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Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups

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Abstract

We give three equivalent conditions for weak convergence of almost orbits of an asymptotically nonexpansive commutative semigroup acting on a nonempty bounded closed convex subset of a uniformly convex Banach space whose dual has the Kadec property.

Keywords: Almost orbits; Asymptotically nonexpansive commutative semigroups; Duality mappings; Kadec–Klee property; Kadec property; Uniformly convex Banach spaces

1. Introduction

This paper is devoted to the study of weak convergence of almost orbits of asymptotically nonexpansive semigroups of mappings. The main result is strictly connected with theorems due to Lin [15] and Oka [16].

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2. Basic theorem

Let us start with some basic definitions and notations. Throughout this paper we assume that X is a real Banach space, X^* is the dual space of X, and $J : X \to 2^{X^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f \in X^*\colon \langle x, f\rangle = \|x\|^2,\ \|x\| = \|f\| \right\},$$

where $x \in X$ and $\langle\cdot,\cdot\rangle$ denotes the pairing of X and X^*. We will denote by $\omega_w(\{x_n\})$ $(\omega_w(\{x_\alpha\}_{\alpha\in I}))$ the set of all weak subsequential limits (all limits of weakly convergent subnets) of a bounded sequence $\{x_n\}$ (of a bounded net $\{x_\alpha\}_{\alpha\in I}$) in X. The fact that $\{x_\alpha\}_{\alpha\in I}$ tends weakly to x will be denoted either by $x_\alpha \rightharpoonup x$ or by $w\text{-}\lim_{t\in I} x_\alpha = x$.

Now we recall notions of the Kadec property and the Kadec–Klee property.

Definition 1 [5]. A Banach space X is said to have the Kadec property if for every net $\{x_\alpha\}_{\alpha\in I}$ in X the following implication holds:

$$\left.\begin{array}{l} x_\alpha \rightharpoonup x \\ \|x_\alpha\| \to \|x\| \end{array}\right\} \quad\Rightarrow\quad \|x_\alpha - x\| \to 0.$$

If we restrict this definition to sequences we get the definition of the Kadec–Klee property.

Definition 2 [5]. A Banach space X is said to have the Kadec–Klee property if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X

$$\left.\begin{array}{l} x_n \rightharpoonup x \\ \|x_n\| \to \|x\| \end{array}\right\} \quad\Rightarrow\quad \|x_n - x\| \to 0.$$

Clearly, the Kadec property means that the relative weak and norm topologies agree on the unit sphere.

It is known (see [1, p. 113] and [11]) that within the class of reflexive spaces the Kadec–Klee property is equivalent to the Kadec property.

The following lemma will be very useful.

Lemma 1 [3]. *Let X be a real Banach space. Then for $x, y \in X$*

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\big\langle y, j(x + y)\big\rangle$$

for all $j(x + y) \in J(x + y)$.

The next lemma is due to Li and Sims [14] and is a generalization of the García Falset lemma [4].

Lemma 2. *Let X be a real reflexive Banach space such that its dual X^* has the Kadec–Klee property. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in X and $x, y \in \omega_w\{x_n\}$. Suppose*

$$\lim_{n\to\infty}\|ax_n + (1-a)x - y\|$$

exists for all $a \in [0,1]$. Then $x = y$.

Remark 1. In [4,7–9] one can find various applications of Lemma 2 in nonlinear functional analysis.

Now we are ready to prove our basic theorem.

Theorem 2.1. *Let X be a real reflexive Banach space such that its dual X^* has the Kadec–Klee property. Let $\{x_\alpha\}_{\alpha\in I}$ be a bounded net in X and $x, y \in \omega_w\{x_\alpha\}_{\alpha\in I}$. Suppose*

$$\lim_{\alpha\in I}\|ax_\alpha + (1-a)x - y\|$$

exists for all $a \in [0,1]$. Then $x = y$.

