Uniform asymptotics of the Stieltjes–Wigert polynomials via the Riemann–Hilbert approach

Z. Wang, R. Wong *

Department of Mathematics, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong

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Abstract

It has been known for some time that the existing asymptotic methods for integrals and differential equations are not applicable in the case of Stieltjes–Wigert polynomials with degree going to infinity. Using the recently introduced nonlinear steepest descent method for Riemann–Hilbert problems, here we not only derive an asymptotic expansion for these polynomials, but we also show that the result holds uniformly in the complex plane except for a sector containing the real axis from $-\infty$ to $1/4$. Furthermore, we give an asymptotic formula for the zeros of these polynomials, which approximates the true values of the zeros closely.

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1. Introduction

The Stieltjes–Wigert polynomials, given in (1.5) below, were mentioned in the first edition (1939) of the authoritative book “Orthogonal Polynomials” by G. Szegő. Hence, they have been known to the specialists in the field of orthogonal polynomials for a long time. One of the special features of these polynomials is that they provide an example in which different weight functions can give the same moments (i.e., the moment problem is indeterminate). For

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* Corresponding author.

E-mail address: rscwong@cityu.edu.hk (R. Wong).
a recent paper on this topic, see Christiansen [2]. Another special feature is that they belong to what are now known as $q$-orthogonal polynomials. So any new and important method that works for the Stieltjes–Wigert polynomials will have great potential to be applicable to other $q$-orthogonal polynomials, which has been a hot topic in the last quarter century.

Let us first recall the definition of these polynomials. Let $k > 0$ be a fixed number, and

$$q = \exp\left\{- (2k^2)^{-1}\right\}. \tag{1.1}$$

For $0 < \nu < n$, put

$$\left[\begin{array}{c} n \\ \nu \end{array}\right] = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-v+1})}{(1 - q)(1 - q^2) \cdots (1 - q^\nu)} \tag{1.2}$$

with

$$\left[\begin{array}{c} n \\ 0 \end{array}\right] = \left[\begin{array}{c} n \\ n \end{array}\right] = 1. \tag{1.3}$$

The Stieltjes–Wigert polynomials are orthonormal with respect to the weight function

$$w(x) = k \pi^{-1/2} \exp\left(-k^2 \log^2 x\right) \tag{1.4}$$

for $0 < x < \infty$. In terms of the Gauss notation (1.2), these polynomials have the explicit representation

$$p_n(x) = (-1)^n q^{n/2+1/4} \left[\begin{array}{c} n \\ 0 \end{array}\right] q^{n/2} q^{1/2}(q; q)_n x^n, \tag{1.5}$$

from which it follows that the leading coefficient of $p_n(x)$ is

$$\gamma_n = q^{n^2+n+1/4} \left[\begin{array}{c} n \\ 0 \end{array}\right] q^{n/2} (1 - q)^{1/2}. \tag{1.6}$$

The three-term recurrence relation satisfied by the Stieltjes–Wigert polynomials is given by

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + A_{n-1} p_{n-1}(x), \tag{1.7}$$

where

$$A_n = \frac{\gamma_n}{\gamma_{n+1}} = q^{-(2n+2)} \left(1 - q^{n+1}\right)^{1/2} \tag{1.8}$$

and

$$B_n = q^{-2n-3/2} + q^{-2n-1/2} - q^{-n-1/2}. \tag{1.9}$$

All these properties can be found in Szegö [10] and Chihara [1]; see also Koekoek and Swarttouw [8].

One of the major areas of research in the study of orthogonal polynomials is to investigate the asymptotic behavior of the polynomials, and of their zeros, as the degree of the polynomials grows to infinity. For instance, in 1923 Wigert [11] proved that

$$(-1)^n q^{-n/2} p_n(x) \rightarrow \frac{q^{1/4}}{\sqrt{(q; q)_\infty}} \sum_{k=0}^{\infty} (-1)^k q^{k^2+k/2} (q; q)_k x^k \tag{1.10}$$

as $n \rightarrow \infty$. In (1.10), we have used the notation

$$(q; q)_k := \prod_{j=1}^{k} (1 - q^j). \tag{1.11}$$

The result in (1.10) holds only for fixed $x > 0$. To obtain a more detailed description of the asymptotic behavior of the Stieltjes–Wigert polynomials, the problem becomes extremely difficult. The difficulty lies in the fact that none of the existing asymptotic methods developed over a long period of time for differential equations (see Olver [9]) and for integrals (see Wong [14]) is applicable, not even the recently discovered method for difference equations with transition points [12,13]. In this paper, we show that with some modifications and improvements, the (nonlinear)
steepest descent method for 2-dimensional oscillatory Riemann–Hilbert problems introduced by Deift and Zhou [4] in 1993, further explored and extended in [3] and [5], can be used to give a very nice result for the polynomial \( p_n(z) \) in (1.5). More precisely, we derive a \textit{globally uniform} asymptotic expansion for these polynomials in the complex \( z \)-plane as \( n \to \infty \), except for a sector containing the real axis from \(-\infty\) to \(1/4\). In particular, the region of validity of the expansion includes the infinite interval \((1/4, \infty)\) on the positive real axis, where all the zeros lie. As a consequence of our main theorem, we also give an asymptotic formula for the zeros of the Stieltjes–Wigert polynomials. Numerical computation shows that our formula approximates the true values of the zeros closely. The exact statements of our results are summarized in Section 2, and the proofs of the results are given in the subsequent sections. In connection with our investigation, mention should be made of a recent paper by Ismail [7], in which he has also studied the asymptotics of the Stieltjes–Wigert polynomials \( p_n(x) \) with \( x = tq^{-2n} \), but for fixed values of \( t \).

2. Statement of the results

Let \( \alpha_n \) and \( \beta_n \) be positive numbers with \( \alpha_n < \beta_n \), and let \( \mu_n(s) \) be a probability density function supported on \([\alpha_n, \beta_n]\), i.e., \( \mu_n(s) \geq 0 \) and

\[
\int_{\alpha_n}^{\beta_n} \mu_n(s) \, ds = 1.
\]

(2.1)

In Section 3, it will be shown that the above \textit{Mhaskar–Rakhmanov–Saff numbers} \( \alpha_n \) and \( \beta_n \) are given by

\[
\alpha_n = 2e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2} - 2e^{(n+1/2)/2k^2} \sqrt{e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2}},
\]

(2.2)

\[
\beta_n = 2e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2} + 2e^{(n+1/2)/2k^2} \sqrt{e^{(n+1/2)/k^2} - e^{(n+1/2)/2k^2}}.
\]

(2.3)

Note that as \( n \to \infty \), we have

\[
\alpha_n \to \frac{1}{4} \quad \text{and} \quad \frac{\beta_n}{4e^{(n+1/2)/k^2}} \to 1.
\]

(2.4)

Next, we introduce the function

\[
\phi_n(z) = \frac{k^2}{N} \int_{\beta_n}^{\beta_n} \frac{1}{\zeta} \left\{ 2\log[\zeta + \sqrt{\alpha_n \beta_n} + \sqrt{(\zeta - \alpha_n)(\zeta - \beta_n)}] - \log[(\sqrt{\alpha_n} + \sqrt{\beta_n})^2 \zeta] \right\} \, d\zeta
\]

(2.5)

for \( z \in \mathbb{C} \setminus (-\infty, \beta_n] \), where \( N = n + 1/2 \). This function is connected with the \( g \)-function

\[
g(z) = \int_{\alpha_n}^{\beta_n} \log(z - s) \, d\mu_n(s),
\]

(2.6)

i.e., the logarithmic potential of \( \mu_n(s) \). From (2.5), it is easily verified that

\[
(\phi_n)(+) = (\phi_n)(-) \quad \text{for } x > \beta_n
\]

(2.7)

and

\[
(\phi_n)(+) = (\phi_n)(-) - 2\pi i \quad \text{for } 0 < x < \alpha_n.
\]

(2.8)

Put

\[
\zeta_n(z) := \left[ \frac{3}{2} \phi_n(z) \right]^{2/3}.
\]

(2.9)

This function plays an important role in the theory of uniform asymptotic expansions. It arises in the Liouville transformation for differential equations [9, p. 398], and in the cubic transformation for integrals [14, p. 367]. Since \( (\zeta_n)(+) = (\zeta_n)(-) \) for \( x \in (\alpha_n, \beta_n) \), \( \zeta_n(z) \) can be analytically continued to \( \mathbb{C} \setminus (-\infty, \alpha_n] \).
Finally, we mention the constant
\[ l_n = 2 \left(n + \frac{1}{2}\right) g(\beta_n) + \log w(\beta_n) - \log \frac{\beta_n - \alpha_n}{4}. \] (2.10)

How we arrive at this number will be shown in the next section.

The following theorem is the main result of this paper. It provides a globally uniform asymptotic expansion for the Stieltjes–Wigert polynomial \( p_n(x) \) as \( n \to \infty \).

