Classification of Algebraic Function Fields with Divisor Class Number Two

DOMINIQUE LE BRIGAND

Institut de Mathématiques de Jussieu—Equipe Analyse Algébrique, Université Pierre et Marie Curie, Paris VI, Case 247, 4, place Jussieu F-75252 Paris, France

E-mail: lebrigan@kali.inria.fr

Communicated by Michael Tsfasman

Received December 29, 1994; revised November 20, 1995

In a previous paper we proved that there are 11 quadratic algebraic function fields with divisor class number two. Here we complete the classification of algebraic function fields with divisor class number two giving all non-quadratic solutions. Our result is the following. Let us denote by \( k \) the finite field with \( q \) elements. Up to isomorphism, there are exactly 8 non-quadratic algebraic function fields of one variable \( K/k \) having \( k \) for full constant field and with a divisor class number equal to two.

1. INTRODUCTION

Let \( k = \mathbb{F}_q \) be the finite field with \( q \) elements and let \( K/k \) be an algebraic function field of one variable having \( k \) for full constant field and of genus \( g \geq 1 \). We denote by \( h \) the divisor class number of \( K/k \). We recall that \( h = L(1) \), where \( L \) is the numerator polynomial of the zeta function of \( K/k \). The classification of function fields \( K/k \) such that \( h = 1 \) is done in [6] and [4]. Previously MacRae [5] gave the quadratic solutions which have a place of degree one. The divisor class number two problem for algebraic function fields has been studied previously in [4], but we do not agree with some results contained in that paper. In [3] we proved that up to isomorphism there are exactly 11 quadratic algebraic function fields such that \( h = 2 \). We quote this result in Section 3. It remains to give the non-quadratic solutions. This is the object of Sections 4 and 5. We prove that up to isomorphism there are only 8 non-quadratic function fields such that \( h = 2 \). These solutions are given in Theorem 4.1.
2. Preliminaries

We refer to [1] or [7] for precise definitions. For any positive integer $i$, we denote by $n_i$ the number of places of $K/k$ of degree $i$, by $\mathcal{D}_i$ the set of all positive divisors of $K/k$ of degree $i$ and by $A_i$, the cardinality of $\mathcal{D}_i$. The zeta function $\zeta(t)$ of $K/k$ is

$$\zeta(t) = \sum_{i=1}^{\infty} A_i T^i.$$ 

Further

$$\zeta(t) = \frac{L(t)}{(1-t)(1-qt)},$$

where $L(t)$ is a polynomial of $\mathbb{Z}[t]$. Precisely

$$L(t) = \sum_{i=1}^{2g} a_i t^i,$$

where $a_0 = 1$ and $a_{2g-i} = q^{g-i}a_i$ for $i = 0, \ldots, g$. The roots of $L(t)$ have square absolute value equal to $1/q$. The divisor class number of $K/k$ is $h = L(1)$. To obtain all the solutions for which $h = 2$, we use [4, Theorem 4, p. 26].

**Proposition 2.1.** Let $K/k$ be a function field of genus $g > 0$ having the finite field $k$ with $q$ elements as full constant field. Then $h = 2$ if and only if

- $g = 1, \ 2 \leq q \leq 5, \ n_1 = 2$;
- $g = 2, \ q = 2, \ n_1 = 0, n_2 = 4$ or $n_1 = 1, n_2 = 3$ or $n_1 = 2, n_2 = 1$;
- $g = 2, \ q = 3, \ n_1 = 0, n_2 = 5$;
- $g = 3, \ q = 2, \ n_1 = 0, n_3 = 2$ or $n_1 = 1, n_3 = 3 - n_2$;
- $g = 4, \ q = 2, \ n_1 = n_2 = 0, 0 \leq n_3 \leq 6, n_4 = 2$ or $n_1 = 0, n_2 = 1, 0 \leq n_3 \leq 3, n_4 = 3$ or $n_1 = n_3 = 0, n_2 = 2, n_4 = 3$;
- $g = 5, \ q = 2, \ n_1 = n_2 = n_4 = 0, n_3 = 1, n_5 = 4$ or $n_1 = n_2 = n_3 = 0, 0 \leq n_4 \leq 3, n_5 = 2$.

In fact, it is stated in [4, Theorem 4] that for $g = 4, q = 2$, and $n_2 = 1$ we must have $0 \leq n_3 \leq 2$ but the proof is not correct. Actually in that case,
$h = 2$ if and only if $n_1 = 0$ and $n_2^2 = 3n_2 - 2n_4 + 4$. Let us denote by $\omega_j = \sqrt[q]{a^{n_j}}$ and $\bar{\omega}_j$, for $j = 1, \ldots, g$, the reciprocal roots of $L$. It is easy to compute the polynomial $f(x) \in \mathbb{R}[x]$ of degree $g$ whose roots are $\cos \theta_j$ for all $j = 1, \ldots, g$. If for particular values of $n_1, \ldots, n_g$ we have $f(1) < 0$ we are sure that there are no function fields having these numbers of places. Here we obtain

$$64f(1) = (18 - 12\sqrt{2}) + (8 - 6\sqrt{2})n_2 + (2\sqrt{2} - 3)n_3.$$  

So for $n_2 = 1$ we have to consider $0 \leq n_3 \leq 3$ and in fact there is a solution such that $n_3 = 3$ (see Theorem 4.1).

### 3. Quadratic Case

In this section we quote the result obtained in [3]. Recall that a function field $K/k$ is quadratic if there exists $x \in K$ transcendental over $k$ such that $[K:k(x)]=2$. So $K/k$ is quadratic if it is elliptic ($g = 1$) or hyperelliptic ($g \geq 2$).

**Proposition 3.1.** Up to isomorphism there are only 11 quadratic function fields $K/k$ of genus $g \geq 1$, having the finite field $k$ with $q$ elements as full constant field and such that $h = 2$. They are obtained for $K = k(x, y)$ with

<table>
<thead>
<tr>
<th>$g$</th>
<th>$q$</th>
<th>$y^2 + xy = x^3 + x^2 + 1$</th>
<th>$n_1$</th>
<th>$n_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>$y^2 + y = x^3 + 2x^2 + 2$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$y^2 + y = (x^3 + x + 1)/(x^2 + x + 1)$</td>
<td>1</td>
<td>2, 3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>$y^2 + y = (x^4 + x + 1)/(x^2 + x + 1)$</td>
<td>1</td>
<td>2, 3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y^2 + y = (x^4 + x^3 + 1)/(x^4 + x + 1)$</td>
<td>1</td>
<td>2, 3</td>
</tr>
</tbody>
</table>

$(*)$ where $a$ is a generator of $GF(4)$. 