Proof. For each $\varepsilon > 0$ there exists $\overline{\alpha} \in I$ such that

$$\|ax_\alpha + (1-a)x - y\| \leqslant \lim_{\alpha\in I}\|ax_\alpha + (1-a)x - y\| + \varepsilon$$

for all $\alpha \geqslant \overline{\alpha}$. It follows that for all $\alpha \geqslant \overline{\alpha}$ and $j(x-y) \in J(x-y)$ we have

$$\begin{aligned}&\langle ax_\alpha + (1-a)x - y, j(x-y)\rangle\\ &\quad\leqslant \|x-y\|\Big(\lim_{\alpha\in I}\|ax_\alpha + (1-a)x - y\| + \varepsilon\Big).\end{aligned}$$

Since $x \in \omega_w\{x_\alpha\}_{\alpha\in I}$, we obtain

$$\begin{aligned}\|x-y\|^2 &= \langle ax + (1-a)x - y, j(x-y)\rangle\\ &\leqslant \|x-y\|\Big(\lim_{\alpha\in I}\|ax_\alpha + (1-a)x - y\| + \varepsilon\Big),\end{aligned}$$

and, letting $\varepsilon \to 0$, we get

$$\|x-y\| \leqslant \lim_{\alpha\in I}\|ax_\alpha + (1-a)x - y\|. \tag{2.1}$$

By Lemma 1,

$$\|ax_\alpha + (1-a)x - y\|^2 \leqslant \|x-y\|^2 + 2a\langle x_\alpha - x, j\big(ax_\alpha + (1-a)x - y\big)\rangle$$

for all $a \in (0,1]$ and $j(ax_\alpha + (1-a)x - y) \in J(ax_\alpha + (1-a)x - y)$. Hence, by (2.1), we see that

$$\liminf_{\alpha\in I}\langle x_\alpha - x, j\big(ax_\alpha + (1-a)x - y\big)\rangle \geqslant 0.$$

Therefore there exists a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_m \leqslant \alpha_n$ for $m \leqslant n$ and

$$\left\langle x_\alpha - x, j\left(\frac{1}{n}x_\alpha + \left(1-\frac{1}{n}\right)x - y\right)\right\rangle \geqslant -\frac{1}{n} \tag{2.2}$$

for each $n \in \mathbb{N}$ and $\alpha \geqslant \alpha_n$. Put

$$I_1 = \{\alpha\colon \alpha \geqslant \alpha_1\}.$$

Without loss of generality we can assume that $I = I_1$ since

$$\omega_w\{x_\alpha\}_{\alpha\in I} = \omega_w\{x_\alpha\}_{\alpha\in I_1}$$

and

$$\lim_{\alpha\in I}\left\|ax_\alpha + (1-a)x - y\right\| = \lim_{\alpha\in I_1}\left\|ax_\alpha + (1-a)x - y\right\|$$

for all $a \in [0,1]$. Next, for each $\alpha \in I$, we set

$$a_\alpha = \inf\left\{\frac{1}{n}\colon \alpha \geqslant \alpha_n\right\}$$

and consider two cases.

Case 1. $\alpha \in I$ and $a_\alpha > 0$. Putting

$$j_\alpha = j\bigl(a_\alpha x_\alpha + (1-a_\alpha)x - y\bigr)$$

we obtain

$$\langle x-y, j_\alpha\rangle = \left\|a_\alpha x_\alpha + (1-a_\alpha)x - y\right\|^2 - a_\alpha\langle x_\alpha - x, j_\alpha\rangle \tag{2.3}$$

and

$$\|j_\alpha\| = \left\|a_\alpha x_\alpha + (1-a_\alpha)x - y\right\|. \tag{2.4}$$

