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On finite-dimensional maps II

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Abstract

Let $f : X \to Y$ be a perfect n-dimensional surjective map of paracompact spaces and Y a C-space. We consider the following property of continuous maps $g : X \to \mathbb{I}^k = [0,1]^k$, where $1 \leqslant k \leqslant \omega$: each $g(f^{-1}(y))$, $y \in Y$, is at most n-dimensional. It is shown that all maps $g \in C(X, \mathbb{I}^{n+1})$ with the above property form a dense G_δ-set in the function space $C(X, \mathbb{I}^{n+1})$ equipped with the source limitation topology. Moreover, for every $n+1 \leqslant m \leqslant \omega$ the space $C(X, \mathbb{I}^m)$ contains a dense G_δ-set of maps having this property.

Keywords: Finite-dimensional maps; Set-valued maps; Selections; C-space

MSC: primary 54F45; secondary 55M10, 54C65

1. Introduction

This note is inspired by a result of Uspenskij [15, Theorem 1]. Answering a question of R. Pol, Uspenskij proved the following theorem: Let $f : X \to Y$ be a light map (i.e., every fiber $f^{-1}(y)$ is 0-dimensional) between compact spaces and $\mathcal{A}$ be the set of all functions $g : X \to \mathbb{I} = [0,1]$ such that $g(f^{-1}(y))$ is 0-dimensional for all $y \in Y$. Then $\mathcal{A}$ is a dense G_δ-subset of the function space $C(X, \mathbb{I})$ provided Y is a C-space (the case when Y is countable-dimensional was established earlier by Torunczyk). We extend this result as follows:

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Theorem 1.1. *Let $f : X \to Y$ be a σ-perfect surjection such that* $\dim f \leqslant n$ *and Y is a paracompact C-space. Let $\mathcal{H} = \{g \in C(X, \mathbb{I}^{n+1})\colon \dim g(f^{-1}(y)) \leqslant n$ for each $y \in Y\}$. Then $\mathcal{H}$ is dense and G_δ in $C(X, \mathbb{I}^{n+1})$ with respect to the source limitation topology.*

Corollary 1.2. *Let X, Y and f satisfy the hypotheses of Theorem* 1.1 *and* $n+1 \leqslant m \leqslant \omega$. *Then, there exists a dense G_δ-subset $\mathcal{H}_m$ of $C(X, \mathbb{I}^m)$ with respect to the source limitation topology such that* $\dim g(f^{-1}(y)) \leqslant n$ *for every $g \in \mathcal{H}_m$ and $y \in Y$.*

Here, $\dim f = \sup\{\dim f^{-1}(y)\colon y \in Y\}$ and f is said to be σ-perfect if there exists a sequence $\{X_i\}$ of closed subsets of X such that each restriction map $f|X_i$ is perfect and the sets $f(X_i)$ are closed in Y. The C-space property was introduced by Haver [7] for compact metric spaces and then extended by Addis and Gresham [1] for general spaces (see [4] for the definition and some properties of C-spaces). Every countable-dimensional (in particular, every finite-dimensional) paracompact space has property C, but there exists a compact metric C-space which is not countable-dimensional [13]. For any spaces X and Y by $C(X, Y)$ we denote the set of all continuous maps from X into Y. If (Y, d) is a metric space, then the source limitation topology on $C(X, Y)$ is defined in the following way: a subset $U \subset C(X, Y)$ is open in $C(X, Y)$ with respect to the source limitation topology provided for every $g \in U$ there exists a continuous function $\alpha : X \to (0, \infty)$ such that $\overline{B}(g, \alpha) \subset U$, where $\overline{B}(g, \alpha)$ denotes the set $\{h \in C(X, Y)\colon d(g(x), h(x)) \leqslant \alpha(x)$ for each $x \in X\}$. The source limitation topology is also known as the fine topology and $C(X, Y)$ with this topology has Baire property provided (Y, d) is a complete metric space [12]. Moreover, the source limitation topology on $C(X, Y)$ does not depend on the metric of Y when X is paracompact [8].