4. Non-quadratic Case

The non-quadratic solutions to the class number two problem are given by the following theorem.

**Theorem 4.1.** Up to isomorphism there are only 8 non-quadratic function fields $K/k$ of genus $g \geq 1$, having the finite field $k$ with $q$ elements as full constant field and with divisor class number 2. They are obtained for $k = \mathbb{F}_2$ and $K = k(x, y)$ with

- $g = 3$
  \begin{align*}
  y^4 + xy^3 + (x + 1)y + (x^4 + x^2 + 1) &= 0, \\
  &n_1 = 0, n_2 = 2, n_3 = 2; \\
  y^3 + (x^2 + x + 1)y + (x^4 + x^3 + 1) &= 0, \\
  &n_1 = 1, n_2 = 0, n_3 = 3; \\
  y^3 + y + (x^4 + x^3 + 1) &= 0, \\
  &n_1 = 1, n_2 = 1, n_3 = 2;
  \end{align*}

- $g = 4$
  \begin{align*}
  y^3 + (x^4 + x^3 + 1)y + (x^6 + x^3 + 1) &= 0, \\
  &n_1 = 0, n_2 = 0, n_3 = 4, n_4 = 2; \\
  y^3 + (x^4 + x^2 + 1)y + (x^6 + x^5 + 1) &= 0, \\
  &n_1 = 0, n_2 = 0, n_3 = 4, n_4 = 2; \\
  y^3 + (x^4 + x^3 + 1)y + (x^6 + x + 1) &= 0, \\
  &n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 3; \\
  y^6 + xy^5 + (x + 1)y^4 + (x^3 + x)y^3 + x^3y^2 + (x^5 + x^3 + x^2)y + x^6 + x^5 + x^3 + x + 1 &= 0, \\
  &n_1 = 0, n_2 = 1, n_3 = 1, n_4 = 3; \\
  y^6 + xy^5 + (x + 1)y^4 + x^3y^3 + x^2y^2 + (x^5 + x^3 + x^2 + x)y + x^6 + x^5 + x^3 + x + 1 &= 0, \\
  &n_1 = 0, n_2 = 1, n_3 = 2, n_4 = 3.
  \end{align*}

5. Proof of Theorem 4.1

Let $K/k$ be a non-quadratic (or non-hyperelliptic) function field of genus $g \geq 3$ having the finite field $k$ with $q$ elements as full constant field and let $\mathcal{X}$ be a non-hyperelliptic smooth projective curve defined and absolutely irreducible over $k$ whose function field is $K/k$. We denote by $\kappa$ any positive divisor of the canonical class of $K/k$. The set $|\kappa|$ of all positive canonical divisors is a complete linear system of type $g_{2g-2}$ with no base points. It is very ample since $K/k$ is non-hyperelliptic. For any divisor $D$, we denoted by $|D|$ the set of all the positive divisors which are equivalent to $D$. This is a complete linear system. $|D|$ has no base points if and only if for any point $P$ of $\mathcal{X}$

\[ l(D - P) = l(D) - 1. \]
It is *very ample* if and only if for any points $P$ and $Q$ of $X$

\[ l(D - P - Q) = l(D) - 2. \]

In particular, if $D$ is any positive divisor of degree 2 of $K/k$ then $l(k - D) = g - 2$.

Since $K/k$ is not hyperelliptic the linear system $|k|$ defines a closed immersion

\[ \Psi_k : X \rightarrow \mathbf{P}^{g-1}(k) \]

and the curve $\mathcal{C} = \Psi_k(X)$ is a normal smooth curve of degree $2g - 2$ whose function field is $K/k$. We will say that $\mathcal{C}$ is the canonical model of $K/k$. Two non-hyperelliptic function fields are isomorphic if and only if their canonical models are isomorphic under an automorphism of the projective plane $\mathbf{P}^{g-1}(k)$.

We recall that for any divisor $D$ of $K/k$ we have

\[ \text{card } |D| = q^{\nu(D)} - 1. \]

If $K/k$ has a divisor $D$ of degree $m$ such that $l(D) = 2, |D|$ is a complete linear system of type $g^1_m$ and $\text{card } |D| = q + 1$. Further, if $|D|$ has no base points, there exists $x$ in $K$ transcendental over $k$ such that $[K:k(x)] = m$. Then there exists $y \in K$ such that $K = k(x, y)$, where $y$ satisfies a monic equation of degree $m$ with coefficients in $k[x]$. If there exists a $g^1_m$, this linear system contains $q + 1$ distinct positive divisors of degree $m$ and then $A_m \approx q + 1$.

If $h = 2$, the divisor class group is isomorphic to the integers modulo 2. Thus if $D_1$ and $D_2$ are two distinct divisors with the same degree, then $2D_1$ is equivalent to $2D_2$ (we will write $\sim$ for equivalent and $\not\sim$ for not equivalent). The set $D^+_1$ can be divided into at most two disjoint sets $D^+_1$ and $D^+_2$. The divisors of $D^+_1$ are mutually equivalent and a divisor of $D^+_1$ is never equivalent to a divisor of $D^+_2$. If $i = 2$, a set $D^+_2$ contains at most one element since $K/k$ is non-hyperelliptic. So $A_2 \leq 2$, and if $n_1 = 0$ and $n_2 = 2$ then the two places of degree 2 are not equivalent.

To solve our problem, according to Proposition 2.1 we have to consider only $k = F_2$ and the following cases:

(a) \quad $g = 3$

   (i) \quad $n_1 = 0, n_3 = 2$;

   (ii) \quad $n_1 = 1, n_3 = 3 - n_2$ so $0 \leq n_2 \leq 3$.  


We study these possible cases.