So, by (2.2), we have

$$\langle x_\alpha - x, j_\alpha\rangle \geqslant -a_\alpha. \tag{2.5}$$

Case 2. $\alpha \in I$ and $a_\alpha = 0$. In this case we can choose a subsequence $\{j((1/n_k)x_\alpha + (1-1/n_k)x - y)\}_{k\in\mathbb{N}}$ which is weakly convergent, say to j, and set

$$j_\alpha = j.$$

It then follows from (2.2) that

$$\langle x_\alpha - x, j_\alpha\rangle \geqslant 0. \tag{2.6}$$

Next observe that

$$\begin{aligned}\|j_\alpha\| &\leqslant \liminf_{k\to\infty}\left\|j\left(\frac{1}{n_k}x_\alpha + \left(1-\frac{1}{n_k}\right)x - y\right)\right\| \\ &= \lim_{k\to\infty}\left\|\frac{1}{n_k}x_\alpha + \left(1-\frac{1}{n_k}\right)x - y\right\| = \|x-y\|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle x-y, j_\alpha\rangle &= \lim_{k\to\infty}\left\langle x-y, j\left(\frac{1}{n_k}x_\alpha+\left(1-\frac{1}{n_k}\right)x-y\right)\right\rangle \\ &= \lim_{k\to\infty}\left(\left\|\frac{1}{n_k}x_\alpha+\left(1-\frac{1}{n_k}\right)x-y\right\|^2\right. \\ &\quad \left.-\frac{1}{n_k}\left\langle x_\alpha-x, j\left(\frac{1}{n_k}x_\alpha+\left(1-\frac{1}{n_k}\right)x-y\right)\right\rangle\right) \\ &= \|x-y\|^2. \end{aligned} \tag{2.7}$$

Therefore

$$\|j_\alpha\| = \|x-y\| \tag{2.8}$$

and $j_\alpha \in J(x-y)$. (Let us mention here that by the Kadec–Klee property of X^*, the sequence $\{j((1/n_k)x_\alpha + (1-1/n_k)x-y)\}_{k\in\mathbb{N}}$ tends strongly to j_α.)

Now from the net I we choose a subnet $\{\alpha_\beta\}_{\beta\in\tilde{I}}$ such that $\{x_{\alpha_\beta}\}_{\beta\in\tilde{I}}$ converges weakly to $y \in \omega_w\{x_\alpha\}_{\alpha\in I}$ and $\{j_{\alpha_\beta}\}_{\beta\in\tilde{I}}$ tends weakly to $\tilde{j}$. Then by (2.4) and (2.8), we get

$$\|\tilde{j}\| \leqslant \|x-y\|$$

and, by (2.3) and (2.7),

$$\langle x-y, \tilde{j}\rangle = \|x-y\|^2.$$

Hence $\tilde{j} \in J(x-y)$. Since X is reflexive and X^* has the Kadec–Klee property, the space X^* has also the Kadec property and this implies that $\{j_{\alpha_\beta}\}_{\beta\in\tilde{I}}$ converges strongly to $\tilde{j}$. It then follows from (2.5) and (2.6) that

$$\langle y-x, \tilde{j}\rangle \geqslant 0.$$

That is,

$$\|x-y\|^2 \leqslant 0,$$

which gives $x = y$. This completes the proof. □

The assumption that the dual X^* of the reflexive Banach space X has the Kadec–Klee property is essential and cannot be dropped from the above theorem as the following example shows.

Example 1. Let X be a Cartesian product $\mathbb{R}\times l^2$ furnished with the l^1-norm. The dual space X^* can be identified with a Cartesian product $\mathbb{R}\times l^2$ furnished with the maximum norm and therefore does not have the Kadec–Klee property. Taking

$$x_n = \begin{cases} (1, e_n) & \text{if } n \text{ is odd,} \\ (0,0) & \text{if } n \text{ is even} \end{cases}$$

and $x=(0,0)$, $y=(1,0)$, we see that $x,y\in\omega_w\{x_n\}$ and $x\neq y$ but

$$\|ax_n-y\|=1$$

for each $n\in\mathbb{N}$ and all $a\in[0,1]$.

An application of the above theorem will be presented in the next section.

3. Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups

Let G be a commutative semigroup with 0. Define a binary relation $\leqslant$ on G by: $a\leqslant b$ if and only if $b=a+c$ for some $c\in G$. Then $(G,\leqslant)$ is a directed system and applying this system we get a limit $\lim_{t\in G}$ in the sense of Rodé [20].

In this section X is always a uniformly convex Banach space and C a nonempty closed convex subset of X.

Now let $\mathcal{J}=\{T(t)\colon t\in G\}$ be a family of self-mappings of C. Recall that $\mathcal{J}$ is said to be an asymptotically nonexpansive semigroup acting on C if the following conditions are satisfied:

(i) $T(t):C\to C$ for each $t\in G$;
(ii) $T(s+t)x=T(s)T(t)x$ for all $s,t\in G$ and $x\in C$;
(iii) $T(0)=I$;
(iv) There exists a net $\{k_t\}_{t\in G}$ of positive numbers with

$$\lim_{t\in G}k_t=1$$

such that

$$\big\|T(t)x-T(t)y\big\|\leqslant k_t\|x-y\|$$

for all $x,y\in C$ and $t\in G$.

If $k_t=1$ for every $t\in G$, then $\mathcal{J}$ is called a nonexpansive semigroup. The nonempty set of common fixed points of $\mathcal{J}$ (see [10]) is denoted by $F(\mathcal{J})$.

The notion of an almost orbit of a nonexpansive mapping was introduced by Bruck [2]. Kobayashi and Miyadera [12] extended the notion to the case of a one-parameter semigroup of nonexpansive mappings. Later, Park and Takahashi [18] extended it to the case of a general commutative semigroup.

We say that a function $u:G\to C$ is an almost orbit of $\mathcal{J}$ if

$$\lim_{s\in G}\Big(\sup_{t\in G}\big\|u(t+s)-T(t)u(s)\big\|\Big)=0.$$

Let $\omega_w(u)$ denote the set of all weak subsequential limits of $\{u(t)\}_{t\in G}$.

We also need the following auxiliary lemmas.

Lemma 3 [16]. *Suppose that u_i, $i = 1, 2, \ldots,$ are almost orbits of $\mathcal{J}$. Then, for any $\varepsilon > 0$ and $n \geqslant 1$ there exist $t_0(\varepsilon) \in G$ and $s_0(\varepsilon, n) \in G$, where $t_0(\varepsilon)$ is independent of n and u_i $(i = 1, 2, \ldots, n)$, such that*

$$\left\| T(t)\left(\sum_{i=1}^{n} \lambda_i u_i(s)\right) - \sum_{i=1}^{n} \lambda_i T(t) u_i(s) \right\| < \varepsilon$$

for each $t \geqslant t_0(\varepsilon)$, $s \geqslant s_0(\varepsilon, n)$, and all nonnegative $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

Lemma 4. *Under the above definitions and assumptions the limit*

$$\lim_{t \in G} \left\| \alpha u(t) + (1-\alpha) f_1 - f_2 \right\|$$

exists for every almost orbit u of $\mathcal{J}$ in a bounded closed convex subset C of a uniformly convex Banach space X, $f_1, f_2 \in F(\mathcal{J})$ and for each $\alpha \in [0, 1]$.

Proof. By our assumption about the semigroup $\mathcal{J}$ and by Lemma 3, given $\varepsilon > 0$, there exist $t_0 \in G$ and $s_0 \in G$ such that

$$k_t - 1 < \frac{\varepsilon}{\operatorname{diam} C + 1},$$
$$\left\| T(t)\big(\alpha u(s) + (1-\alpha) f_1\big) - \alpha T(t) u(s) - (1-\alpha) f_1 \right\| < \varepsilon,$$