**Theorem 1.** Let \( \pi_n(z) = p_n(z)/\gamma_n \). With \( \zeta_n(z) \) and \( \ln \) defined as above, we have
\[ \pi_n(z) = \frac{\sqrt{\pi} e^{\ln/2}}{\sqrt{w(z)}} \left\{ \left( n + \frac{1}{2} \right)^{1/6} \text{Ai} \left( \left( n + \frac{1}{2} \right)^{2/3} \zeta_n \right) A(z, n) - \left( n + \frac{1}{2} \right)^{-1/6} \text{Ai}' \left( \left( n + \frac{1}{2} \right)^{2/3} \zeta_n \right) B(z, n) \right\}, \] (2.11)
where \( A(z, n) \) and \( B(z, n) \) are analytic functions of \( z \) in \( \mathbb{C} \setminus \mathbb{S}_\delta \), where \( \mathbb{S}_\delta = \{ z : 2\pi/3 \leq \arg[z - (1/4 + \delta)] \leq 4\pi/3 \} \) and \( \delta \) is any small positive number. Furthermore, the asymptotic expansions
\[ A(z, n) \sim \zeta_n^{-1/4} \left( \beta_n - \alpha_n \right)^{1/2} \sum_{k=1}^\infty A_k(z) n^k, \] (2.12)
\[ B(z, n) \sim \zeta_n^{-1/4} \left( \beta_n - \alpha_n \right)^{1/2} \sum_{k=1}^\infty B_k(z) n^k, \] (2.13)
hold uniformly, and the coefficient functions \( A_k(z) \) and \( B_k(z) \) are analytic functions in \( \mathbb{C} \setminus \mathbb{S}_\delta \).

To describe the behavior of the zeros, we make the change of variable
\[ t \mapsto x_n(t) = \sqrt{\alpha_n \beta_n} \exp \left[ \frac{t}{2} \log(\beta_n/\alpha_n) \right], \] (2.14)
which takes the interval \([-1, 1]\) onto \([\alpha_n, \beta_n]\). Let \( t_v = t_v(n) \) denote the zeros of \( \pi_n(x_n(t)) \), and arrange them in the order
\[ -1 < t_v < t_{v-1} < \cdots < t_2 < t_1 < 1. \] (2.15)

A consequence of Theorem 1 is the following result concerning the behavior of the zeros \( t_v(n) \) as \( n \to \infty \). Numerical evidence on the accuracy of this result is given in Table 1 at the end of the paper.

**Theorem 2.** Let \( a := \frac{1}{2} \log(\beta_n/\alpha_n) \), and
\[ F(x) := \int_0^x \arctan \sqrt{e^\tau - 1} \, d\tau. \] (2.16)
The zeros in (2.15) satisfy the asymptotic formula
\[ F(a(1 - t_v)) = \frac{2}{3} (-a_v)^{3/2} \log \left( \frac{1}{q} \right) + O \left( \frac{1}{N^{1/3}} \right), \] (2.17)
where \( a_v \) denotes the \( v \)th negative zero of the Airy function \( \text{Ai}(x) \) and the \( O \)-term is uniform with respect to \( 1 \leq v \leq (1 - \delta)n, \delta > 0 \). If \( v \) is fixed, then the \( O \)-term in (2.17) can be replaced by \( O(1/N^{4/3}) \).

The behavior of the corresponding zeros of \( \pi_n(x) \), denoted by \( x_{n,v} = x_n(t_v) \), can be obtained from (2.14). Since
\[ \begin{bmatrix} n \\ v \end{bmatrix} = \begin{bmatrix} n \\ n - v \end{bmatrix}, \] (2.18)
by using (1.5) it can be easily verified that
\[ p_n(x) = (-1)^n q^{n^2 + n/2} x^n p_n \left( 1/(xq^{2n+1}) \right), \] (2.19)
from which it follows
\[ x_{n,v} \cdot x_{n,n-v+1} = q^{-(2n+1)}, \quad \nu = 1, \ldots, n. \]  
(2.20)

As a result, one can provide asymptotic formula for all zeros of the Stieltjes–Wigert polynomial \( p_n(x) \).

3. Riemann–Hilbert problems

We begin with the Riemann–Hilbert problem (RHP) of finding a \( 2 \times 2 \) matrix-valued function \( Y : \mathbb{C} \setminus [0, \infty) \rightarrow \mathbb{C}^2 \times \mathbb{C}^2 \) satisfying

\((Y_a)\) \( Y(z) \) is analytic in \( \mathbb{C} \setminus [0, \infty) \);
\((Y_b)\) for \( x \in (0, \infty) \),
\[ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \]
where \( w(x) \) is the weight function given in (1.4);
\((Y_c)\) for \( z \in \mathbb{C} \setminus [0, \infty) \),
\[ Y(z) = \left[ I + O \left( \frac{1}{z} \right) \right] \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as} \quad z \rightarrow \infty; \]
\((Y_d)\) as \( z \rightarrow 0 \),
\[ Y(z) = O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \]
where the notation on the right-hand side of the equality is used to indicate that all entries in the \( 2 \times 2 \) matrix \( Y(z) \) are \( O(1) \), i.e., bounded.

By the now well-known theorem of Fokas, Its and Kitaev [6], the unique solution to the above problem is given by
\[ Y(z) = \begin{pmatrix} \pi_n(z) & C[\pi_n w](z) \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & -2\pi i \gamma_{n-1}^2 C[\pi_{n-1} w](z) \end{pmatrix} \]
for \( z \in \mathbb{C} \setminus [0, \infty) \), where \( C[f](z) \) denotes the Cauchy transform of \( f \) defined by
\[ C[f](z) := \frac{1}{2\pi i} \int_0^\infty \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} \setminus [0, \infty). \]

In order to normalize condition \((Y_c)\) in the RHP for \( Y \), we first note that the \( g \)-function in (2.6) satisfies the jump conditions
\[ g_+(x) - g_-(x) = 2\pi i, \quad x < \alpha_n, \]  
(3.1)
and
\[ g_+(x) - g_-(x) = 2\pi i \int_{x}^{\beta_n} \mu_n(s) ds, \quad \alpha_n < x < \beta_n. \]  
(3.2)

On account of (2.6) and (3.1), one readily sees that \( e^{\mu g(z)} \) can be analytically extended to \( \mathbb{C} \setminus [\alpha_n, \beta_n] \), and
\[ e^{\mu g(z)} = z^n \left[ 1 + O \left( \frac{1}{z} \right) \right] \quad \text{as} \quad z \rightarrow \infty. \]  
(3.3)

As usual, we let \( \sigma_3 \) denote the Pauli matrix
\[ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
(3.4)
For convenience, we also set
\[ \varphi(z) := z - \frac{\alpha_n + \beta_n}{2} + \sqrt{(z - \alpha_n)(z - \beta_n)}, \] (3.5)
which is analytic in \( \mathbb{C} \setminus [\alpha_n, \beta_n] \). Now we define the first transformation
\[ T(z) = e^{-\frac{i}{2} l_n \sigma_3} Y(z) e^{\left[ -N \varphi + \frac{1}{2} \log(\varphi/2) \right] \sigma_3} e^{\frac{i}{2} l_n \sigma_3}, \] (3.6)
where \( N := n + 1/2 \) and \( l_n \) is a constant to be determined. (We have already mentioned the constant \( l_n \) in (2.10).)

From the RHP for \( Y \), it follows by a straightforward calculation that \( T \) is the unique solution of the following Riemann–Hilbert problem:

- \( (T_a) \) \( T(z) \) is analytic in \( \mathbb{C} \setminus [0, \infty) \);
- \( (T_b) \) for \( x \in (0, \infty) \),
  \[ T_+ (x) = T_- (x) \begin{pmatrix} J_{11} (x) & J_{12} (x) \\ J_{21} (x) & J_{22} (x) \end{pmatrix}, \]
  where
  \[ J_{11} (x) = e^{-N \left( g_+ (x) - g_- (x) \right) + \frac{1}{2} \log(\varphi_+/\varphi_-)}, \]
  \[ J_{12} (x) = e^{N g_+ (x) + N g_- (x) + \log w - \frac{1}{2} \log(\varphi_+/\varphi_-) - l_n}, \]
  \[ J_{22} (x) = e^{-N \left( g_+ (x) - g_- (x) \right) - \frac{1}{2} \log(\varphi_+/\varphi_-)}, \]
  and \( J_{21} (x) = 0 \);
- \( (T_c) \) \( T(z) \) behaves like the identity matrix at infinity
  \[ T(z) = I + O \left( \frac{1}{z} \right), \quad \text{as } z \to \infty, \]
  for \( z \in \mathbb{C} \setminus [0, \infty) \);
- \( (T_d) \) as \( z \to 0 \),
  \[ T(z) = O \left( \frac{1}{z} \right); \]
  see \( (Y_d) \) for an explanation of the notation.