5.1. Genus 3

If \( g = 3 \), the canonical model \( C \) of \( K/k \) is a smooth plane quartic. So there exist \( x \) and \( y \) in \( K \) such that \( K = k(x, y) \) with \( F(x, y) = 0 \) and \( F = 0 \) is the affine equation of \( C \). We recall that two non-quadratic function fields of genus 3 are \( k \)-isomorphic if and only if their canonical models are transformed one to the other under an automorphism of the projective plane \( \mathbb{P}^2(k) \) and that the group of automorphisms of \( \mathbb{P}^2(k) \) is \( \text{PGL}(3, k) = \text{PSL}(3, k) \). We have to consider smooth quartics defined over \( k = \mathbb{F}_2 \) and having 0 or 1 rational point. Using a computer, it is easy to classify up to isomorphism all smooth quartics defined over \( k = \mathbb{F}_2 \). This has been done by Gaëtan Hache of INRIA-Rocquencourt (France). In particular, there are only 4 algebraic function fields \( K/k \) having 0 rational points and 7 having only 1 rational point. Let us give their respective plane model, number of places \( n_1, n_2, n_3, n_4 \), and divisor class number.

<table>
<thead>
<tr>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( n_3 )</th>
<th>( n_4 )</th>
<th>( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The first two function fields are such that \( h = 1 \) and they were given in [4]. From this classification we know that there are only 3 solutions to our problem. We are going to show how one can find this result without a computer. We mainly use the fact that the canonical class is cut out on the curve by the rational lines of \( \mathbb{P}^2(k) \).
5.1.1. Case (a)(i) $n_1 = 0, n_3 = 2$

If $n_1 = 0$ and $n_3 = 2$ we have

$$L(t) = 1 - 3t + (n_2 + 2)t^2 + (2 - 3n_2)t^3 + 2(n_2 + 2)t^4 - 12t^5 + 8t^6.$$  

We can exclude $n_2 = 0$ or 1, since for these values $L(t)$ has 4 zeros with square absolute value not equal to $\frac{1}{3}$. We can exclude $n_2 = 3$ since $h = 5$ would imply that two distinct places of degree 2 are equivalent and then $K/k$ would be hyperelliptic.

The only remaining case is $n_1 = 0, n_2 = 2$ and $n_3 = 2$. Let us denote by $Q_1$ and $Q_2$ the two places of degree 2. We have $Q_1 \prec Q_2$ and $2Q_1 \prec 2Q_2 \prec Q_1 + Q_2$. Thus the canonical class contains $2Q_i, i = 1, 2$, or $Q_1 + Q_2$. In both cases the canonical class contains at least five distinct places of degree 4. Let $F_1$ be one of them and let $\{1, x, y\}$ be a basis for $L(F_1)$. Following [4, p. 16] we have a relation

$$ey^4 + \varphi_1(x)y^3 + \varphi_2(x)y^2 + \varphi_3(x)y + \varphi_4(x) = 0,$$  

(1)

where $\varphi_i(x) \in k[x]$ and $\deg(\varphi_i) = i$, for $i = 1, \ldots, 4$.

The homogeneous polynomial associated with (1) is

$$f(X, Y, Z) = eY^4 + (a_1X + a_0Z)Y^3 + (b_2X^2 + b_1XZ + b_0Z^2)Y^2 + (c_3X^3 + c_2X^2Z + c_1XZ^2 + c_0Z^3) + (d_4X^4 + d_3X^3Z + d_2X^2Z^2 + d_1XZ^3 + d_0Z^4) = 0.$$  

(2)

This is the equation of a smooth projective quartic. We may chose $x, y \in K$ such that $(x) = (X) - (Z) = F_2 - F_1$ and $(y) = (Y) - (Z) = F - F_1$, where $F_2$ is a place of degree 4 and $F$ is either $2Q_1$ or $Q_1 + Q_2$. Since $(Z) = F_1$ and

$$f(X, Y, 0) = eY^4 + a_1XY^3 + b_2X^2Y^2 + c_3X^3Y + d_4X^4,$$

we may assume that $e = 1 = d_4, b_2 = 0$ and $a_1 + c_3 = 1$. Since $(X) = F_2$ and

$$f(0, Y, Z) = Y^4 + a_0ZY^3 + b_0Z^2Y^2 + c_0Z^3Y + d_0Z^4,$$

we may assume that $d_0 = 1$ and $a_0 + b_0 + c_0 = 1$. We easily prove that
there are no solutions for $F = Q_1 + Q_2$. If $F = 2Q_1$ there is a unique solution, up to isomorphism, which is $K = k(x, y)$ with

$$y^4 + xy^3 + (x + 1)y + (x^4 + x^2 + 1) = 0.$$  

5.1.2. Case (a)(ii) $n_1 = 1, n_3 = 3 - n_2, 0 \leq n_2 \leq 3$

We have

$$L(t) = 1 - 2t + nt^2 + (3 - 3n_2)t^3 + 2nt^4 - 8t^5 + 8t^6.$$  

As before we exclude cases $n_2 = 2$ or 3. We are left with

$$n_1 = 1, n_2 = 0, \text{ and } n_3 = 3 \quad \text{or} \quad n_1 = 1, n_2 = 1, \text{ and } n_3 = 2.$$  

We denote by $P$ the unique rational point of the curve. Since $K/k$ is non-hyperelliptic, divisor $\kappa$ is very ample. Therefore $l(\kappa - P) = 2$. Thus there exist two distinct positive divisors, $D_1$ and $D_2$, of degree 3 mutually equivalent such that $\kappa \sim D_1 + P \sim D_2 + P$. Moreover $l(\kappa - 2P) = 1$ and there exists a positive divisor of degree $2D$ such that $\kappa \sim D + 2P$.

* $n_1 = 1, n_2 = 0$ and $n_3 = 3$. Since $n_2 = 0$, the only $D$ such that $\kappa \sim D + 2P$ is $D = 2P$ and we have $l(4P) = 3$ and $l(3P) = 2$. Further $|\kappa - P| = \{3P, T_1, T_2\}$, where $T_1, T_2$ are places of degree 3. We can choose $x \in K$ such that $(x) = T_1 - 3P$; thus $\{1, x, y\}$ is a basis for $\mathcal{L}(3P)$. Let $\{1, x, y\}$ be a basis for $\mathcal{L}(4P)$. The element $y$ has a pole at $P$ of order exactly 4. Since $l(12P) = 10$ and the 11 following functions $\{1, x, x^2, x^3, x^4, y, xy, x^2y, y^2, xy^2, y^3\}$ are in $\mathcal{L}(12P)$, there is a relation

