \text{in} \, \Gamma.
\]

**Proof.** We must show that the equation for the \( \eta = (e_1, e_2, e_3 = w) \) can be written component-wise (in a local coordinate system) as \( \Delta_\Gamma^2 e_i = L(\eta) \) where the lower-order terms

\[
|L(\eta)|^2 \leq C (|D_\Gamma^3 e_\Gamma|^2 + |D_\Gamma^3 w|^2).
\]
From (20), we have that the equation on the tangential components $e_Γ$ is of the form

$$\lambda \nabla_Γ \text{div}_Γ e_Γ + \mu \left( \nabla_Γ \text{div}_Γ D_Γ e_Γ + \text{div}_Γ * D_Γ e_Γ + (D_Γ^2 b)^2 e_Γ \right) = \rho \zeta^2 e_Γ + P_2(e_Γ, w).$$

(134)

We apply the operator $\nabla_Γ \text{div}_Γ$ onto Eq. (134):

$$\lambda \nabla_Γ \text{div}_Γ \nabla_Γ \text{div}_Γ e_Γ + \mu \left( \nabla_Γ \text{div}_Γ \text{div}_Γ D_Γ e_Γ + \text{div}_Γ \text{div}_Γ * D_Γ e_Γ \right) = L(\eta).$$

(135)

Next, we apply the operator $\text{div}_Γ * D_Γ$:

$$\lambda \text{div}_Γ * D_Γ \nabla_Γ \text{div}_Γ e_Γ + \mu \left( \text{div}_Γ * D_Γ \text{div}_Γ D_Γ e_Γ + \text{div}_Γ * D_Γ \text{div}_Γ * D_Γ e_Γ \right) = L(\eta).$$

(136)

Inspection shows that the expression $\text{div}_Γ * D_Γ \text{div}_Γ \nabla_Γ e_Γ - \nabla_Γ \text{div}_Γ \text{div}_Γ e_Γ = L(\eta)$ and

$$\nabla_Γ \text{div}_Γ \text{div}_Γ D_Γ e_Γ - \text{div}_Γ * D_Γ \text{div}_Γ D_Γ e_Γ = L(\eta).$$

However this is not possible for the third terms in Eqs. (135) and (136) because of the specific form of the operator. Instead, applying Hypothesis 2 gives that Eq. (134) is equivalent to

$$\mu \Delta_Γ^2 e_Γ = L(\eta).$$

Of course, the second equation of (20), that for $w$, is already of the form $\Delta_Γ^2 w = L(\eta)$; so that we have the desired result. Now all that remains is to consider the boundary conditions. It suffices to show that $D_Γ^2 e_Γ|_Γ = 0$ and $D_Γ^3 e_Γ|_Γ = 0$. In fact, once we have that $e_Γ = D_Γ e_Γ \nu = 0$, this follows quickly by application of Propositions 8, 10 and Corollary 9. As $D_Γ^2 e_Γ$ is a third-order tensor, the calculation is lengthy. We omit the details as it essentially follows the method already shown in detail for the expression $* D_Γ^P e_Γ \nu$ in Proposition 16. We pick a suitable basis for the space of third-order tensors over $\mathbb{R}^3$, built on the vectors $(\nu, \tau, \nabla b)$ as in Proposition 16. Using these expansions, Corollary 9, and the original equation (134), we can show that $D_Γ^2 e_Γ|_Γ = 0$. A similar calculation gives that $D_Γ^3 e_Γ|_Γ = 0$. With this information we have enough Cauchy data on $Γ$ to show that problem (133) is equivalent to three coupled fourth-order equations with the same principal part $\Delta_Γ^2$ and thus is covered by [22].

Finally, by applying Proposition 20 to estimate (118) (Lemma 19), we achieve the continuous observability estimate (21).

Appendix A. Background material

A.1. Overview of the oriented distance function and the intrinsic geometry

In order to improve readability we here include a brief discussion of the oriented distance function and the intrinsic geometric methods of Delfour and Zolésio. Since by necessity this overview will lack detail, the reader is referred to [10,13] for a definitive exposition on this topic.

Consider a domain $O \subset \mathbb{R}^3$ whose nonempty boundary $\partial O$ is a $C^1$ two-dimensional submanifold of $\mathbb{R}^3$. Define the oriented (or signed) distance function to $O$ as
\[ b(x) = d_O(x) - d_{\mathbb{R}^3 \setminus O}(x) \]  
(A.1)

where \( d \) is the Euclidean distance from the point \( x \) to the domain \( O \). In other words, \( b(x) \) is simply the positive or negative distance to the boundary \( \partial O \), depending on if we are outside or inside the domain \( O \). It can be shown that for every \( x \in \partial O \), there exists a neighborhood where the function \( \nabla b = \nu \), the unit outward external normal to \( \partial O \) [10].