All single-valued maps under discussion are continuous, and all function spaces, if not explicitly stated otherwise, are equipped with the source limitation topology.

2. Proofs

Let show first that the proof of Theorem 1.1 can be reduced to the case when f is perfect. Indeed, we fix a sequence $\{X_i\}$ of closed subsets of X such that each map $f_i = f|X_i : X_i \to Y_i = f(X_i)$ is perfect and $Y_i \subset Y$ is closed. Consider the maps $\pi_i : C(X, \mathbb{I}^{n+1}) \to C(X_i, \mathbb{I}^{n+1})$ defined by $\pi(g) = g|X_i$ and the sets $\mathcal{H}_i = \{g \in C(X_i, \mathbb{I}^{n+1})\colon \dim g(f_i^{-1}(y)) \leqslant n$ for each $y \in Y_i\}$. If Theorem 1.1 holds for perfect maps, then every $\mathcal{H}_i$ is dense and G_δ in $C(X_i, \mathbb{I}^{n+1})$, so are the sets $\pi_i^{-1}(\mathcal{H}_i)$ in $C(X, \mathbb{I}^{n+1})$ because π_i are open and surjective maps. Finally, observe that $\mathcal{H}$ is the intersection of all $\mathcal{H}_i$ and since $C(X, \mathbb{I}^{n+1})$ has Baire property, we are done.

Everywhere in this section X, Y, f and $\mathcal{H}$ are fixed and satisfy the hypotheses of Theorem 1.1 with f being perfect. Any finite-dimensional cube $\mathbb{I}^k$ is considered with the Euclidean metric. We say that a set-valued map $\theta : H \to \mathcal{F}(Z)$, where $\mathcal{F}(Z)$ denotes the family of all closed subsets of the space Z, is upper semi-continuous (br. u.s.c.) if $\{y \in H\colon \theta(y) \subset W\}$ is open in H for every open $W \subset Z$. In the above notation, θ is called lower semi-continuous if $\{y \in H\colon \theta(y) \cap W \neq \emptyset\}$ is open in H whenever W is open in Z.

Proof of Theorem 1.1. For every open set V in $\mathbb{I}^{n+1}$ let $\mathcal{H}_V$ be the set of all $g \in C(X, \mathbb{I}^{n+1})$ such that V is not contained in any $g(f^{-1}(y))$, $y \in Y$. Following the Uspenskij idea from [15], it suffices to show that each set $\mathcal{H}_V$ is dense and open in $C(X, \mathbb{I}^{n+1})$. Indeed, choose a countable base $\mathcal{B}$ in $\mathbb{I}^{n+1}$. Since a subset of $\mathbb{I}^{n+1}$ is at most n-dimensional if and only if it does not contain any $V \in \mathcal{B}$, we have that $\mathcal{H}$ is the intersection of all $\mathcal{H}_V$, $V \in \mathcal{B}$. But $C(X, \mathbb{I}^{n+1})$ has the Baire property, so $\mathcal{H}$ is dense and G_δ in $C(X, \mathbb{I}^{n+1})$ and we are done. □

Lemma 2.1. *The set $\mathcal{H}_V$ is open in $C(X, \mathbb{I}^{n+1})$ for every open $V \subset \mathbb{I}^{n+1}$.*