$$y^3 + \varphi_1(x)y^2 + \varphi_2(x)y + \varphi_3(x) = 0,$$

where $\varphi_i(x) \in k[x]$, for $i = 1, 2, 4$, $\deg(\varphi_1) \leq 1$, $\deg(\varphi_2) \leq 2$, and $\deg(\varphi_3) = 4$, since the functions $y^3$ and $x^4$ are the only ones having a pole of order 12 at $P$. We can set $v = y + \varphi_1(x)$ and we obtain

$$v^3 + (a_2x^2 + a_3x + a_0)v + (x^4 + b_3x^3 + b_2x^2 + b_1x + b_0) = 0.$$  

We have $(x) = T_1 - 3P$, where $T_1$ is a place of degree 3; thus $a_0 = b_0 = 1$. We have $(v) = F - 4P$, where $F$ is a place of degree 4; thus we may assume w.l.o.g. that $x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 = x^4 + x^3 + 1$ or $x^4 + x + 1$. Up to isomorphism there is a unique solution $K = k(x, y)$ with

$$y^3 + (x^2 + x + 1)y + (x^4 + x^3 + 1) = 0, \quad n_1 = 1, n_2 = 0, n_3 = 3.$$
 Finally up to isomorphism there are 3 solutions for $g = 3$: $K = k(x, y)$ with

$$y^3 + y + (x^4 + x^3 + 1) = 0, \quad n_1 = 1, n_2 = 1, n_3 = 2.$$ 

Finally up to isomorphism there are 3 solutions for $g = 3$: $K = k(x, y)$ with

$$y^4 + xy^3 + (x + 1)y + (x^4 + x^2 + 1) = 0, \quad n_1 = 0, n_2 = 2, n_3 = 2;$$
$$y^3 + (x^2 + x + 1)y + (x^4 + x^3 + 1) = 0, \quad n_1 = 1, n_2 = 0, n_3 = 3;$$
$$y^3 + y + (x^4 + x^3 + 1) = 0, \quad n_1 = 1, n_2 = 1, n_3 = 2.$$

5.2. Genus 4

If $g = 4$, the complete linear system $|\kappa|$ is of type $g^1_6$. The canonical model $C$ of $K/k$ is a smooth curve of degree 6 embedded in $\mathbf{P}^3(k)$. If we project $C$ to $\mathbf{P}^3(k)$ from a rational point off the curve we obtain a plane singular model of the curve of degree 6.

Since $h = 2$ is equivalent to $n_1 = 0$ and $n_2^3 = 3n_2 - 2n_4 + 4$, we have

$$L(t) = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + 2a_5t^5 + 4a_6t^6 + 8a_7t^7 + 16t^8,$$
where

\[ a_1 = -3 \]
\[ a_2 = n_2 + 2 \]
\[ a_3 = n_3 - 3n_2 \]
\[ a_4 = 4n_2 - 3n_3 + 2. \]

In the following cases \( L(t) \) has 4 zeros with square absolute value not equal to \( \frac{1}{2} \):

\[ n_1 = n_2 = 0, \quad 0 \leq n_3 \leq 2, \quad n_4 = 2 \quad \text{and} \quad n_1 = 0, \quad n_2 = 1, \quad n_3 = 0, \quad n_4 = 3. \]

The possible cases reduce to:

(b)(i) \[ n_1 = n_2 = 0, \quad 3 \leq n_3 \leq 6, \quad n_4 = 2. \]
(b)(ii) \[ n_1 = 0, \quad n_2 = 1, \quad 1 \leq n_3 \leq 3, \quad n_4 = 3. \]
(b)(iii) \[ n_1 = 0, \quad n_2 = 2, \quad n_3 = 0, \quad n_4 = 3. \]

The zeta function is equal to

\[ Z(t) = 1 + n_2t^2 + n_3t^3 + \frac{n_4 + n_2(n_2 + 1)}{2}t^4 + 6t^5 + 22t^6 + O(t^7). \quad (3) \]

We copy the argument of [1, p. 346]. There the field \( k \) is supposed to be algebraically closed but the result remains true in our case since the objects are \( k \)-rational. The curve \( 6 \) is the complete intersection of a unique absolutely irreducible quadric \( \mathcal{Q} \) and a cubic surface \( S \) which can be chosen rationally over \( k \). We denote by \( (x:y:z:t) \) the homogeneous projective coordinates in \( \mathbb{P}^4(k) \). For \( k = F_2 \), the equation of an absolutely irreducible quadric surface \( \mathcal{Q} \) is of one of the following forms (up to isomorphism) see [2, p. 14]:

- \( x^2 + yz = 0 \), \( \mathcal{Q} \) is a cone. It is a singular surface with \( (q^2 + q + 1) \) rational points over \( k \). There is a unique set of rational generators, the rational lines passing through the vertex \( P \) of the cone.

- \( xy + \gamma(z, t) = 0 \), where \( \gamma(z, t) \) is an irreducible binary form, \( \mathcal{Q} \) is an elliptic quadric. If \( k = F_2 \) we have \( \gamma(z, t) = z^2 + zt + t^2 \). \( \mathcal{Q} \) is a smooth surface with \( (q^2 + 1) \) rational points over \( k \). It contains no rational lines.

- \( xy + zt = 0 \), \( \mathcal{Q} \) is a hyperbolic quadric. It is a smooth surface with \( (q + 1)^2 \) rational points over \( k \). It has two sets of rational generators.

We have the following result.
LEMMA 5.1. Let $K/k$ be a non-hyperelliptic function field of genus 4 and let $\mathcal{C}$ be the canonical model of $K/k$. The curve $\mathcal{C}$ is the complete intersection in $\mathbf{P}^3(k)$ of a unique rational absolutely irreducible quadric surface $\mathcal{Q}$ and a rational cubic surface $\mathcal{S}$. If we denote by $N$ the number of linear systems $g_1$ of $K/k$, then $0 \leq N \leq 2$ and

1. $N = 0$ if and only if $\mathcal{C}$ lies on an elliptic quadric.
2. $N = 1$ if and only if $\mathcal{C}$ lies on a cone.
3. $N = 2$ if and only if $\mathcal{C}$ lies on a hyperbolic quadric.