We wish to choose the probability function \( \mu_n (s) \) in (2.1) and the constant \( l_n \) mentioned in (2.10) so that the entry \( J_{12} (x) \) in the jump matrix in condition \( (T_b) \) becomes 1 for \( x \in (\alpha_n, \beta_n) \). Thus, we set
\[ g_+ (x) + g_- (x) - \frac{k^2}{N} \log^2 x + \frac{1}{N} \log(k/\sqrt{\pi}) - \frac{1}{N} \log \frac{\beta_n - \alpha_n}{4} - \frac{1}{N} l_n = 0 \] (3.7)
for \( x \in (\alpha_n, \beta_n) \). Since \( g_+ (\beta_n) = g_- (\beta_n) \) by (3.2), formula (2.10) now follows. Differentiating (3.7) gives
\[ G_+ (x) + G_- (x) = \frac{2k^2 i}{\pi N} \frac{\log x}{x}, \quad x \in (\alpha_n, \beta_n), \]
where
\[ G(z) := \frac{1}{\pi i} \int_{\alpha_n}^{\beta_n} \frac{1}{s - z} \mu_n (s) \, ds, \quad z \in \mathbb{C} \setminus [\alpha_n, \beta_n]. \]
(3.9)

For convenience, we put
\[ \tilde{G}(z) := \frac{G(z)}{\sqrt{(z - \alpha_n)(z - \beta_n)}}. \]
Since
\[ \tilde{G}_+(x) - \tilde{G}_-(x) = \frac{2k^2 \log x}{\pi N x \sqrt{(x - \alpha_n)(\beta_n - x)}}, \quad x \in (\alpha_n, \beta_n), \]
we can solve this scalar Riemann–Hilbert problem to give
\[ \tilde{G}(z) = \frac{1}{2\pi i} \int_{\alpha_n}^{\beta_n} \frac{2k^2 \log s}{\pi N s \sqrt{(s - \alpha_n)(\beta_n - s)}} \sqrt{(s - \alpha_n)(\beta_n - s)} ds \]
or, equivalently,
\[ G(z) = \frac{\sqrt{(z - \alpha_n)(z - \beta_n)}}{2\pi iz} \int_{\alpha_n}^{\beta_n} \frac{2k^2 \log s}{\pi N \sqrt{(s - \alpha_n)(\beta_n - s)}} \left( \frac{1}{s - z} - \frac{1}{s} \right) ds. \tag{3.10} \]
From (3.9) and (2.1), it is easily seen that \( G(z) \to 0 \) and \( zG(z) \to -1/(\pi i) \) as \( z \to \infty \). Therefore we have from (3.10)
\[ \int_{\alpha_n}^{\beta_n} \frac{\log s}{s \sqrt{(s - \alpha_n)(\beta_n - s)}} ds = 0 \tag{3.11} \]
and
\[ \int_{\alpha_n}^{\beta_n} \frac{\log s}{\sqrt{(s - \alpha_n)(\beta_n - s)}} ds = \frac{N\pi}{k^2}. \tag{3.12} \]
The last two equations provide a system of two equations in two unknowns \( \alpha_n \) and \( \beta_n \). To solve for these two unknowns, we need the following result:

**Lemma 1.** For any \( 0 < a < b < \infty \), we have
\[ \int_a^b \frac{\log s}{\sqrt{(s - a)(b - s)}} ds = 2\pi \log \frac{\sqrt{a} + \sqrt{b}}{2} \tag{3.13} \]
and
\[ \int_a^b \frac{\log s}{s \sqrt{(s - a)(b - s)}} ds = -\frac{2\pi}{\sqrt{ab}} \log \frac{\sqrt{a} + \sqrt{b}}{2\sqrt{ab}}. \tag{3.14} \]

**Proof.** Let \( I(a, b) \) denote the integral in (3.13). The change of variable \( s = a \cos^2 \theta + b \sin^2 \theta \) gives
\[ I(a, b) = 2 \int_0^{\pi/2} \log(a \cos^2 \theta + b \sin^2 \theta) \, d\theta, \]
which, upon making the further change of variable \( \theta = \frac{1}{2} \phi \), becomes
\[ I(a, b) = \int_0^\pi \log \left( \frac{a + b}{2} - \frac{b - a}{2} \cos \phi \right) d\phi. \]
Let \( r = (\sqrt{b} - \sqrt{a})/(\sqrt{a} + \sqrt{b}) \). It can be verified by straightforward calculation that the last equation is equivalent to
\[ I(a, b) = \int_0^\pi \log \left( \frac{\sqrt{a} + \sqrt{b}}{2} \right)^2 + \log(1 + r^2 - 2r \cos \theta) \]
Expanding $\log(1 + r^2 - 2r \cos \theta)$ into a Fourier cosine series shows that the second integral on the right-hand side is actually equal to zero. Hence

$$I(a, b) = 2\pi \log \left( \frac{\sqrt{a} + \sqrt{b}}{2} \right),$$

(i.e., (3.13) is proved). To establish (3.14), we simply note that the change of variable $s = 1/t$ gives

$$\int_a^b \frac{\log s}{s \sqrt{(s-a)(b-s)}} \, ds = -\frac{1}{\sqrt{ab}} I\left(1/b, 1/a\right).$$

(3.16)

Coupling (3.15) and (3.16) yields (3.14). □

Applying the results in Lemma 1 to (3.11) and (3.12), we obtain

$$2\sqrt{\alpha_n\beta_n} = \sqrt{\alpha_n} + \sqrt{\beta_n}$$

(3.17)

and

$$\log \frac{\sqrt{\alpha_n} + \sqrt{\beta_n}}{2} = \frac{N}{2k^2},$$

(3.18)

from which it follows that

$$\alpha_n = 2e^{N/k^2} - e^{N/2k^2} - 2e^{N/2k^2} \sqrt{e^{N/k^2} - e^{N/2k^2}}$$

(3.19)

and

$$\beta_n = e^{N/k^2} - e^{N/2k^2} + 2e^{N/2k^2} \sqrt{e^{N/k^2} - e^{N/2k^2}}$$

(3.20)

see (2.2) and (2.3).

**Lemma 2.** For any $0 < a < b < \infty$ and $z \in \mathbb{C} \setminus (-\infty, 0] \cup [a, b]$, we have

$$\int_0^\infty \frac{1}{\sqrt{(a+s)(b+s)}} \frac{ds}{s + z} = \frac{1}{\sqrt{(z-a)(z-b)}} \log \left[ \frac{z + \sqrt{ab} + \sqrt{(z-a)(z-b)}}{(\sqrt{a} + \sqrt{b})^2z} \right].$$

(3.21)

**Proof.** Let $I(z)$ denote the integral in (3.21), and make the change of variable

$$s = \frac{b-a}{4} \left( \frac{t + 1}{t} \right) - \frac{a + b}{2}.$$

This transformation takes the $s$-interval $[0, +\infty)$ onto the $t$-interval $[1/r, +\infty)$, where $r = (\sqrt{b} - \sqrt{a})/(\sqrt{b} + \sqrt{a})$. A simple calculation gives

$$I(z) = \int_{1/r}^\infty \frac{dt}{1/4(b-a)(t^2 + 1) + (z - \frac{1}{2}(b+a))t}.$$  

Let

$$t_+ = \frac{2}{b-a} \left[ -\left( \frac{z - b + a}{2} \right) \pm \sqrt{(z-a)(z-b)} \right],$$

and note that

$$t_+ - t_-= \frac{4}{b-a} \sqrt{(z-a)(z-b)}.$$

By partial fraction,

\[ I(z) = \frac{1}{\sqrt{(z - a)(z - b)}} \int_{1/r}^{\infty} \left( \frac{1}{t - t_1} - \frac{1}{t - t_2} \right) \, dt. \]

An integration yields

\[ I(z) = \frac{1}{\sqrt{(z - a)(z - b)}} \left( \log z + \sqrt{ab} + \sqrt{(z - a)(z - b)} \right) \frac{\sqrt{z}}{z} + \left( \sqrt{ab} - \sqrt{(z - a)(z - b)} \right), \]

which is equivalent to (3.21). □

The above result is used to give an explicit formula for the probability density function \( \mu_n(s) \) in (2.1). To see this, we consider a contour \( \Gamma \) which consists of a large circle \( \Gamma_R \), a loop embracing the negative real-axis and a closed curve \( \Gamma_1 \) surrounding the interval \((\alpha_n, \beta_n)\); see Fig. 1. By Cauchy’s theorem, we have

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\log \zeta}{\sqrt{(\zeta - \alpha_n)(\zeta - \beta_n)}} \frac{d\zeta}{\zeta - z} = \frac{\log z}{\sqrt{(z - \alpha_n)(z - \beta_n)}},
\]

for \( z \in \mathbb{C} \setminus (-\infty, 0] \cup [\alpha_n, \beta_n] \). It is readily seen that the integral on \( \Gamma_R \) tends to zero as \( R \to \infty \), the integral on \( \Gamma_1 \) is equal to

\[
\frac{2}{i} \int_{\alpha_n}^{\beta_n} \frac{\log s}{\sqrt{(s - \alpha_n)(\beta_n - s)}} \frac{ds}{s - z},
\]

and the integral on \( \Sigma_+ \cup \Sigma_- \) is equal to \( 2\pi i f(z) \), where \( f(z) \) is the integral given in (3.21) with \( a = \alpha_n \) and \( b = \beta_n \). Therefore, it follows from (3.10), (3.11) and (3.18) that

\[
G(z) = \frac{i}{\pi z} - \frac{2k^2 i}{N \pi z} \log \frac{z + \sqrt{\alpha_n \beta_n} + \sqrt{(z - \alpha_n)(z - \beta_n)}}{2z}. \tag{3.22}
\]