Proof. Fix an open set V in $\mathbb{I}^{n+1}$ and $g_0 \in \mathcal{H}_V$. We are going to find a continuous function $\alpha : X \to (0, \infty)$ such that $\overline{B}(g_0, \alpha) \subset \mathcal{H}_V$. To this end, let $p : Z \to Y$ be a perfect surjection with $\dim Z = 0$ and define $\psi : Y \to \mathcal{F}(\mathbb{I}^{n+1})$ by $\psi(y) = g_0(f^{-1}(y))$, $y \in Y$. Since f is perfect, ψ is upper semi-continuous and compact-valued. Now, consider the set-valued map $\psi_1 : Z \to \mathcal{F}(\mathbb{I}^{n+1})$, $\psi_1 = \psi \circ p$. Obviously, $g_0 \in \mathcal{H}_V$ implies $\overline{V} \backslash \psi_1(z) \neq \emptyset$ for every $z \in Z$. Moreover, ψ_1 is also upper semi-continuous, in particular it has a closed graph. Then, by a result of Michael [10, Theorem 5.3], there exists a continuous map $h : Z \to \mathbb{I}^{n+1}$ such that $h(z) \in \overline{V} \backslash \psi_1(z)$, $z \in Z$. Next, consider the u.s.c. compact-valued map $\theta : Y \to \mathcal{F}(\mathbb{I}^{n+1})$, $\theta(y) = h(p^{-1}(y))$, $y \in Y$. We have $\emptyset \neq \theta(y) \subset \overline{V}$ and $\theta(y) \cap \psi(y) = \emptyset$ for all $y \in Y$. Hence, the function $\alpha_1 : Y \to \mathbb{R}$, $\alpha_1(y) = d(\theta(y), \psi(y))$, is positive, where d is the Euclidean metric on $\mathbb{I}^{n+1}$. Since, both θ and ψ are upper semi-continuous, α_1 has the following property: $\alpha_1^{-1}(a, \infty)$ is open in Y for every $a \in \mathbb{R}$. Finally, take a continuous function $\alpha_2 : Y \to (0, \infty)$ with $\alpha_2(y) < \alpha_1(y)$ for every $y \in Y$ (see, for example, [3]) and define $\alpha = \alpha_2 \circ f$. It remains to observe that, if $g \in \overline{B}(g_0, \alpha)$ and $y \in Y$, then $\theta(y) \subset \overline{V} \backslash g(f^{-1}(y))$. So, $g(f^{-1}(y))$ does not contain V for all $y \in Y$, i.e., $\overline{B}(g_0, \alpha) \subset \mathcal{H}_V$. □

Remark. Analyzing the proof of Lemma 2.1, one can see that we proved the following more general statement: Let $h : \overline{X} \to \overline{Y}$ be a perfect surjection between paracompact spaces and K a complete metric space. Then, for every open $V \subset K$ the set of all maps $g \in C(\overline{X}, K)$ with $V \not\subset g(h^{-1}(y))$ for any $y \in \overline{Y}$ is open in $C(\overline{X}, K)$.

The remaining part of this section is devoted to the proof that each $\mathcal{H}_V$ is dense in $C(X, \mathbb{I}^{n+1})$, which is finally accomplished by Lemma 2.6.

Lemma 2.2. *Let Z and K be compact spaces and $K_0 = \bigcup_{i=1}^{\infty} K_i$ with each K_i being a closed 0-dimensional subset of K. Then the set $\mathcal{A} = \{g \in C(Z \times K, \mathbb{I})$: $\dim g(\{z\} \times K_0) = 0$ for every $z \in Z\}$ is dense and G_δ in $C(Z \times K, \mathbb{I})$.*

Proof. Since, for every i, the restriction map $p_i : C(Z \times K, \mathbb{I}) \to C(Z \times K_i, \mathbb{I})$ is a continuous open surjection, we can assume that $K_0 = K$ and $\dim K = 0$. Then $\mathcal{A}$ is the intersection of the sets $\mathcal{A}_V$, $V \in \mathcal{B}$, where $\mathcal{B}$ is a countable base of $\mathbb{I}$ and $\mathcal{A}_V$ consists of all $g \in C(Z \times K, \mathbb{I})$ such that $V \not\subseteq g(\{z\} \times K)$ for every $z \in Z$. By the remark after Lemma 1.1, every $\mathcal{A}_V \subset C(Z \times K, \mathbb{I})$ is open, so $\mathcal{A}$ is G_δ. It remains only to show that $\mathcal{A}$ is dense in $C(Z \times K, \mathbb{I})$. Since K is 0-dimensional, the set $C_K = \{h \in C(K, \mathbb{R})$: $h(K)$