Proof. We write $\mathcal{C} = \mathcal{Q} \cap \mathcal{S}$. A set of rational generators of $\mathcal{Q}$ cuts out on $\mathcal{C}$ a linear system $g_1$. Conversely let us show that a linear system $g_1$ is cut out on $\mathcal{C}$ by a set of rational generators lying on $\mathcal{Q}$. Assume that there exists a $g_1$. This system has no base points since $K/k$ is non-hyperelliptic. Let $D$ be any divisor of this $g_1$. We have $l(D) = l(k - D) = 2$. So there are two canonical divisors of the form $D + D_i$, for $i = 1, 2$, where $D_i$ is a positive divisor of degree 3. Since the canonical class $|\mathcal{C}|$ is cut out on $\mathcal{C}$ by the planes of $\mathbf{P}^3(k)$, $D$ is equal to $P_1 + P_2 + P_3$, where the $P_i$'s are points of the intersection line $\mathcal{L}$ of two distinct planes ($\mathcal{L}$ is a trisecant to the curve). A line which is not contained in $\mathcal{Q}$ cuts $\mathcal{Q}$ (and also the curve) in at most two points, so it cannot cut out $D$ on $\mathcal{C}$.

Let $K_1/k$ and $K_2/k$ be two function fields of genus 4, and $\mathcal{C}_1$ and $\mathcal{C}_2$ be their respective canonical models. Assume that $\mathcal{C}_1 = \mathcal{Q} \cap \mathcal{S}_1$ and $\mathcal{C}_2 = \mathcal{Q} \cap \mathcal{S}_2$ for some quadric surface $\mathcal{Q}$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are canonical, their function fields are $k$-isomorphic if and only if there exists a projective automorphism $\sigma$ of $\mathbf{P}^3(k)$ fixing the quadric surface $\mathcal{Q}$ and such that $\sigma(\mathcal{S}_1) = \mathcal{S}_2$.

We divide our study into two parts:

- $K/k$ has a linear system of type $g_1$. This implies that there exists $x \in K$ transcendental over $k$ such that $[K:k(x)] = 3$.
- $K/k$ has no linear system of type $g_1$. This implies that for all $x \in K$ transcendental over $k$ we have $[K:k(x)] > 3$.

5.2.1. There Exists a $g_1$

We have the following result:

LEMMA 5.2. Let $K/k$ be a non-hyperelliptic function field of genus 4 such that $h = 2$. If there exists at least a linear system of type $g_1$, then there is a place $T$ of degree 3 such that $l(T) = 2$ and $l(2T) = 4$. Further there is exactly one linear system of type $g_1$. 

Proof. Since $h = 2$, the $n_i$'s verify the conditions (b) and the existence of a $g^3$ is equivalent to $n_3 \geq 3$. Then we are in case (b)(i) or in case (b)(ii) with $n_3 = 3$. The $g^3$ contains 3 distinct places of degree 3 mutually equivalent and $g^1 = \mathcal{D}_3 = \{T_1, T_2, T_3\}$ with $l(T_i) = 2$ for $i = 1, 2, 3$. If $n_3 \geq 4$, there is another set

$$\mathcal{D}_3 = \{T_i, i = 4, \ldots, n_3 - 3\}$$

of places of degree 3 mutually equivalent and not equivalent to $T_i$, for $i = 1, 2, 3$. So if $n_3 \geq 5$ we have $l(T_4) = 2$ and there is another $g^3$ so we must have $n_3 = 6$. Let us show that this last case is impossible. The number of positive divisors of degree 6 is equal (see (3)) to

$$A_6 = 22 = n_3(n_3 + 1)/2 + n_6.$$  

We obtain $n_6 = 1$. Among the 21 divisors of degree 6 and of the form $T_i + T_j$, we have three groups of mutually equivalent divisors

1. $T_i + T_j$, $i, j = 1, 2, 3$; there are 6 of these.
2. $T_i + T_j$, $i, j = 4, 5, 6$; there are 6 of these.
3. $T_i + T_j$, $i = 1, 2, 3$ and $j = 4, 5, 6$; there are 9 of these.

A divisor of group 1 or 2 is never equivalent to a divisor of group 3. The canonical class contains 15 positive divisors of degree 6 mutually equivalent. This is impossible. We are left with the cases $k = F_2$ and

$$n_1 = n_2 = 0, n_3 = 3, 4, n_4 = 2 \quad \text{or} \quad n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 3.$$  

Divisors $2T_i$ are equivalent to $T_i + T_j$, for $i, j = 1, 2, 3$, and mutually equivalent since $h = 2$. There is a unique $g^1$, so the curve lies on a cone. We denote by $L_i$, for $i = 1, 2, 3$, the rational line passing by the vertex of the cone and containing $T_i$ and by $\mathcal{P}_{ij}$ the rational plane containing $L_i$ and $L_j$. Then $\mathcal{P}_{ij}$ cuts the curve in $T_i + T_j$ which is a canonical divisor and we have $l(2T_i) = 4$ for $i = 1, 2, 3$.  

We give now a plane model of the function field. Let $T$ be a place of degree 3 of $K/k$ such that $l(T) = 2$ and $l(2T) = 4$ and let $\{1, x\}$ be a basis for $\mathcal{L}(T)$. Then $[K : k(x)] = 3$ and the place $(1/x)$ of $k(x)/k$ is inert in $K/k$, since $(x)_a = T$. Since $l(2T) = 4$, there exists $y \in \mathcal{L}(2T) \setminus \mathcal{L}(T)$ such that $\{1, x, x^2, y\}$ is a basis for $\mathcal{L}(2T)$. We have $(y)_a = 2T$ and $k(x, y) = K$. Further the 16 following functions are in $\mathcal{L}(6T)$:

$$\{1, x, x^2, \ldots, x^6, y, xy, \ldots, x^4y, y^2, xy^2, x^2y^2, y^3\}. $$
Since \( l(6T) = 15 \) and

\[
\{1, x, x^2, \ldots, x^6, y, xy, \ldots, x^4y, x^2y^2, x^3y^2\}
\]

is a free system, we obtain a relation

\[
F(x, y) := y^3 + \phi_2(x)y^2 + \phi_4(x)y + \phi_6(x) = 0, \quad (4)
\]

where \( \phi_i(x) \in k[x] \) and \( \deg(\phi_i) \leq i \) for \( i = 2, 4, 6 \). Setting \( v = y + \phi_2(x) \), we have \( k(x, y) = k(x, v) \) and

\[
v^3 + \psi_4(x)v + \psi_6(x) = 0, \quad (5)
\]

where \( \psi_i(x) \in k[x] \) and \( \deg(\psi_i) \leq i \) for \( i = 2, 4, 6 \). We set

\[
\psi_4(x) = a_4x^4 + \cdots + a_0 \\
\psi_6(x) = b_6x^6 + \cdots + b_0.
\]