Since \( G_+(x) = \lim_{\epsilon \to 0} G(x + i\epsilon) \), by (3.9)

\[
G_+(x) = \lim_{\epsilon \to 0} \frac{\beta_n}{i \pi} \int_{\alpha_n}^{\beta_n} \frac{(s - x) + i\epsilon}{(s - x)^2 + \epsilon^2} \mu_n(s) \, ds = \mu_n(x) + \frac{i}{\pi} \text{P.V.} \int_{\alpha_n}^{\beta_n} \frac{1}{x - s} \mu_n(s) \, ds.
\]
for \( x \in (\alpha_n, \beta_n) \). Hence
\[
\mu_n(x) = \Re G_+(x) = \frac{2k^2}{N\pi x} \Im \left\{ \log \frac{x + \sqrt{\alpha_n\beta_n} + i\sqrt{(x - \alpha_n)(\beta_n - x)}}{2x} \right\}
\]
or, equivalently,
\[
\mu_n(x) = \frac{2k^2}{N\pi x} \arctan \frac{\sqrt{(x - \alpha_n)(\beta_n - x)}}{x + \sqrt{\alpha_n\beta_n}}.
\] (3.23)

We next derive an asymptotic formula for the constant \( \ln l_n \) in (2.10). From (3.1) and (2.5), we note that the function
\[ g(z) = \log(z - \alpha_n) \]
is analytic in \( \mathbb{C} \setminus [\alpha_n, \beta_n] \), and
\[
g(z) - \log(z - \alpha_n) = O \left( \frac{1}{z} \right) \quad \text{as} \quad z \to \infty.
\] (3.24)

Define the function
\[ H(z) := N \frac{g(z) - \log(z - \alpha_n)}{\sqrt{(z - \alpha_n)(z - \beta_n)}}. \]

It can be readily verified that
\[
H_+(x) - H_-(x) = \frac{N[g_+(x) + g_-(x)] - 2N \log(x - \alpha_n)}{i\sqrt{(x - \alpha_n)(\beta_n - x)}}.
\]

By the Plemelj formula,
\[
g(z) - \log(z - \alpha_n) = \frac{\sqrt{(z - \alpha_n)(z - \beta_n)}}{2N} \frac{1}{2\pi i} \int_{\alpha_n}^{\beta_n} \frac{H_+(x) - H_-(x)}{x - z} \, dx.
\]

From (3.7), it follows that
\[
g(z) - \log(z - \alpha_n) = -\frac{\sqrt{(z - \alpha_n)(z - \beta_n)}}{2\pi N}
\times \int_{\alpha_n}^{\beta_n} \frac{k^2 \log^2 x - 2N \log(x - \alpha_n) + \left[ l_n + \log \left( 1 + \frac{1}{4} (\beta_n - \alpha_n) \right) \right] - \log(k/\sqrt{\pi})}{(x - z) \sqrt{(x - \alpha_n)(\beta_n - x)}} \, dx.
\]

Now let \( z \to \infty \). In view of (3.24), we have
\[
\int_{\alpha_n}^{\beta_n} \frac{k^2 \log^2 x}{\sqrt{(x - \alpha_n)(\beta_n - x)}} \, dx - 2N \int_{\alpha_n}^{\beta_n} \frac{\log(x - \alpha_n)}{\sqrt{(x - \alpha_n)(\beta_n - x)}} \, dx
\]
\[
+ \left[ l_n + \log \frac{\beta_n - \alpha_n}{4} - \log \frac{k}{\sqrt{\pi}} \right] \int_{\alpha_n}^{\beta_n} \frac{dx}{\sqrt{(x - \alpha_n)(\beta_n - x)}} = 0.
\] (3.25)

Let the three integrals on the left-hand side be denoted by \( I_1, I_2 \) and \( I_3 \), respectively. As in Lemma 1, one can show that
\[
I_1 = k^2 \int_0^\pi \left[ \log \left( \frac{\sqrt{\alpha_n} + \sqrt{\beta_n}}{2} \right)^2 + \log(1 + r^2 - 2r \cos \theta) \right]^2 \, d\theta,
\]
where \( r = (\sqrt{\beta_n} - \sqrt{\alpha_n})/(\sqrt{\beta_n} + \sqrt{\alpha_n}) \). Expanding
\[
\log(1 + r^2 - 2r \cos \theta) = \log \left[ (1 - r e^{i\theta})(1 - r e^{-i\theta}) \right]
\]
into a Fourier cosine series
\[ -\sum_{l=1}^{\infty} \frac{e^{il\theta} + e^{-il\theta}}{l} r^l = -2 \sum_{l=1}^{\infty} \frac{\cos l\theta}{l} r^l, \]
one obtains, upon termwise integration,
\[ I_1 = 4k^2 \left[ \pi \log^2 \left( \sqrt{\alpha_n} + \sqrt{\beta_n} \right) + \frac{\pi}{2} \sum_{l=1}^{\infty} \frac{r^{2l}}{l^2} \right]; \]
cf. (3.15). On account of (3.18), (3.19) and (3.20), we have \( r = 1 + O(q^N) \), and
\[ I_1 = \frac{\pi N^2}{k^2} + \frac{k^2 \pi^3}{3} + O(Nq^N). \]
(3.26)

Similarly, one can show that
\[ I_2 = 2 \int_0^{\pi/2} \log\left( (\beta_n - \alpha_n) \sin^2 \theta \right) d\theta = \pi \log(\beta_n - \alpha_n) + 4 \int_0^{\pi/2} \log \sin \theta d\theta \]
\[ = \pi \log(\beta_n - \alpha_n) - 2\pi \log 2 = \pi \log \frac{\beta_n - \alpha_n}{4} \]
(3.27)
and
\[ I_3 = \int_{\alpha_n}^{\beta_n} \frac{1}{\sqrt{(x - \alpha_n)(\beta_n - x)}} dx = \pi. \]
(3.28)

Inserting (3.26)–(3.28) in (3.25), and observing the behavior
\[ \frac{\beta_n - \alpha_n}{4} = e^{N/k^2} \left[ 1 + O(q^N) \right], \]
we obtain
\[ l_n = \frac{N(N - 1)}{k^2} - \frac{k^2 \pi^2}{3} + \log \left( \frac{k}{\sqrt{\pi}} \right) + O(Nq^N). \]
(3.29)

Let \( v_n(z) := \pi i G(z) + k^2 \log z/(Nz) \). Clearly, \( (v_n)_{\pm}(x) = \pm \pi i \mu_n(x) \) for \( x \in [\alpha_n, \beta_n] \). By (3.22) and (3.18), it can be shown that
\[ v_n(z) = \frac{k^2}{Nz} \left\{ -\log\left[ \sqrt{\alpha_n} + \sqrt{\beta_n} \right]^2 z \right\} + 2 \log(z + \sqrt{\alpha_n \beta_n} + \sqrt{(z - \alpha_n)(z - \beta_n)}) \}
(3.30)
Define
\[ \phi_n(z) := \int_{\beta_n}^{z} v_n(\zeta) d\zeta, \quad z \in \mathbb{C} \setminus (-\infty, \beta_n], \]
(3.31)
and
\[ \tilde{\phi}_n(z) := \int_{\alpha_n}^{z} v_n(\zeta) d\zeta, \quad z \in \mathbb{C} \setminus (-\infty, 0] \cup [\alpha_n, \infty); \]
(3.32)
cf. (2.5). If \( f(x) \) denotes the expression inside the curly brackets in (3.30) for \( x \in (\beta_n, \infty) \), then it is easily seen that this function vanishes at \( \beta_n \) and its derivative is positive in \( \beta_n < x < \infty \) (since \( x > \sqrt{(x - \alpha_n)(x - \beta_n)} + \sqrt{\alpha_n \beta_n} \)). Thus
\[ \phi_n(x) > 0 \quad \text{for} \quad \beta_n < x < \infty. \]
Similarly, one can show that
\[ \tilde{\phi}_n(x) > 0 \quad \text{for } 0 < x < \alpha_n. \] (3.34)

From (3.1) and (3.2), we have
\[ g_+(x) - g_-(x) = -\left[ (\phi_n)_+(x) - (\phi_n)_-(x) \right] \] (3.35)
for \( x \in (0, \beta_n) \); see also (2.7) and (2.8). On account of this, it follows that \( g(z) + \phi_n(z) \) can be analytically extended to \( \mathbb{C} \setminus (-\infty, 0] \). Since \( (\nu_n)_+(x) = -(\nu_n)_-(x) \) for \( x \in (\alpha_n, \beta_n) \), by (3.31)
\[ (\phi_n)_+(x) + (\phi_n)_-(x) = 0, \quad x \in (\alpha_n, \beta_n), \] (3.36)
and
\[ g(x) + \phi_n(x) = \frac{1}{2} \left[ (g + \phi_n)_+(x) + (g + \phi_n)_-(x) \right] = \frac{1}{2} \left[ g_+(x) + g_-(x) \right], \quad x \in (\alpha_n, \beta_n). \] (3.37)

Using analytic continuation, we obtain from (3.7)
\[ g(z) + \phi_n(z) = -\frac{1}{2N} \log w(z) + \frac{1}{2N} \log \frac{\beta_n - \alpha_n}{4} + \frac{1}{2N} I_n \] (3.38)
for \( z \in \mathbb{C} \setminus (-\infty, 0] \).