is finite} is dense in $C(K,\mathbb{R})$. Hence, by the Stone–Weierstrass theorem, all polynomials of elements of the family $\gamma = \{t \cdot h\colon\ t \in C(Z,\mathbb{R}),\ h \in C_K\}$ form a dense subset $\mathcal{P}$ of $C(Z \times K,\mathbb{R})$. We fix a retraction $r:\mathbb{R} \to \mathbb{I}$ and define $u_r : C(Z \times K,\mathbb{R}) \to C(Z \times K,\mathbb{I})$, $u_r(h) = r \circ h$. Then $u_r(\mathcal{P})$ is dense in $C(Z \times K,\mathbb{I})$. It is easily seen that every $g \in u_r(\mathcal{P})$ has the following property: $g(\{z\} \times K)$ is finite for every $z \in Z$. So, $u_r(\mathcal{P}) \subset \mathcal{A}$, i.e., $\mathcal{A}$ is dense in $C(Z \times K,\mathbb{I})$. □

Lemma 2.3. *Let M and K be compact spaces with $\dim K \leqslant n$ and M metrizable. If $V \subset \mathbb{I}^{n+1}$ is open, then the set of all maps $g \in C(M \times K,\mathbb{I}^{n+1})$ such that $V \not\subset g(\{y\} \times K)$ for each $y \in M$ is dense in $C(M \times K,\mathbb{I}^{n+1})$.*

Proof. We are going to prove this lemma by induction with respect to the dimension of K. According to Lemma 2.2, it is true if $\dim K = 0$. Suppose the lemma holds for any K with $\dim K \leqslant m - 1$ for some $m \geqslant 1$ and let K be a fixed compact space with $\dim K = m$. For $g^0 \in C(M \times K,\mathbb{I}^{m+1})$ and $\varepsilon > 0$ we need to find a function $g \in C(M \times K,\mathbb{I}^{m+1})$ which is ε-close to g^0 and $V \not\subset g(\{y\} \times K)$ for every $y \in M$. If K is not metrizable, we represent it as the limit space of a σ-complete inverse system $\mathcal{S} = \{K_\lambda, p_\lambda^{\lambda+1}\colon\ \lambda \in \Lambda\}$ such that each K_λ is a metrizable compactum with $\dim K_\lambda \leqslant m$. Then $M \times K$ is the limit of the system $\{M \times K_\lambda, \mathrm{id} \times p_\lambda^{\lambda+1}\colon\ \lambda \in \Lambda\}$, where id is the identity map on M. Applying standard inverse spectra arguments (see [2]), we can find $\lambda(0) \in \Lambda$ and $g_{\lambda(0)} \in C(M \times K_{\lambda(0)},\mathbb{I}^{m+1})$ such that $g_{\lambda(0)} \circ (\mathrm{id} \times p_{\lambda(0)}) = g^0$, where $p_{\lambda(0)} : K \to K_{\lambda(0)}$ denotes the $\lambda(0)$th limit projection of $\mathcal{S}$. Therefore, the proof is reduced to the case when K is metrizable.