The functions \( v^3, x^4v, \) and \( x^6 \) are the only ones having a pole of order 6 in \( T \). Then \( a_4 \neq 0 \) or \( b_6 \neq 0 \). We set \( u = 1/x \) in (5) and multiplying by \( u^6 \) we obtain

\[
(u^2v)^3 + (a_4 + \cdots + a_6u^4)(u^2v) + (b_6 + \cdots + b_6u^6) = 0. \quad (6)
\]

Setting \( t = u^2v \) in (6) we obtain the equation

\[
F_1(u, t) := t^3 + (a_4 + \cdots + a_6u^4)t + (b_6 + \cdots + b_6u^6) = 0. \quad (7)
\]

The function fields \( k(x, y)/k \) with \( F(x, y) = 0 \) and \( k(u, t)/k \) with \( F_1(u, t) = 0 \) are equal and the infinite place \( \alpha \) of \( k(x)/k \) corresponds to the finite place \( (u) \) of \( k(u)/k \). Then \( (u) \) inert implies that the polynomial \( t^2 + a_4t + b_6 \in k[t] \) is irreducible over \( k \). Since \( k = F_2 \), we must have \( a_4 = b_6 = 1 \). The plane curve \( C \) with affine equation (7) has a singular point \( P \) at infinity (which is the only point at infinity) and its invariant delta is \( \delta_P = 6 \). Since the genus has to be 4, \( C \) must have no affine singular point.

There are 12 possible function fields in each of the cases (b)(i) with \( n_3 = 4 \), (b)(ii) with \( n_3 = 3 \), and no solutions for (b)(i) with \( n_3 = 3 \). Up to isomorphisms of type \( x \mapsto (x + 1) \) and \( x \mapsto 1/x \), there are 3 solutions
\[ K = k(x, y) \text{ with} \]

\[
y^3 + (x^4 + x^3 + 1)y + (x^6 + x^3 + 1) = 0, \quad n_1 = 0, n_2 = 0, n_3 = 4, n_4 = 2; \\
y^3 + (x^2 + x + 1)^2y + (x^4 + x^5 + 1) = 0, \quad n_1 = 0, n_2 = 0, n_3 = 4, n_4 = 2; \\
y^3 + (x^4 + x^3 + 1)y + (x^6 + x + 1) = 0, \quad n_1 = 0, n_2 = 1, n_3 = 3, n_4 = 3.
\]

Let us show that the first two function fields are not isomorphic. They can be viewed as the function fields of canonical curves \( C_1 \) and \( C_2 \) which are complete intersections in \( \mathbb{P}^4(k) \) of a quadric cone and a cubic surface. Their equations are respectively

\[
\begin{align*}
&f(x, y, z, t) = x^2 + zt = 0 \\
&s_1(x, y, z, t) = y^3 + (t^2 + xt + z^2)y + t^3 + xzt + z^3 = 0 \\
&f(x, y, z, t) = x^2 + zt = 0 \\
&s_2(x, y, z, t) = y^3 + (t^2 + tz + z^2)y + t^3 + xt^2 + z^3 = 0.
\end{align*}
\]

The group of automorphisms of \( \mathbb{P}^4(k) \) fixing the cone defined by \( f(x, y, z, t) = x^2 + zt = 0 \) has 48 elements, and the general form of such an automorphism is

\[
\sigma = \begin{pmatrix}
1 & 0 & c_1c_3 & c_2c_4 \\
a_1 & 1 & a_2 & a_3 \\
0 & 0 & c_1 & c_2 \\
0 & 0 & c_3 & c_4
\end{pmatrix},
\]

where the coefficients \( a_i \) and \( c_j \) belong to \( k \) and \( c_1c_4 + c_2c_3 = 1 \). There is no automorphism of \( \mathbb{P}^4(k) \) fixing the cone and sending \( s_1 \) on \( s_2 + l \times f \), where \( l \) is a form of degree 1. The corresponding function fields are not isomorphic.

5.2.2. \textit{There Exists No} \( g_1^3 \)

As we have seen before we must have \( n_3 < 3 \). Then we are in one of the remaining cases \( k = \mathbb{F}_2 \) and

- (b)(ii) \( n_1 = 0, n_2 = 1, 1 \leq n_3 \leq 2, n_4 = 3; \)
- (b)(iii) \( n_1 = 0, n_2 = 2, n_3 = 0, n_4 = 3. \)

- Case \( n_1 = 0, n_2 = 1, 1 \leq n_3 \leq 2, n_4 = 3. \) Let \( \ell = \mathcal{D} \cap \mathcal{L} \) be the canonical
model of $K/k$. We can choose the projective homogeneous coordinates in $\mathbb{P}^3(k)$, $(x:y:z:t)$, such that the equation of $\mathcal{Q}$ is

$$f(x, y, z, t) = xy + z^2 + zt + t^2 = 0.$$  

A general cubic surface $\mathcal{S}$ has an equation of the form

$$s_0(x, y, z, t) = \sum_{i+j+k+l=3} d_{i,j,k,l} x^i y^j z^k t^l = 0.$$  

There are 20 distinct monomials of degree 3 in four indeterminates. It is easy to see that there is a unique cubic form $s$ in the four indeterminates $x, y, z, t$ which does not contain the monomials $x^2 y, xy^2, xyz, xyt$ and such that

$$s_0(x, y, z, t) = s(x, y, z, t) + l(x, y, z, t) \times f(x, y, z, t),$$  

where $l$ is a form of degree 1. So we can take the equation of $\mathcal{S}$ in the form

$$s(x, y, z, t) = c_1 x^2 z + c_2 x^2 t + c_3 x z^2 + c_4 x t^2 + c_5 x z t + c_6 y^2 z + c_7 y^2 t + c_8 y z^2 + c_9 y z t + c_{10} y t^2 + c_{11} z^2 t + c_{12} z^2 t + c_{13} z^3 + c_{14} t^3 + c_{15} x^3 + c_{16} y^3 = 0.$$  