The jump matrix for \( T \) in condition \((T_b)\) can be written as follows:
\[ T_+(x) = T_-(x) \begin{pmatrix} e^{-2N(\phi_n)_-(x)} + \frac{1}{2} \log(\varphi_+ / \varphi_-) & 0 \\ 0 & e^{-2N(\phi_n)_+(x)} - \frac{1}{2} \log(\varphi_+ / \varphi_-) \end{pmatrix} \] (3.39)
for \( x \in (\alpha_n, \beta_n) \),
\[ T_+(x) = T_-(x) \begin{pmatrix} 1 & e^{-2N\phi_n(x)} \\ 0 & 1 \end{pmatrix}, \quad x > \beta_n, \] (3.40)
\[ T_+(x) = T_-(x) \begin{pmatrix} 1 & e^{-2N\tilde{\phi}_n(x)} \\ 0 & 1 \end{pmatrix}, \quad 0 < x < \alpha_n. \] (3.41)

Let \( \Sigma_S = \bigcup_{i=1}^5 \Sigma_i \) denote the contour shown in Fig. 2, and set
\[ S(z) := T(z) \quad \text{for } z \text{ outside the lens-shaped region,} \] (3.42)
\[ S(z) := T(z) \begin{pmatrix} 1 & 0 \\ -e^{2N\phi_n + \log[2\varphi/(\beta_n - \alpha_n)]} & 1 \end{pmatrix} \] in the upper lens region, (3.43)
\[ S(z) := T(z) \begin{pmatrix} 1 & 0 \\ e^{2N\phi_n + \log[2\varphi/(\beta_n - \alpha_n)]} & 1 \end{pmatrix} \] in the lower lens region. (3.44)

Furthermore, we define the jump matrix
\[ J_S(z) = \begin{pmatrix} 1 & 0 \\ e^{2N\phi_n + \log[2\varphi/(\beta_n - \alpha_n)]} & 1 \end{pmatrix}, \quad z \in \Sigma_1, \] (3.45)
\[ J_S(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x \in \Sigma_2, \] (3.46)
\[ J_S(z) = \begin{pmatrix} \frac{1}{e^{2N\phi_n + \log[2\phi/(\beta_n - \alpha_n)]}} & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in \Sigma_3, \quad (3.47) \]

\[ J_S(x) = \begin{pmatrix} 1 & e^{-2N\phi_n(x)} \\ 0 & 1 \end{pmatrix}, \quad x \in \Sigma_3, \quad (3.48) \]

and

\[ J_S(x) = \begin{pmatrix} 1 & e^{-2N\phi_n(x)} \\ 0 & 1 \end{pmatrix}, \quad x \in \Sigma_5. \quad (3.49) \]

It is readily verified that \( S \) satisfies the following conditions:

\( (S_a) \) \( S(z) \) is analytic in \( \mathbb{C} \setminus \Sigma_S \);

\( (S_b) \) for \( z \in \Sigma_S \),

\[ S_+(z) = S_-(z)J_S(z); \]

\( (S_c) \) as \( |z| \to \infty \),

\[ S(z) = I + O\left(\frac{1}{z}\right); \]

\( (S_d) \) as \( |z| \to 0 \),

\[ S(z) = O\left(\frac{1}{1/z}\right). \]

From (3.31), it can be shown that

\[ \text{Re} \, \phi_n(z) < 0 \quad \text{in the upper lens region.} \quad (3.50) \]

In fact, we also have

\[ \text{Re} \, \phi_n(z) < 0 \quad \text{in the lower lens region.} \quad (3.51) \]

These together with (3.33) and (3.34) imply that the jump matrix \( J_S(z) \) tends to the identity matrix as \( n \to \infty \), for \( z \in \Sigma_S/\Sigma_2 \). For \( z \in \Sigma_2 \), \( J_S(z) \) is the constant matrix given in (3.46). It is therefore natural to suggest that for large \( n \), the solution of the RHP for \( S \) may behave asymptotically like the solution of the following RHP for \( N \):

\( (N_a) \) \( N(z) \) is analytic in \( \mathbb{C} \setminus \{\alpha_n, \beta_n\} \);

\( (N_b) \) for \( x \in (\alpha_n, \beta_n) \),

\[ N_+(x) = N_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \]

\( (N_c) \) as \( |z| \to \infty \),

\[ N(z) = I + O\left(\frac{1}{z}\right). \]

This problem can be solved explicitly, and its solution is

\[ N(z) = \frac{1}{2} \begin{pmatrix} a(z) + a(z)^{-1} & i(a(z)^{-1} - a(z)) \\ i(a(z) - a(z)^{-1}) & a(z) + a(z)^{-1} \end{pmatrix}, \quad (3.52) \]

where

\[ a(z) = \left(\frac{z - \beta_n}{z - \alpha_n}\right)^{1/4}. \quad (3.53) \]

By (3.50) and (3.51), the equations in (3.42)–(3.44) imply that \( T \sim S \) as \( n \to \infty \). Since \( S \sim N \) as \( n \to \infty \), it follows

\[ T(z) \sim N(z) \quad \text{for} \quad z \in \mathbb{C} \setminus \{\alpha_n, \beta_n\}. \quad (3.54) \]
On account of (3.6), we also have

\[ Y(z) \sim e^{\frac{1}{2} \int_0^1 N(z) e^{(Ng(z) - \frac{1}{2} \log(\phi(z)/2) - \frac{1}{2} N_0) \sigma_3}. \]

(3.55)

The argument in this paragraph is only formal, we give it here just as a motivation for the asymptotic formula which we shall establish rigorously.

Note that the function \( \phi(z) \) in (3.5) can be written as

\[ \phi(z) = \frac{1}{2} \left( \sqrt{z - \alpha_n} + \sqrt{z - \beta_n} \right)^2. \]

(3.56)

Hence, the matrix \( N(z) \) in (3.52) can be expressed as

\[ N(z) = \frac{1}{\left( \frac{(\phi/2) - \frac{1}{2} \beta_n - \frac{1}{2} \sigma_3}{\phi/2} - \frac{1}{2} \log(\phi(z)/2) - \frac{1}{2} \ln \right)} \]

(3.57)

Also, observe that the exponential function in (3.55) is equal to

\[ e^{-\frac{1}{2} \int_0^1 N(z) e^{\phi(z) \frac{1}{2} \sigma_3}} \]

and that the first matrix above can be factored as

\[ \left( \begin{array}{c} \sqrt{\beta_n - \alpha_n} (\phi/2) - \frac{1}{2} \beta_n - \frac{1}{2} \sigma_3 \\ 0 \end{array} \right) \]

\[ \left( \begin{array}{c} \frac{1}{2} \sigma_3 \\ 0 \end{array} \right) \]

(3.58)

Therefore, it follows from (3.55) and (3.52) that

\[ e^{-\frac{1}{2} \int_0^1 N(z) e^{\phi(z) \frac{1}{2} \sigma_3}} \]

(3.59)

cf. (2.9). From the identities

\[ \text{Ai}(z) + \omega \text{Ai}(\omega z) + \omega^2 \text{Ai}(\omega^2 z) = 0 \]

and

\[ \text{Bi}(z) = i\omega^2 \text{Ai}(\omega^2 z) - i\omega \text{Ai}(\omega z), \]

we also have

\[ \omega \text{Ai}(\omega z) = -\frac{1}{2} \text{Ai}(z) + i \frac{1}{2} \text{Bi}(z) \]

and

\[ \omega^2 \text{Ai}(\omega^2 z) = -\frac{1}{2} \text{Ai}(z) - i \frac{1}{2} \text{Bi}(z). \]

It is now easily verified that

\[
\begin{pmatrix}
\text{Ai}(z) & -\omega^2 \text{Ai}(\omega^{-1} z) \\
\text{Ai}'(z) & -\omega \text{Ai}'(\omega^{-1} z)
\end{pmatrix}
= \begin{pmatrix}
\text{Ai}(z) & \text{Bi}(z) \\
\text{Ai}'(z) & \text{Bi}'(z)
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-\frac{1}{2} & \frac{1}{2}i
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\text{Ai}(z) & \omega \text{Ai}(\omega z) \\
\text{Ai}'(z) & \omega^2 \text{Ai}'(\omega z)
\end{pmatrix}
= \begin{pmatrix}
\text{Ai}(z) & \text{Bi}(z) \\
\text{Ai}'(z) & \text{Bi}'(z)
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-\frac{1}{2} & \frac{1}{2}i
\end{pmatrix}.
\]
Recall the well-known asymptotic formulas
\[ \text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}}, \quad \text{Ai}'(z) \sim -\frac{1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3}z^{3/2}}, \]
asymptotez \rightarrow \infty in |\text{arg } z| < \pi, and note that \( \xi_n^{1/4}(z)/[(z - \alpha_n)(z - \beta_n)]^{1/4} \) can be analytically continued to \( \mathbb{C} \setminus (-\infty, \alpha_n) \).