Let K be metrizable and $K = K_1 \cup K_2$ such that K_1 is a 0-dimensional σ-compact subset of K and $\dim K_2 \leqslant m - 1$ (this is possible because K is metrizable and m-dimensional, see [4]). Let $g^0 = g_1^0 \times g_2^0$, where g_1^0 is a function from $M \times K$ into $\mathbb{I}$, and $g_2^0 : M \times K \to \mathbb{I}^m$. We can assume that $V = V_1 \times V_2$ with both $V_1 \subset \mathbb{I}$ and $V_2 \subset \mathbb{I}^m$ open. According to Lemma 2.2, there exists a function $g_1 : M \times K \to \mathbb{I}$ which is $\varepsilon/\sqrt{2}$-close to g_1^0 and such that $\dim g_1(\{y\} \times K_1) = 0$ for every $y \in M$. Hence, V_1 is not contained in any of the sets $g_1(\{y\} \times K_1)$, $y \in M$.

Claim. *There exists an open set $A_1 \subset K$ containing K_1 such that $\overline{V}_1 \not\subset g_1(\{y\} \times A_1)$ for any $y \in M$.*

To prove the claim, we represent K_1 as the union of countably many compact 0-dimensional sets K_{1i} and consider the upper semi-continuous compact-valued maps $\psi_i : M \to \mathcal{F}(\mathbb{I})$ defined by $\psi_i(y) = g_1(\{y\} \times K_{1i})$. As in the proof of Lemma 2.1, we fix a 0-dimensional space Z, a surjective perfect map $p : Z \to M$ and define the set-valued maps $\bar{\psi}_i : Z \to \mathcal{F}(\mathbb{I})$, $\bar{\psi}_i = \psi_i \circ p$. It follows from our construction that each $\bar{\psi}_i(z)$, $z \in Z$, $i \in \mathbb{N}$, is 0-dimensional. By [10, Theorem 5.5] (see also [6, Theorem 1.1]), there is $h \in C(Z,\mathbb{I})$ such that $h(z) \in \overline{V}_1 \setminus \bigcup_{i=1}^{\infty} \bar{\psi}_i(z)$, $z \in Z$. Then $\theta : M \to \mathcal{F}(\mathbb{I})$, $\theta(y) = h(p^{-1}(y))$, is u.s.c. with $\emptyset \neq \theta(y) \subset \overline{V}_1 \setminus g_1(\{y\} \times K_1)$ for every $y \in M$. Since the graph G_θ of θ is closed in $M \times \mathbb{I}$, the set $U = \{(y,x) \in M \times K\colon\ (y, g_1(y,x)) \notin G_\theta\}$ is open in $M \times K$ and contains $M \times K_1$. So, $A_1 = \{x \in K\colon\ M \times \{x\} \subset U\}$ is open in K and contains K_1. Moreover, $\theta(y) \subset \overline{V}_1 \setminus g_1(\{y\} \times A_1)$ for every $y \in M$, which completes the proof of the claim.

Now, let $A_2 = K \setminus A_1$. Obviously, A_2 is a compact subset of K_2, so $\dim A_2 \leqslant m-1$. According to the assumption that the lemma is true for any space of dimension $\leqslant m-1$, there exists a map $h_2 : M \times A_2 \to \mathbb{I}^m$ which is $\varepsilon/\sqrt{2}$-close to $g_2^0|(M \times A_2)$ and such that $\overline{V}_2 \not\subset h_2(\{y\} \times A_2)$ for any $y \in M$. We finally extend h_2 to a map g_2: $M \times K$ such that g_2 is $\varepsilon/\sqrt{2}$-close to g_2^0. Hence,

$$K = A_1 \cup A_2 \quad \text{and} \quad \overline{V}_j \not\subset g_j\big(\{y\} \times A_j\big) \quad \text{for any } y \in M,\ j = 1, 2. \tag{1}$$

Then the map $g = g_1 \times g_2 : M \times K \to \mathbb{I}^{m+1}$ is ε-close to g_0. It follows from (1) that $\overline{V} \not\subset (g\{y\} \times K)$ for any $y \in M$. □

For any open $V \subset \mathbb{I}^{n+1}$ we consider the set-valued map ψ_V from Y into $C(X, \mathbb{I}^{n+1})$, given by $\psi_V(y) = \{g \in C(X, \mathbb{I}^{n+1})\colon\ V \subset g(f^{-1}(y))\}$, $y \in Y$.