The coordinates of the 5 $k$-rational points of $\mathcal{Q}$ are

$$P_1 = (1:0:0:0)$$  
$$P_2 = (0:1:0:0)$$  
$$P_3 = (1:1:0:1)$$  
$$P_4 = (1:1:1:0)$$  
$$P_5 = (1:1:1:1).$$  

Since $n_1 = 0$, we write that $\mathcal{Q}$ does not pass by these points and obtain the linear relations

$$
\begin{align*}
c_{16} &= 1 \\
c_{15} &= 1 \\
c_{14} &= 1 + c_{10} + c_7 + c_4 + c_2 \\
c_{13} &= 1 + c_8 + c_6 + c_3 + c_1 \\
c_{12} &= 1 + c_{11} + c_9 + c_5.
\end{align*}
$$
Let $\alpha$ be a generator of $F_4$. The 10 places of degree 2 of $\mathcal{O}$ are

\[
Q_1 = (\alpha : \alpha^2 : 0 : 1) + (\alpha^2 : \alpha : 0 : 1)
\]
\[
Q_2 = (\alpha : \alpha^2 : 1 : 0) + (\alpha^2 : \alpha : 1 : 0)
\]
\[
Q_3 = (\alpha : \alpha^2 : 1 : 1) + (\alpha^2 : \alpha : 1 : 1)
\]
\[
Q_4 = (0 : 1 : 1 : \alpha) + (0 : 1 : 1 : \alpha^2)
\]
\[
Q_5 = (1 : 0 : 1 : \alpha) + (1 : 0 : 1 : \alpha^2)
\]
\[
Q_6 = (0 : 0 : 1 : \alpha) + (0 : 0 : 1 : \alpha^2)
\]
\[
Q_7 = (0 : 1 : \alpha : 1) + (0 : 1 : \alpha^2 : 1)
\]
\[
Q_8 = (0 : \alpha^2 : \alpha : 1) + (0 : \alpha : \alpha^2 : 1)
\]
\[
Q_9 = (1 : 0 : \alpha : 1) + (1 : 0 : \alpha^2 : 1)
\]
\[
Q_{10} = (\alpha^2 : 0 : \alpha : 1) + (\alpha : 0 : \alpha^2 : 1).
\]

We have for instance

\[
Q_1 \in \mathcal{O} \iff \begin{cases} c_4 + c_7 = 1 \\ c_{10} + c_2 = 1. \end{cases}
\]

The subgroup $\mathcal{O}$ of $\text{PSL}(4,2)$ fixing $\mathcal{O}$ is $\text{PSL}(4,2)$ and it is triply transitive on $\mathcal{O}$ (see [2, p. 19]). Moreover $\mathcal{O}$ is isomorphic to $S_5$, group of permutations of $(1, 2, 3, 4, 5)$. This isomorphism can be made explicit in the following way: each $\sigma \in \mathcal{O}$ corresponds to a permutation of the five rational points $P_i, i = 1, \ldots, 5$. The cardinality of $\mathcal{O}$ equals 120. All places $Q_i$ belong to the same orbit under $\mathcal{O}$. The subgroup $\mathcal{O}_1$ of $\mathcal{O}$ fixing the quadric and the place $Q_1$ (for instance) contains 12 elements. Precisely $\mathcal{O}_1 = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$, where $\sigma_1$ and $\sigma_2$ correspond to transpositions of $S_5$ and $\sigma_3$ to a 3-cycle. We have

\[
\sigma_1 = (1, 2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = (4, 5) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},
\]
\[
\sigma_3 = (1, 4, 2) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.
\]
If we look for the cubic surfaces which contain $Q_i$ but not $Q_j$, for $2 \leq j \leq 10$, we have 12 isomorphic solutions in each cases $n_1 = 0$, $n_2 = 1$, $n_3 = 1$, $n_4 = 3$ and $n_1 = 0$, $n_2 = 1$, $n_3 = 2$, $n_4 = 3$. Thus up to isomorphism there are only 2 solutions, which are

1. $n_1 = 0$, $n_2 = 1$, $n_3 = 1$, $n_4 = 3$

$$
\begin{align*}
  &f(x, y, z, t) = xy + z^2 + zt + t^2 = 0 \\
  &s_1(x, y, z, t) = x^3 + x^2t + xz^2 + x^2t + y^3 + zt^2 + t^3 = 0.
\end{align*}
$$

2. $n_1 = 0$, $n_2 = 1$, $n_3 = 2$, $n_4 = 3$

$$
\begin{align*}
  &f(x, y, z, t) = xy + z^2 + zt + t^2 = 0 \\
  &s_2(x, y, z, t) = x^3 + x^2t + xz^2 + x^2t + y^3 + yzt + zt^2 + t^3 = 0.
\end{align*}
$$

Let $U$ be the open set of $\mathbf{P}^3(k)$ defined by $y \neq 0$. Let $(u:v:w)$ be projective coordinates in $\mathbf{P}^2(k)$ and $V$ be the open set defined by $w \neq 0$. To obtain a plane model, we consider the birational morphism from $V$ to $U$ defined by

$$(u:v:1) \mapsto (u^2 + uw + v^2:1:v:u).$$

Then $K = k(u, v)$ with

for $n_1 = 0$, $n_2 = 1$, $n_3 = 1$, $n_4 = 3$:

$$v^6 + uw^5 + (u + 1)v^4 + (u^3 + u)v^3 + u^3v^2 + (u^5 + u^3 + u^2)v
+ u^6 + u^5 + u^3 + u + 1 = 0;$$

for $n_1 = 0$, $n_2 = 1$, $n_3 = 2$, $n_4 = 3$:

$$v^6 + uw^5 + (u + 1)v^4 + u^3v^3 + u^3v^2 + (u^5 + u^3 + u^2 + u)v
+ u^6 + u^5 + u^3 + u^2 + u = 0.$$