Define the matrix \( P(z) \) by
\[
P(z) := \sqrt{\pi} \begin{pmatrix} \frac{1}{\beta_n - \alpha_n} & 0 \\ -i(2z - \alpha_n - \beta_n) & -2i \end{pmatrix} \begin{pmatrix} (\frac{\beta_n - \alpha_n}{z - \alpha_n} N^{2/3}(\xi_n))^{1/4} & 0 \\ (\frac{\alpha_n - \beta_n}{z - \beta_n})^{1/4} \end{pmatrix}
\]

for \( 0 < \text{arg}[z - (\frac{1}{4} + \delta)] < \frac{2\pi}{5} \), where \( \delta \) is any positive number, and
\[
P(z) := \sqrt{\pi} \begin{pmatrix} \frac{1}{\beta_n - \alpha_n} & 0 \\ -i(2z - \alpha_n - \beta_n) & -2i \end{pmatrix} \begin{pmatrix} (\frac{\beta_n - \alpha_n}{z - \alpha_n} N^{2/3}(\xi_n))^{1/4} & 0 \\ (\frac{\alpha_n - \beta_n}{z - \beta_n})^{1/4} \end{pmatrix}
\]

for \( -\frac{2\pi}{3} < \text{arg}[z - (\frac{1}{4} + \delta)] < 0 \).

Introduce the function \( \tilde{\xi}_n(z) \) defined by
\[
\frac{2}{3} \xi_n^{1/3}(z) := \tilde{\phi}_n(z), \quad z \in \mathbb{C} \setminus (-\infty, 0] \cup [\alpha_n, \infty);
\]
see (3.32) and (3.59). For \( z \) in the sectors \( \frac{2\pi}{3} < \text{arg}[z - (\frac{1}{4} + \delta)] < \pi \) and \( -\pi < \text{arg}[z - (\frac{1}{4} + \delta)] < -\frac{2\pi}{3} \), we define \( P(z) \) respectively, as in (3.60a) and (3.60b) with \( \xi \) replaced by \( \tilde{\xi} \) and the quantity on the right-hand of the equations multiplied by \( (-1)^n \). Here, we take the branch of the function \( (z - \alpha_n)(z - \beta_n) \) such that it is analytic on the interval \( (-\infty, \alpha_n) \) and is positive there. Formula (3.58) becomes
\[
Y(z) \sim e^{\frac{1}{2} \ln \sigma_3} P(z) \begin{pmatrix} \frac{1}{\sqrt{w(z)}} & 0 \\ 0 & \sqrt{w(z)} \end{pmatrix}
\]
for \( z \in \mathbb{C} \setminus \mathbb{R} \cup L_+ \cup L_- \), where \( L_+ \) and \( L_- \) are the radial lines defined by
\[
L_{\pm} := \left\{ z : \text{arg} \left[ z - \left( \frac{1}{4} + \delta \right) \right] = \pm \frac{2\pi}{3} \right\}.
\]

Put
\[
\tilde{Y}(z) := P(z) \begin{pmatrix} \frac{1}{\sqrt{w(z)}} & 0 \\ 0 & \sqrt{w(z)} \end{pmatrix}.
\]

We shall show that the matrix \( \tilde{Y}(z) \) in (3.63) is the leading term in an asymptotic expansion of \( Y(z) \) multiplied by a simple diagonal matrix independent of \( z \), which holds uniformly for all \( z \in \mathbb{C} \) bounded away from the radial lines \( \text{arg}[z - (\frac{1}{4} + \delta)] = \pi \) and \( L_{\pm} \).

4. Riemann–Hilbert problem for \( \tilde{Y} \)

In this section, we verify that the matrix \( \tilde{Y}(z) \) defined in (3.63) is a solution of the Riemann–Hilbert problem:

(\( \tilde{Y}_a \)) \( \tilde{Y}(z) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \cup L_+ \cup L_- \);

(\( \tilde{Y}_b \)) \( \tilde{Y}(z) \) satisfies the jump conditions
\begin{align*}
\tilde{Y}_+(x) &= \tilde{Y}_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (0, \infty), \\
\tilde{Y}_+(z) &= M_1(z) \tilde{Y}_-(z), \quad z \in L_+, 
\end{align*} 
\tag{4.1}

\text{and}

\begin{align*}
\tilde{Y}_+(z) &= M_2(z) \tilde{Y}_-(z), \quad z \in (-\infty, 0), 
\end{align*} 
\tag{4.3}

where the jump matrices \( M_1(z) \) and \( M_2(z) \) have uniform asymptotic expansions of the form

\begin{align*}
M_k(z) &\sim I + \sum_{j=1}^{\infty} \frac{M_{k,j}(z)}{n_j}, \quad k = 1, 2, 
\end{align*} 
\tag{4.4}

and the coefficient matrices satisfy

\begin{align*}
\| M_{k,j}(z) \| \leq M_{k,j} / \log^2 |z|, \quad j = 1, 2, \ldots; \quad k = 1, 2, 
\end{align*} 
\tag{4.5}

for \( z \in L_+ \cup L_- \cup (-\infty, 0) \) with \( \| \cdot \| \) being a matrix norm;

\((\tilde{Y}_c)\) as \( z \to \infty, \)

\begin{align*}
\tilde{Y}(z) &\sim e^{-\frac{3}{2} n \sigma_1} \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}; 
\end{align*} 
\tag{4.6}

\((\tilde{Y}_d)\) as \( z \to 0, \)

\begin{align*}
\tilde{Y}(z) &= O \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. 
\end{align*} 
\tag{4.7}

We first establish the jump condition (4.1). From (3.60), it is easy to show that

\begin{align*}
P_+(x) = P_-(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} 
\end{align*} 
\tag{4.8}

for \( x \in (0, \infty) \). Hence

\begin{align*}
\tilde{Y}^{-1}_-(x) \tilde{Y}_+(x) &= \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix};
\end{align*}

i.e., (4.1) holds. Next, we verify the jump condition (4.3). Put

\begin{align*}
M_2(x) := \tilde{Y}_+(x) \tilde{Y}^{-1}_-(x). 
\end{align*} 
\tag{4.9}

Inserting (3.60) into (4.9) gives

\begin{align*}
M_2(x) &= \left( \frac{1}{2i(z^2 - \alpha_n^2)} \right) \begin{pmatrix} 0 & (\beta_n - \alpha_n)^{1/2} \\ -2i & \frac{1}{[(z - \alpha_n)(z - \beta_n)]^{1/2}} \end{pmatrix} \sigma_3 \begin{pmatrix} m_{11} \\ m_{21} \end{pmatrix} \\
&\quad \times \left( \frac{1}{[(z - \alpha_n)(z - \beta_n)]^{1/2}} \right)^{\frac{1}{2}} \begin{pmatrix} \frac{1}{2(z - \alpha_n - \beta_n)} & 0 \\ \xi_n & \frac{1}{z} - i \end{pmatrix}, 
\end{align*} 
\tag{4.10}

where

\begin{align*}
m_{11} &= \pi \xi_n^{1/4} \xi_n^{-1/4} \left[ A_i(N^2/3 \xi_n) B_i(N^2/3 \xi_n) \sqrt{w_+ / w_-} - B_i(N^2/3 \xi_n) A_i(N^2/3 \xi_n) \sqrt{w_- / w_+} \right] \\
&\quad + i\pi \left( \sqrt{\frac{w_+}{w_-}} + \sqrt{\frac{w_-}{w_+}} \right) \xi_n^{1/4} \xi_n^{-1/4} \left[ A_i(N^2/3 \xi_n) + B_i(N^2/3 \xi_n) \right], \\
m_{12} &= \pi n^{1/3} \xi_n^{1/3} \xi_n^{-1/4} \left[ A_i(N^2/3 \xi_n) B_i(N^2/3 \xi_n) \sqrt{w_+ / w_-} + B_i(N^2/3 \xi_n) A_i(N^2/3 \xi_n) \sqrt{w_- / w_+} \right] \\
&\quad + i\pi n^{1/3} \left( \sqrt{\frac{w_+}{w_-}} + \sqrt{\frac{w_-}{w_+}} \right) \xi_n^{1/4} \xi_n^{-1/4} \left[ A_i(N^2/3 \xi_n) A_i(N^2/3 \xi_n) \right],
\end{align*}
There exists a sequence of matrices \( m_{21} = \pi n^{-1/3} \zeta^{-1/4} \zeta_-^{-1/4} \left[ \text{Ai}'(N^2/3 \zeta_+) \text{Bi}'(N^2/3 \zeta_-) \sqrt{w_+/w_-} - \text{Bi}'(N^2/3 \zeta_+) \text{Ai}'(N^2/3 \zeta_-) \sqrt{w_-/w_+} \right] \)