Lemma 2.4. *If $V \subset \mathbb{I}^{n+1}$ is open and $C(X, \mathbb{I}^{n+1})$ is equipped with the uniform convergence topology, then ψ_V has a closed graph.*

Proof. Let $G_V \subset Y \times C(X, \mathbb{I}^{n+1})$ be the graph of ψ_V and $(y_0, g_0) \notin G_V$. Then $g_0 \notin \psi_V(y_0)$, so $g_0(f^{-1}(y_0))$ does not contain V. Consequently, there exists $z_0 \in V \setminus g_0(f^{-1}(y_0))$ and let $\varepsilon = d(z_0, g_0(f^{-1}(y_0)))$. Since f is a closed map, there exists a neighborhood U of y_0 in Y with $d(z_0, g_0(f^{-1}(y))) > 2^{-1}\varepsilon$ for every $y \in U$. It is easily seen that $U \times B_{4^{-1}\varepsilon}(g_0)$ is a neighborhood of (y_0, g_0) in $Y \times C(X, \mathbb{I}^{n+1})$ which does not meet G_V (here $B_{4^{-1}\varepsilon}(g_0)$ is the $4^{-1}\varepsilon$-neighborhood of g_0 in $C(X, \mathbb{I}^{n+1})$ with the uniform metric). Therefore $G_V \subset Y \times C(X, \mathbb{I}^{n+1})$ is closed. □

Recall that a closed subset F of the metrizable space M is said to be a Z-set in M [11], if the set $C(Q, M \backslash F)$ is dense in $C(Q, M)$ with respect to the uniform convergence topology, where Q denotes the Hilbert cube.

Lemma 2.5. *Let $\alpha : X \to (0, \infty)$ be a positive continuous function, $V \subset \mathbb{I}^{n+1}$ open and $g_0 \in C(X, \mathbb{I}^{n+1})$. Then $\psi_V(y) \cap \overline{B}(g_0, \alpha)$ is a Z-set in $\overline{B}(g_0, \alpha)$ for every $y \in Y$, where $\overline{B}(g_0, \alpha)$ is considered as a subspace of $C(X, \mathbb{I}^{n+1})$ with the uniform convergence topology.*

Proof. The proof of this lemma follows very closely the proof of [14, Lemma 2.8]. For sake of completeness we provide a sketch. In this proof all function spaces are equipped with the uniform convergence topology generated by the Euclidean metric d on $\mathbb{I}^{n+1}$. Since, by Lemma 2.4, ψ_V has a closed graph, each $\psi_V(y)$ is closed $\overline{B}(g_0, \alpha)$. We need to show that, for fixed $y \in Y$, $\delta > 0$ and a map $u : Q \to \overline{B}(g_0, \alpha)$ there exists a map $v : Q \to \overline{B}(g_0, \alpha) \backslash \psi_V(y)$ which is δ-close to u. Observe first that u generates $h \in C(Q \times X, \mathbb{I}^{n+1})$, $h(z, x) = u(z)(x)$, such that $d(h(z, x), g_0(x)) \leqslant \alpha(x)$ for any $(z, x) \in Q \times X$. Since $f^{-1}(y)$ is compact, take $\lambda \in (0, 1)$ such that $\lambda \sup\{\alpha(x)\colon\ x \in f^{-1}(y)\} < \delta/2$ and define $h_1 \in C(Q \times f^{-1}(y), \mathbb{I}^{n+1})$ by $h_1(z, x) = (1-\lambda)h(z, x) + \lambda g_0(x)$. Then, for every $(z, x) \in Q \times f^{-1}(y)$, we have

$$d\big(h_1(z, x), g_0(x)\big) \leqslant (1-\lambda)\alpha(x) < \alpha(x) \tag{2}$$

and

$$d\big(h_1(z,x),h(z,x)\big)\leqslant \lambda\alpha(x)<\frac{\delta}{2}. \tag{3}$$