$\bullet$ $n_1 = 0$, $n_2 = 2$, $n_3 = 0$, $n_4 = 3$. Following a procedure suggested by Ruud Pellikaan, we show that this case is impossible. We have

$$22 = A_6 = n_2(n_2 + 1)(n_2 + 2)/6 + n_4n_6 = 4 + 6 + n_6.$$  

Thus $n_6 = 12$. Let $Q_1$ and $Q_2$ be the two places of degree 2. Then $2Q_1 \sim 2Q_2$ and $Q_1$ is not equivalent with $Q_2$, since the curve is not hyperelliptic. Therefore $3Q_1 \sim Q_1 + 2Q_2$ and $3Q_2 \sim 2Q_1 + Q_2$, and $3Q_1$ and $3Q_2$ are not equivalent so either one is canonical, say $3Q_1$, since $h = 2$. Let the curve $\mathcal{C}$ be embedded in $\mathbf{P}^3(k)$ by the canonical embedding. Let $L_i$ be the
rational line going through \( Q_i \), for \( i = 1, 2 \). Let \( V_1 \) be the plane which intersects the curve \( C \) in \( 3Q_1 \), and \( V_2 \) the plane which intersects \( C \) in \( Q_1 + 2Q_2 \). Then \( V_1 \) and \( V_2 \) intersect in the line \( L_1 \). So both \( L_1 \) and \( L_2 \) lie in the plane \( V_2 \) and intersect in a point say \( P \). Projecting \( C \) from \( P \) to a plane, we get a plane curve \( C \) of degree 6 with two singular points, both of multiplicity 2. We may suppose that the projection of the plane \( V_1 \) becomes the line with equation \( x = 0 \) in the projective plane, and the projection of the plane with equation \( V_2 \) becomes the line with equation \( y = 0 \), so the projection of \( Q_1 \) becomes the point \( P_1 = (0:0:1) \) and we may assume that the projection of \( Q_2 \) becomes the point \( P_2 = (1:0:0) \). Let \( F(x, y, z) = 0 \) be the homogeneous polynomial of degree 6 which defines the plane curve \( C \). Then \( F(0, y, z) = y^6 \), since \( V_1 \) intersects \( C \) in \( 3Q_1 \), and \( F(x, 0, z) = x^2 z^4 \), since \( V_2 \) intersects \( C \) in \( Q_1 + 2Q_2 \). Thus

\[
F(x, y, z) = x^2 z^4 + y^6 + xyH(x, y, z).
\]

where \( H \) is a homogeneous form of degree 4. The points \( P_1 \) and \( P_2 \) are singular points of \( C \). Let us show that the invariant delta is equal to 3 at \( P_1 \) and at \( P_2 \). We have

\[
F(x, y, z) = x^2 z^4 + y^6 + a_1 x y z^4 + a_2 x^2 y z^3 + a_3 x y^2 z^3 + a_4 x^3 y z^2 + a_5 x^2 y^2 z^2 + a_6 x y^3 z^2 + a_7 x^4 y z + a_8 x^3 y^2 z + a_9 x^2 y^3 z + a_{10} x y^4 z + a_{11} x^4 y z + a_{12} x^3 y^2 z + a_{13} x^2 y^3 z + a_{14} x y^4 z + a_{15} x y^5.
\]

The local equation of the plane curve at \( P_1 \) is

\[
F(x, y, 1) = x^2 + xyH(x, y, 1) + y^6
\]

and \( P_1 \) is of multiplicity 2. Since there are no rational points we must have \( H(0, 0, 1) = 0 \) (so \( a_1 = 0 \)); otherwise \( P_1 \) would be an ordinary point and we would obtain two rational points by blowing up. So

\[
F(x, y, 1) = x^2 + a_2 x^2 y + a_3 x y^2 + a_6 x y^3 + \cdots + y^6.
\]

We blow up \( P_1 \):

\[
F(xy, y, 1) = y^2 (x^2 + a_2 x^2 y + a_3 x y^2 + a_6 x y^3 + \cdots + y^4) = y^2 g(x, y).
\]

There is a unique infinitely near point of multiplicity 2. We must have \( a_3 = 0 \) for the same reason as previously. Then

\[
g(xy, y) = y^2 (x^2 + a_2 x^2 y + a_6 x y \cdots + y^3) = y^2 g_1(x, y).
\]
Once again there is a unique infinitely near point of multiplicity 2 and we must have $a_6 = 1$. This infinitely close point is ordinary. By blowing up we obtain $Q_1$ and so $\delta_1 = 3$. The local equation of the plane curve at $P_2$ is

$$F(1, y, z) = a_{11}y + a_{12}y^2 + a_2yz + a_3y^2z + a_4y^3 + \cdots + z^4 + y^6$$

and $P_2$ is singular, so $a_1 = 0$. The point $P_2$ is of multiplicity 2 and it cannot be ordinary, so we must have $a_{12} = 1$ and $a_7 = 0$. We blow up $P_2$

$$F(1, yz, z) = z^2(y^2 + a_4yz + \cdots + z^2 + y^6z^4) = z^2g_3(y, z).$$

As before we must have $a_6 = 1$. The unique infinitely close point is ordinary. Blowing up once again we obtain $Q_2$ and $\delta_2 = 2$.

There must be another singular point $P_3$ with $d_3 = 1$. This point is rational of multiplicity 2 and the curve is desingularized in a neighbourhood of $P_3$ after only one blowing up. If $P_3$ is an ordinary double point, we obtain a third place of degree 2 and otherwise a rational point. Both cases are impossible.

5.3. Genus 5

We have

$$L(t) = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + 2a_6t^6 + 4a_7t^7 + 8a_8t^8 + 16a_9t^9 + 32t^{10},$$

where

$$\begin{cases}
   a_1 = -3 \\
   a_2 = 2 \\
   a_3 = n_3 \\
   a_4 = n_4 - 3n_3 \\
   a_5 = 4n_3 - 3n_4 + 2.
\end{cases}$$

(c)(i) If $n_1 = n_2 = n_4 = 0$, $n_3 = 1$ and $n_5 = 4$

$$L(t) = 1 - 3t + 2t^2 + t^3 - 3t^4 + 6t^5 - 6t^6 + 4t^7 + 16t^8 - 48t^9 + 32t^{10}. \quad (8)$$

(c)(ii) If $n_1 = n_2 = n_3 = 0$, $0 \leq n_4 \leq 3$ and $n_5 = 2$

$$L(t) = 1 - 3t + 2t^2 + n_4t^4 + (2 - 3n_4)t^5 + 2n_4t^6 + 16t^8 - 48t^9 + 32t^{10}. \quad (9)$$
In all cases $L(t)$ has 4 zeros with square absolute value not equal $\frac{1}{2}$, so there are no solutions for $g = 5$.

6. Conclusion

In this paper we complete the classification of algebraic function fields with $h = 2$. Up to isomorphism, there are 11 quadratic solutions and 8 non-quadratic ones.

Acknowledgments

The author is deeply grateful to Ruud Pellikaan for helpful discussions and to Gaétan Haché for his computations.

References