\[ + i \pi n^{-1/3} \left( \sqrt{w_+/w_-} + \sqrt{w_-/w_+} \right) \zeta^{-1/4} \zeta_-^{-1/4} \text{Ai}'(N^2/3 \zeta_+) \text{Ai}'(N^2/3 \zeta_-), \]

\[ m_{22} = \pi \zeta_-^{-1/4} \zeta_+^{-1/4} \left[ - \text{Ai}'(N^2/3 \zeta_+) \text{Bi}'(N^2/3 \zeta_-) \sqrt{w_+/w_-} + \text{Bi}'(N^2/3 \zeta_+) \text{Ai}(N^2/3 \zeta_-) \sqrt{w_-/w_+} \right] \]

\[ - i \pi \left( \sqrt{w_+/w_-} + \sqrt{w_-/w_+} \right) \zeta_+^{-1/4} \zeta_-^{-1/4} \text{Ai}'(N^2/3 \zeta_+) \text{Ai}(N^2/3 \zeta_-). \] (4.11)

For convenience, in (4.11) we have dropped the dependence of \( \zeta \) on \( n \), i.e., \( \zeta(z) = \tilde{\zeta}_n(z) \). Now we recall the well-known asymptotic expansions [9, pp. 392–393]

\[
\text{Ai}(z) \sim \frac{z^{-1/4}}{2\sqrt{\pi}} e^{-\eta} \sum_{s=0}^{\infty} \frac{(-1)^s U_s}{\eta^s}, \quad \text{Ai}'(z) \sim -\frac{z^{1/4}}{2\sqrt{\pi}} e^{-\eta} \sum_{s=0}^{\infty} (-1)^s V_s/\eta^s, \]

\[
\text{Bi}(z) \sim \frac{z^{-1/4}}{\pi} e^{\eta} \sum_{s=0}^{\infty} \frac{U_s}{\eta^s}, \quad \text{Bi}'(z) \sim \frac{z^{1/4}}{\pi} e^{\eta} \sum_{s=0}^{\infty} \frac{V_s}{\eta^s}, \]

where \( \eta = 2 \zeta^{-3/2} \) and \( u_s, v_s \) are constants with \( u_0 = v_0 = 1 \); see the formulas immediately after (3.59). By a combination of (3.38), (3.59), (4.11) and (4.12), we can easily verify that the asymptotic expansion in (4.4) holds with \( k = 2 \), thus demonstrating (4.3). In a similar manner, one can prove (4.2).

5. Asymptotic expansion of \( Y(z) \)

Let \( R(z) := e^{-\frac{1}{2} \int_{[0, 1]} Y(z) Y^{-1}(z) \, dz} \). The jump conditions \( (Y_b) \) and \( (\tilde{Y}_b) \) imply that \( Y(z) \) and \( \tilde{Y}(z) \) have the same jump matrix on \((0, \infty)\). Hence, \( R_+(x) = R_-(x) \) for \( x \in (0, \infty) \), and \( R(z) \) can be analytically extended to \( \mathbb{C} \setminus L \), where \( L := L_+ \cup L_- \cup (-\infty, 0) \). On account of (4.2) and (4.3), it is readily shown that \( R(z) \) is a solution of the Riemann–Hilbert problem:

\( (R_a) \) \( R(z) \) is analytic for \( z \in \mathbb{C} \setminus L \),

\( (R_b) \) for \( z \in L \),

\( R_+(z) = R_-(z) \left[ I + \Delta(z) \right], \) (5.1)

where \( I + \Delta(z) := M_1^{-1}(z) \) for \( z \in L_+ \), \( I + \Delta(z) := M_2^{-1}(z) \) for \( z \in (-\infty, 0) \), and

\( I + \Delta(z) \sim I + \sum_{s=1}^{\infty} \frac{\Delta_s(z)}{n^s}, \) as \( n \to \infty, \) (5.2)

\( (R_c) \) as \( z \to \infty, \)

\( R(z) \to I, \quad z \notin L, \) (5.3)

\( (R_d) \) as \( z \to 0, \)

\( R(z) = O \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right). \) (5.4)

**Lemma 3.** There exists a sequence of matrices \( \{R_k(z)\} \) with the property that for every \( p \geq 1 \), there is a constant \( C_p > 0 \) such that

\[ \left\| R(z) - I - \sum_{k=1}^{p} \frac{R_k(z)}{n^k} \right\| \leq \frac{C_p}{n^{p+1}} \] (5.5)

for all \( z \) strictly bounded away from \( L \), where \( \| \cdot \| \) denotes any matrix norm.
**Proof.** From (5.1), we have \( R_+ = R_- + R_\Delta \) (or, equivalently, \( (R - I)_+ = (R - I)_- + R_\Delta \)). Hence, by the Plemelj formula [6, p. 590],

\[
R(z) = I + \frac{1}{2\pi i} \int_L \frac{R_-(\xi) \Delta(\xi)}{\xi - z} \, d\xi, \quad z \in \mathbb{C} \setminus L.
\] (5.6)

In (5.6), we now insert the formal series

\[
R(z) = I + \sum_{k=1}^{\infty} \frac{R_k(z)}{n^k}
\]

and the asymptotic expansion in (5.2). Collecting terms with like powers of \( 1/n \), we obtain the recursive formula

\[
R_k(z) = \frac{1}{2\pi i} \int_L \sum_{j=1}^{k} (R_{k-j})_-(\xi) \Delta_j(\xi) \frac{d\xi}{\xi - z}, \quad z \in \mathbb{C} \setminus L,
\] (5.7)

\( k = 1, 2, \ldots \), where \( (R_0)_-(\xi) = I \). By induction, it can be verified that for \( z \in \mathbb{C} \setminus L \),

\[
R_k(z) = O\left(\frac{1}{\log^2 |z|}\right), \quad \text{as } z \to \infty.
\] (5.8)

To establish (5.5), we put

\[
E_p(z) := R(z) - I - \sum_{k=1}^{p} \frac{R_k(z)}{n^k}.
\] (5.9)

Using (5.1) and (5.7), it can be shown that

\[
(E_p)_+(z) = (E_p)_-(z) \left[ I + \Delta(z) \right] + F_p(z)
\] (5.10)

for \( z \in L \), where

\[
F_p(z) := \left[ I + \sum_{k=1}^{p} \frac{(R_k)_-(z)}{n^k} \right] \Delta(z) - \sum_{k=1}^{p} \frac{1}{n^k} \left[ \sum_{j=1}^{k} (R_{k-j})_-(z) \Delta_j(z) \right].
\] (5.11)

By the Plemelj formula,

\[
E_p(z) = \frac{1}{2\pi i} \int_L \frac{(E_p)_-(\xi) \Delta(\xi)}{\xi - z} \, d\xi + \frac{1}{2\pi i} \int_L \frac{F_p(\xi)}{\xi - z} \, d\xi.
\] (5.12)

Now define the sequence \( \{E_{p}^{(l)}(z): l = 0, 1, 2, \ldots\} \) successively by \( E_{p}^{(0)}(z) = 0 \), and

\[
E_{p}^{(l)}(z) = \frac{1}{2\pi i} \int_L \frac{(E_{p}^{(l-1)})(\xi) \Delta(\xi)}{\xi - z} \, d\xi + \frac{1}{2\pi i} \int_L \frac{F_p(\xi)}{\xi - z} \, d\xi
\] (5.13)

for \( l = 1, 2, \ldots \). Note that we can rewrite (5.11) as

\[
F_p(z) = \left[ I + \sum_{k=1}^{p} \frac{(R_k)_-(z)}{n^k} \right] \left[ \Delta(z) - \sum_{k=1}^{p} \frac{\Delta_k(z)}{n^k} \right] + \sum_{k=p+1}^{2p} \frac{1}{n^k} \left[ \sum_{j=k-p}^{p} (R_{k-j})_-(z) \Delta_j(z) \right],
\] (5.14)

and that the matrix norm of \( F_p(z) \) is bounded by \( n^{-(p+1)} \), as \( n \to \infty \), uniformly for \( z \in L \). Writing

\[
E_{p}^{(l)}(z) = \sum_{i=0}^{l-1} \left[ E_{p}^{(i+1)}(z) - E_{p}^{(i)}(z) \right],
\] (5.15)

and applying the usual argument used in the method of successive approximation, one can show that the limit

\[
\lim_{l \to \infty} E_{p}^{(l)}(z) = E_p(z)
\] (5.16)
is the unique solution of Eq. (5.12). Furthermore,
\[ \| E_p(z) \| \leq C_p n^{-(p+1)} \]  \hspace{1cm} (5.17)
for all \( z \) bounded away from \( L \). This completes the proof of the lemma. \( \square \)