Let $q<\min\{r,\delta/2\}$, where $r=\inf\{\alpha(x)-d(h_1(z,x),g_0(x))\colon (z,x)\in Q\times f^{-1}(y)\}$. Since $\dim f^{-1}(y)\leqslant n$, by Lemma 2.3 (applied to the product $Q\times f^{-1}(y)$), there is a map $h_2\in C(Q\times f^{-1}(y),\mathbb{I}^{n+1})$ such that $d(h_2(z,x),h_1(z,x))<q$ and $h_2(\{z\}\times f^{-1}(y))$ does not contain V for each $(z,x)\in Q\times f^{-1}(y)$. Then, by (2) and (3), for all $(z,x)\in Q\times f^{-1}(y)$ we have

$$d\big(h_2(z,x),h(z,x)\big)<\delta \quad \text{and} \quad d\big(h_2(z,x),g_0(x)\big)<\alpha(x). \tag{4}$$

Because both Q and $f^{-1}(y)$ are compact, $u_2(z)(x)=h_2(z,x)$ defines the map $u_2\colon Q\to C(f^{-1}(y),\mathbb{I}^{n+1})$. Since the map $\pi\colon \overline{B}(g_0,\alpha)\to C(f^{-1}(y),\mathbb{I}^{n+1})$, $\pi(g)=g|f^{-1}(y)$ is continuous and open (with respect to the uniform convergence topology), we can see that $u_2(z)\in\pi(\overline{B}(g_0,\alpha))$ for every $z\in Q$ and $\theta(z)=\overline{\pi^{-1}(u_2(z))\cap B_\delta(u(z))}$ defines a convex-valued map from Q into $\overline{B}(g_0,\alpha)$ which is lower semi-continuous. By the Michael selection theorem [9, Theorem 3.2$''$], there is a continuous selection $v\colon Q\to C(X,\mathbb{I}^{n+1})$ for θ. Then v maps Q into $\overline{B}(g_0,\alpha)$ and v is δ-close to u. Moreover, for any $z\in Q$ we have $\pi(v(z))=u_2(z)$ and $V\not\subset u_2(z)(f^{-1}(y))$. Hence, $v(z)\notin\psi_V(y)$ for any $z\in Q$, i.e., $v\colon Q\to\overline{B}(g_0,\alpha)\backslash\psi_V(y)$. □

We are now in a position to finish the proof of Theorem 1.1.

Lemma 2.6. *The set $\mathcal{H}_V$ is dense in $C(X,\mathbb{I}^{n+1})$ for every open $V\subset\mathbb{I}^{n+1}$.*

Proof. We need to show that, for fixed $g_0\in C(X,\mathbb{I}^{n+1})$ and a continuous function $\alpha\colon X\to(0,\infty)$, there exists $g\in\overline{B}(g_0,\alpha)\cap\mathcal{H}_V$. The space $C(X,\mathbb{I}^{n+1})$ with the uniform convergence topology is a closed convex subspace of the Banach space E consisting of all bounded continuous maps from X into $\mathbb{R}^{n+1}$. We define the set-valued map ϕ from Y into $C(X,\mathbb{I}^{n+1})$, $\phi(y)=\overline{B}(g_0,\alpha)$, $y\in Y$. According to Lemma 2.5, $\overline{B}(g_0,\alpha)\cap\psi_V(y)$ is a Z-set in $\overline{B}(g_0,\alpha)$ for every $y\in Y$. So, we have a lower semi-continuous closed and convex-valued map $\phi\colon Y\to\mathcal{F}(E)$ and another map $\psi_V\colon Y\to\mathcal{F}(E)$ with a closed graph (see Lemma 2.4) such that $\phi(y)\cap\psi_V(y)$ is a Z-set in $\phi(y)$ for each $y\in Y$. Moreover, Y is a C-space, so we can apply [5, Theorem 1.1] to obtain a continuous map $h\colon Y\to C(X,\mathbb{I}^{n+1})$ with $h(y)\in\phi(y)\backslash\psi_V(y)$ for every $y\in Y$. Then $g(x)=h(f(x))(x)$, $x\in X$, defines a map $g\in\overline{B}(g_0,\alpha)$. On the other hand, $h(y)\notin\psi_V(y)$, $y\in Y$, implies that $g\in\mathcal{H}_V$. □