and

$$\sup_{t \in G} \left\| u(t+s) - T(t) u(s) \right\| < \varepsilon$$

for all $t \geqslant t_0$ and $s \geqslant s_0$. Hence

$$\begin{aligned}
&\left\| \alpha u(t+s) + (1-\alpha) f_1 - f_2 \right\| \\
&\quad \leqslant \alpha \left\| u(t+s) - T(t) u(s) \right\| \\
&\qquad + \left\| T(t)\big(\alpha u(s) + (1-\alpha) f_1\big) - \alpha T(t) u(s) - (1-\alpha) f_1 \right\| \\
&\qquad + k_t \left\| \alpha u(s) + (1-\alpha) f_1 - f_2 \right\| \\
&\quad \leqslant 3\varepsilon + \left\| \alpha u(s) + (1-\alpha) f_1 - f_2 \right\|
\end{aligned}$$

for all $t \geqslant t_0$ and $s \geqslant s_0$. This gives

$$\begin{aligned}
\inf_{t \in G} \sup_{\tau \geqslant t} \left\| \alpha u(\tau) + (1-\alpha) f_1 - f_2 \right\| &\leqslant \sup_{\tau \geqslant t_0} \left\| \alpha u(\tau) + (1-\alpha) f_1 - f_2 \right\| \\
&\leqslant 3\varepsilon + \left\| \alpha u(s) + (1-\alpha) f_1 - f_2 \right\|
\end{aligned}$$

for all $s \geqslant s_0$. Consequently,

$$\inf_{t \in G} \sup_{\tau \geqslant t} \left\| \alpha u(\tau) + (1-\alpha) f_1 - f_2 \right\| \leqslant \sup_{t \in G} \inf_{s \geqslant t} \left\| \alpha u(s) + (1-\alpha) f_1 - f_2 \right\|.$$

Therefore

$$\lim_{t\in G}\left\|\alpha u(t)+(1-\alpha)f_1-f_2\right\|$$

exists. □

Finally, we recall the following theorem due to Oka [16].

Theorem 3.1. *Under the above definitions and assumptions, if u is an almost orbit of $\mathcal{J}$ and*

$$w\text{-}\lim_{t\in G}\bigl(u(t)-u(t+s)\bigr)=0$$

for each $s\in G$, then

$$\omega_w(u)\subset F(\mathcal{J}).$$

Using these facts we are able to prove a theorem on weak convergence of almost orbits.

Theorem 3.2. *Suppose X is a uniformly convex Banach space such that its dual X^* has the Kadec–Klee property, C is a bounded closed convex subset of X, $\mathcal{J}=\{T(t)\colon t\in G\}$ (where G is a commutative semigroup with an identity) is an asymptotically nonexpansive semigroup on C, and u is an almost orbit of $\mathcal{J}$. Then the following conditions are equivalent*:

(1) $\omega_w(u)\subset F(\mathcal{J})$;
(2) $w\text{-}\lim_{t\in G}u(t)=x\in F(\mathcal{J})$;
(3) $w\text{-}\lim_{t\in G}(u(t)-u(t+s))=0$ *for each* $s\in G$.

Proof. (1) ⇒ (2) Let $f_1,f_2\in\omega_w(u)\subset F(\mathcal{J})$ and $\alpha\in[0,1]$. By Lemma 4 we see that

$$\lim_{t\in G}\left\|\alpha u(t)+(1-\alpha)f_1-f_2\right\|$$

exists. So it is sufficient to apply Theorem 2.1 to get $f_1=f_2$. Thus the set $\omega_w(u)$ is a singleton.

(2) ⇒ (3) This is obvious.

(3) ⇒ (1) See Theorem 3.1. □

Remark 2. Several results of a similar type can be found, for example, in [6,10, 13,19,21–23].

It is worth noting here that there exist uniformly convex Banach spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the Kadec–Klee property.

Example 2 [4]. Let us take $X_1 = \mathbb{R}^2$ with the norm defined by

$$|x| = \sqrt{\|x\|_2^2 + \|x\|_1^2}$$

and $X_2 = L^p[0, 1]$ with $1 < p < \infty$ and $p \neq 2$. The Cartesian product of X_1 and X_2 furnished with the l^2-norm is uniformly convex, it does not have the Opial property [5,17], and its norm is not Fréchet differentiable, but its dual does have the Kadec–Klee property.

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