Recall that \( Y(z) = e^{\frac{1}{2} \arg(z)} R(z) \tilde{Y}(z) \). Let \( R_{ij}(z) \), \( i, j = 1, 2 \), denote the entries in the matrix \( R(z) \). From (3.63), we have
\[ Y(z) = \begin{pmatrix} e^{\frac{1}{2} \arg(z)} & 0 \\ \frac{1}{\sqrt{w(z)}} & \sqrt{w(z)} \end{pmatrix} \begin{pmatrix} R_{11}(z) & R_{12}(z) \\ R_{21}(z) & R_{22}(z) \end{pmatrix} P(z) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{w(z)}} \end{pmatrix}. \]  \hspace{1cm} (5.18)
A simple computation shows that the entries in the first column of the matrix \( P(z) \), given in (3.60a), are
\[ P_{11} = \sqrt{\pi} \frac{N^{1/6} \zeta_n^{1/4} (\beta_n - \alpha_n)^{1/2}}{[\alpha_n - \alpha_n](\beta_n - \beta_n)]^{1/4}} A_i, \]
and
\[ P_{21} = -\sqrt{\pi} \frac{i (2z - \alpha_n - \beta_n) N^{1/6} \zeta_n^{1/4}}{(\beta_n - \alpha_n)^{1/2} [(\alpha_n - \alpha_n)(\beta_n - \beta_n)]^{1/4}} \frac{2i [-(\alpha_n - \beta_n)]^{1/4}}{N^{1/6} \zeta_n^{1/4} (\beta_n - \alpha_n)^{1/2}} A_i'. \]
These quantities yield, in particular,
\[ \pi_n(z) = \sqrt{\frac{\pi}{w(z)}} e^{\frac{1}{2} \arg(z)} \left\{ N^{1/6} A_i (N^{2/3} \zeta_n) A(z, n) - N^{-1/6} A_i' (N^{2/3} \zeta_n) B(z, n) \right\}, \]  \hspace{1cm} (5.19)
where
\[ A(z, n) = \frac{\zeta_n^{1/4} (\beta_n - \alpha_n)^{1/2}}{[\alpha_n - \alpha_n)(\beta_n - \beta_n)]^{1/4}} \left[ R_{11}(z) - \frac{i (2z - \alpha_n - \beta_n)}{\beta_n - \alpha_n} R_{12}(z) \right] \]
and
\[ B(z, n) = \frac{2i [-(\alpha_n - \beta_n)]^{1/4}}{\zeta_n^{1/4} (\beta_n - \alpha_n)^{1/2}} - R_{12}(z). \]
Note that \( A(z, n) \) and \( B(z, n) \) are analytic functions of \( z \) in \( |\arg[z - (\frac{1}{2} + \delta)]| < \frac{2\pi}{3} \), and by (5.9) and (5.17) they have the asymptotic expansions given in (2.12) and (2.13). Furthermore, since \( A_i (N^{2/3} \zeta_n) \) and \( A_i' (N^{2/3} \zeta_n) \) are also analytic in \( |\arg[z - (\frac{1}{2} + \delta)]| < \frac{2\pi}{3} \), by analytic continuation the result in (5.19) holds for \( z \) in the sector \( |\arg[z - (\frac{1}{2} + \delta)]| < \frac{2\pi}{3} \). This completes the proof of the first theorem stated in Section 2.

6. Zeros

We first recall the change of variable
\[ t \mapsto x_n(t) = \sqrt{\alpha_n \beta_n} \exp \left[ \frac{t}{2} \log(\beta_n/\alpha_n) \right] \]
in (2.14). From (5.19), we have
\[ \pi_n(x_n(t)) = \sqrt{\frac{\pi}{w(x_n(t))}} e^{\frac{1}{2} \arg(x_n(t))} \left\{ N^{1/6} A_i (N^{2/3} \eta_n(t)) A(x_n(t), n) - N^{-1/6} A_i' (N^{2/3} \eta_n(t)) B(x_n(t), n) \right\}, \]
where \( \eta_n(t) = \zeta_n(x_n(t)), -1 + \delta \leq t \leq 1, \text{ and } \delta > 0. \) For \( t_\nu = t_\nu(n) \) to be a zero of \( \pi_n(x_n(t)) \), \( t_\nu(n) \) must be a root of the equation
\[ A_i (N^{2/3} \eta_n(t)) = \frac{1}{N^{1/3}} \frac{B(x_n(t), n)}{A(x_n(t), n)} A_i' (N^{2/3} \eta_n(t)). \]  \hspace{1cm} (6.1)
As long as \( t \in [-1 + \varepsilon, 1] \) and \( \varepsilon > 0 \), we have from (2.12) and (2.13)
\[
\frac{B(x_n(t))}{A(x_n(t))} = O\left( \frac{1}{N} \right)
\]
uniformly in \( v \), i.e., the O-term is independent of \( v \). Coupling (6.1) and (6.2) gives
\[
N^{2/3} \eta_n(t) - a_v = O\left( \frac{1}{N^{4/3}} \right).
\]
or equivalently
\[
\eta_n(t) = \frac{a_v}{N^{2/3}} + O\left( \frac{1}{N^2} \right). \tag{6.3}
\]
Recall that \( \eta_n(t) = \zeta_n(x_n(t)) \) and \( (v_n)_{\pm}(x) = \pm \pi i \mu_n(x) \) for \( x \in [\alpha_n, \beta_n] \). From (2.9), (3.31) and (3.23), we have
\[
2^{3/2} \left[ -\eta_n(t) \right]^{3/2} = 2^{3/2} \left[ e^{-\pi i} (\zeta_n) + (x_n(t)) \right]^{3/2} = \frac{k^2 \log(\beta_n/\alpha_n)}{N} \int_{t}^{1} \frac{\arctan(\sqrt{(x_n(\tau) - \alpha_n)(\beta_n - x_n(\tau))})}{x_n(\tau) + \sqrt{\alpha_n \beta_n}} d\tau. \tag{6.4}
\]
In the first equality above, we have used \( e^{-\pi i} \), instead of \( e^{\pi i} \), for \(-1\). This is due to the fact that \( \eta_n(t) \) is negative and \( \phi_n(z) \) behaves like \( (z - \beta_n)^{3/2} \) near \( z = \beta_n \); see (3.30). We also recall, from Theorem 2,
\[
a := \frac{1}{2} \log(\beta_n/\alpha_n) = \frac{1}{2} \log(\alpha_n \beta_n) - \log \alpha_n. \tag{6.5}
\]
Using (3.19) and (3.20), it is easily verified that
\[
a = N \log(1/q) + 2 \log 2 + O(q^N). \tag{6.6}
\]
Here we have also made use of (1.1). In terms of \( a \) in (6.5), \( x_n(t) \) can be written as
\[
x_n(t) = \sqrt{\alpha_n \beta_n} e^{at}
\]

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
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<tbody>
<tr>
<td>Numerical values of the zeros</td>
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</table>

<table>
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<tr>
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<th>( q = 0.5 )</th>
<th>( q = 0.7 )</th>
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</table>
and
\[
-\eta_n(t)^{3/2} = \frac{a}{N \log(1/q)} \int_1 \arctan \frac{e^{(r+1)\tau} - e^{-a\tau}}{e^{r\tau} + 1} \, dr. \tag{6.7}
\]

By the addition formula of the arctangent function, one can show that
\[
\arctan \frac{e^{(r+1)\tau} - e^{-a\tau}}{e^{r\tau} + 1} = \arctan \left[ \frac{e^{a\tau} - e^{(r+1)\tau}}{(e^{r\tau} + 1) + \sqrt{(e^{r\tau} - e^{-a\tau})(e^{a\tau} - e^{(r+1)\tau})}} \right].
\]

Since \( \sqrt{1 - e^{-a(1+\tau)}} = 1 + O(e^{-a(1+\tau)}) \), the quantity inside the curly bracket is equal to
\[
\frac{-\sqrt{e^{a(1-\tau)} - \tau} [1 + O(e^{-a\tau})]}{e^{a(1+\tau)}(1 + O(e^{-a(1+\tau)}))} = O(e^{-\frac{1}{2}a(1+\tau)}) = O(q^{\frac{1}{2}N})
\]
for \(-1 + \delta \leq \tau \leq 1\). Hence
\[
\frac{2}{3}[-\eta_n(t)]^{3/2} = \frac{a}{N \log(1/q)} \int_1 \left[ \arctan \frac{e^{a(1-\tau)} - \tau + O(q^{\frac{1}{2}N})}{0} \right] \, dr
\]
\[
= \frac{1}{N \log(1/q)} \int_0^{a(1-\tau)} \arctan \frac{1}{\tau} \, ds + O(q^{\frac{1}{2}N}). \tag{6.8}
\]

The final result (2.17) now follows from (6.3) and (6.8), and this proves Theorem 2.

We conclude this paper with the numerical table, which shows how closely formula (2.17) approximates the true values of the zeros of the Stieltjes–Wigert polynomials (see Table 1).

References