Proof of Corollary 1.2. As in the proof of Theorem 1.1, we can suppose that f is perfect. We first consider the case when m is an integer $\geqslant n+1$. Let $\exp_{n+1}$ be the family of all subsets of $A=\{1,2,\dots,m\}$ having cardinality $n+1$ and let $\pi_B\colon\mathbb{I}^m\to\mathbb{I}^B$ denote the corresponding projections, $B\in\exp_{n+1}$. It can be shown that $C(X,\mathbb{I}^m)=C(X,\mathbb{I}^B]\times C(X,\mathbb{I}^{A\backslash B})$, so each projection $p_B\colon C(X,\mathbb{I}^m)\to C(X,\mathbb{I}^B)$ is open. Since, by Theorem 1.1, every set $\mathcal{H}_B=\{g\in C(X,\mathbb{I}^B)\colon \dim g(f^{-1}(y))\leqslant n \text{ for all } y\in Y\}$ is dense and G_δ in $C(X,\mathbb{I}^B)$, so is the set $p_B^{-1}(\mathcal{H}_B)$ in $C(X,\mathbb{I}^m)$. Consequently, the intersection $\mathcal{H}_m$ of all $\mathcal{H}_B$, $B\in\exp_{n+1}$, is also dense and G_δ in $C(X,\mathbb{I}^m)$. Moreover, if $g\in\mathcal{H}_m$ and

$y \in Y$, then $\dim \pi_B(g(f^{-1}(y))) \leqslant n$ for any $B \in \exp_{n+1}$. The last inequalities, according to a result of Nöbeling [4, Problem 1.8.C], imply $\dim g(f^{-1}(y)) \leqslant n$.

Now, let $m = \omega$ and $\exp_{<\omega}$, denote the family of all finite sets $B \subset \omega$ of cardinality $|B| \geqslant n + 1$. Keeping the above notations, for any $B \in \exp_{<\omega}$, $\pi_B : Q = \mathbb{I}^\omega \to \mathbb{I}^B$ and $p_B : C(X, Q) \to C(X, \mathbb{I}^B)$ stand for the corresponding projections. Then the intersection $\mathcal{H}_\omega$ of all $p_B^{-1}(\mathcal{H}_B)$ is dense and G_δ in $C(X, Q)$. We need only to check that $\dim g(f^{-1}(y)) \leqslant n$ for any $g \in \mathcal{H}_\omega$ and $y \in Y$. And this is certainly true, take an increasing sequence $\{B(k)\}$ in $\exp_{<\omega}$ which covers ω and consider the inverse sequence $\mathcal{S} = \{\pi_{B(k)}(g(f^{-1}(y))), \pi_k^{k+1}\}$, where $\pi_k^{k+1} : \pi_{B(k+1)}(g(f^{-1}(y))) \to \pi_{B(k)}(g(f^{-1}(y)))$ are the natural projections. Obviously, $g(f^{-1}(y))$ is the limit space of $\mathcal{S}$. Moreover, $g \in \mathcal{H}_\omega$ implies that $\pi_{B(k)} \circ g \in \mathcal{H}_{B(k)}$ for any k, so all $\pi_{B(k)}(g(f^{-1}(y)))$ are at most n-dimensional. Hence, $\dim g(f^{-1}(y)) \leqslant n$. □

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