Consider a subset \( \Gamma \subseteq \partial O \) which will eventually become the mid-surface of our shell. We define the projection \( p(x) \) of a point \( x \) onto \( \Gamma \) as
\[ p(x) = x - b(x) \nabla b(x). \]

The orthogonal projection operator \( P(x) \) onto the tangent plane \( T_p(x) \Gamma \) is given by
\[ P(x) = I - \nabla b(x)^* \nabla b(x). \]

Then, we define a shell \( S_h \) of thickness \( h_{SH} \) as
\[ S_h(\Gamma) \equiv \left\{ x \in \mathbb{R}^3 : p(x) \in \Gamma, \quad |b(x)| < \frac{h_{SH}}{2} \right\}. \]  
(A.2)

When \( \Gamma \neq \partial O \), the shell \( S_h \) has a lateral boundary
\[ \Sigma_h(\Gamma) \equiv \left\{ x \in \mathbb{R}^3 : p(x) \in \Upsilon, \quad |b(x)| < \frac{h_{SH}}{2} \right\} \]  
(A.3)

where \( \Upsilon \equiv \partial \Gamma \) denotes the boundary of \( \Gamma \). A natural curvilinear coordinate system \((X, z)\) is thus induced on the shell \( S_h \), where the coordinate vector \( X \) gives the position of a point on the mid-surface \( \Gamma \), and \( z \in (-\frac{h_{SH}}{2}, \frac{h_{SH}}{2}) \) gives the vertical (normal) distance from the mid-surface. Using this notation, we also define the “flow mapping” \( T_z(X) \) as
\[ T_z(X) = X + z \nabla b(X) \]  
(A.4)

for all \( X \) and \( z \) in \( S_h \). This allows us to reconstruct the action at a given height \( z \) of the shell, once we know the action of the mid-surface \( \Gamma \). Define as \( \Gamma^z \) the surface \( T_z(\Gamma) \) at the ‘altitude’ \( z \). Then, one can also describe the shell \( S_h \) as
\[ S_h = \bigcup_{z=-\frac{h_{SH}}{2}}^{\frac{h_{SH}}{2}} \Gamma^z. \]

The curvatures of the shell will be denoted \( H \) and \( K \). These can be reconstructed from the boundary distance function \( b(x) \) by noting that at any point \((X, z)\), the matrix \( D^2 b \) has eigenvalues 0, \( \lambda_1 \), \( \lambda_2 \). The curvatures are then given by \( \text{tr}(D^2 b) = 2H = \lambda_1 + \lambda_2 \) and \( K = \lambda_1 \lambda_2 \).

A.2. Tangential differential calculus

Next, we mention briefly some useful aspects of the tangential differential calculus. Given \( f \in C^1(\Gamma) \), we define the tangential gradient \( \nabla_\Gamma f \) of the scalar function \( f \) by means of the projection as
\[ \nabla_\Gamma f \equiv \nabla (f \circ p)(x)|_{\Gamma}. \]  
(A.5a)

This notion of the tangential gradient is equivalent to the classical definition using an extension \( F \) of \( f \) in the neighborhood of \( \Gamma \), i.e., \( \nabla_\Gamma f = \nabla F|_{\Gamma} - \frac{\partial F}{\partial \nu} \nu \) [10]. Following the same idea we can define the tangential Jacobian matrix of a vector function \( v \in C^1(\Gamma)^3 \) as
\[ D_\Gamma v \equiv D(\nu \circ p)|_{\Gamma} \quad \text{or} \quad (D_\Gamma v)_{ij} = (\nabla_\Gamma v_i)_{j}, \]  
(A.5b)

the tangential divergence as
\[ \text{div}_\Gamma v \equiv \text{div}(v \circ p)|_\Gamma, \] (A.5c)

the Hessian \( D^2_{\Gamma} f \) of \( f \in C^2(\Gamma) \) as
\[ D^2_{\Gamma} f = D_{\Gamma}(\nabla_{\Gamma} f), \] (A.5d)

the Laplace–Beltrami operator of \( f \in C^2(\Gamma) \) as
\[ \Delta_{\Gamma} f \equiv \text{div}_{\Gamma}(\nabla_{\Gamma} f) = \Delta(f \circ p)|_{\Gamma}, \] (A.5e)

the tangential linear strain tensor of elasticity as
\[ \varepsilon_{\Gamma}(v) \equiv \frac{1}{2}(D_{\Gamma} v + * D_{\Gamma} v) = \varepsilon(v \circ p)|_{\Gamma}, \] (A.5f)

and the tangential vectorial divergence of a second-order tensor \( A \) as
\[ \text{div}_{\Gamma} A \equiv \text{div}(A \circ p)|_{\Gamma} = \text{div}_{\Gamma} A_i. \] (A.5g)

References


