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A Coupled System of Rational Difference Equations

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Abstract—We investigate the global stability properties and asymptotic behavior of solutions of the recursive sequence

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}, \qquad n = 0, 1, \dots,$$

where the parameters a, b, c, and d are arbitrary positive numbers, and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

Keywords—Stability, Asymptotic behavior.

1. INTRODUCTION

Consider the recursive sequence

$$u_{n+1} = \frac{u_n}{a + cv_n}, \quad v_{n+1} = \frac{v_n}{b + du_n}, \qquad n = 0, 1, \dots,$$

where the parameters a, b, c, and d are positive numbers and the initial conditions u_0 and v_0 are arbitrary nonnegative numbers. Letting $x_n = du_n$ and $y_n = cv_n$, we obtain

$$x_{n+1} = \frac{x_n}{a + y_n}, \quad y_{n+1} = \frac{y_n}{b + x_n}, \qquad n = 0, 1, \dots. \tag{1}$$

System (1) is deceptively simple. In fact, the authors have used it in "liberal arts math" courses to illustrate the role of computers in mathematical experimentation and how short the distance can be from elementary mathematics to twilight regions where experts are baffled. For an interactive version of this iteration on the web, go to `http://www.uri.edu/artsci/mth/w108/interact.htm`, download a plug-in and then click on Outer Limits. A more significant application of system (1) is indicated at the end of this section.

In a modelling setting, system (1) of nonlinear difference equations represents the rule by which two discrete, competing populations reproduce from one generation to the next. The phase variables x_n and y_n denote population sizes during the n^{th} generation and the sequence or orbit $\{(x_n, y_n) : n = 0, 1, 2, \dots\}$ depicts how the populations evolve over time. Competition between

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the two populations is reflected by the fact that the transition function for each population is a decreasing function of the other population size.

Our aim is to investigate the global stability properties and the global asymptotic behavior of solutions of equation (1). We will exhibit analytically the stable and unstable manifolds for all equilibrium points, for all values of parameters (a, b) on and outside the unit square $(0,1) \times (0,1)$. For these values of a and b, we will show that there are nine cases according as $0 < a < 1$, $a = 1$, or $a > 1$, together with all possibilities $0 < b < 1$, $b = 1$, or $b > 1$. Using a linear change of variables and parameters, nine cases can be reduced to six qualitatively different cases . In this paper, we consider five of the six cases, omitting the case $0 < a < 1$ and $0 < b < 1$. The case $0 < a < 1$ and $0 < b < 1$ leads to different results and requires different techniques and is studied in [1]. We obtain fairly complete information about all possible asymptotic behaviors of solutions to equation (1). In some of the above-mentioned five cases, the behavior is quite simple, such as the case $a > 1$, $b > 1$. In other cases, it is more complicated, for example, the case $0 < a < 1$, $b = 1$.

Our principal tool in resolving the more complex cases is the use of phase plane solutions

$$\left(\frac{y+a}{y^{1-a}}\right)^b = C\left(\frac{x+b}{x^{1-b}}\right)^a, \tag{2}$$

of the differential equation

$$\frac{dy}{dx} = \frac{(x+b-1)(y+a)y}{(x+b)(y+a-1)x} \tag{3}$$

associated with a corresponding system of differential equations

$$x' = \frac{x}{a+y} - x, \qquad y' = \frac{y}{b+x} - y, \tag{4}$$

$$C = \frac{\left((y_0+a)\,/\,\left(y_0^{1-a}\right)\right)^b}{\left((x_0+b)\,/\,\left(x_0^{1-b}\right)\right)^a}.$$

Equation (1) is the Euler discretization of this latter system. Note that the system of differential equations (4) has the same equilibrium points as our system of difference equations, equation (1). The Euler discretization is conventionally used to obtain information about a given differential equation, and it is customary to warn that this discrete approximation is crude, i.e., that the qualitative behavior of its solutions can differ dramatically from the behavior of solutions to the differential equation. In this paper, we reverse the usual direction of analysis. It is a differential equation which underlies a given system of difference equations, and the behavior of their solutions is found to be dramatically similar.

Graphical and numerical experiments revealed that solutions $\{(x_n, y_n)\}$ to our system (1) intersect the solution curves (2) in the direction of increasing or decreasing values of C. This fact becomes crucial (see below) in finding basins of attraction for two qualitatively different types of asymptotic behavior when $0 < a < 1$, $b = 1$. It is again essential in proving that every solution which begins off the coordinate axes has a limit on the positive y-axis when $a > 1$, $b = 1$. To the best of our knowledge, this is the first time that this technique has been used with such effectiveness.

We will use the product representation

$$x_n = \prod_{k=0}^{n-1} \frac{x_0}{a+y_k}, \qquad y_n = \prod_{k=0}^{n-1} \frac{y_0}{b+x_k}$$

of solutions of equation (1) to establish the continuous dependence of finite limits $y_\infty = \lim_{n\to\infty} y_n$ upon initial points (x_0, y_0) which lie in the interior of a basin of attraction of this finite limit.

The special case $a = b = 1$ is interesting in itself. All equilibrium points coincide and their local stability cannot be obtained from the linearization. In this case, we will find an invariant and a closed-form solution.

Observe that the variables x_n and y_n in (1) can be decoupled, and that this system is equivalent to the pair of second-order difference equations

$$x_{n+2} = \frac{x_{n+1}^2 (b + x_n)}{a(b-1)x_{n+1} + (ax_{n+1} + 1) x_n}, \qquad y_{n+2} = \frac{y_{n+1}^2 (a + y_n)}{b(a-1)y_{n+1} + (by_{n+1} + 1) y_n}.$$

The latter equations are essentially equivalent in the sense that x_n can be replaced by y_n and a by b. Thus, all our results can be applied to these two equations.

Systems of rational difference equations have been studied extensively in the literature. They appear in many problems in numerical analysis, such as the application of Newton's method for solving systems of polynomial or rational equations, or solving a polynomial or rational equation in the complex plane, see [2, pp. 317–321]. Systems of rational difference equations are used to model competitive interaction between two biological species, see [3–5] and references therein. The asymptotic behavior and stability of some rational systems is investigated in [2, pp. 168–172], by using Liapunov's method and Lasalle's invariance principle and in [2, pp. 182–184], by using linearized stability analysis.

2. LINEARIZED STABILITY ANALYSIS

The equilibria of equation (1) are $(0,0)$, and $(1-b, 1-a)$, for all values of parameters a and b. In addition, if $a = 1$, then every point on the x-axis is an equilibrium point, and if $b = 1$, then every point on the y-axis is an equilibrium point. Finally, if $a = b = 1$, then the two equilibria $(0,0)$ and $(1-b, 1-a)$ coincide, and every point on each coordinate axis is an equilibrium point.

The characteristic equation of the Jacobian evaluated at $(0,0)$ is

$$\left(\lambda - \frac{1}{a}\right)\left(\lambda - \frac{1}{b}\right) = 0. \tag{5}$$

As is well known (see [6, p. 455]), the equilibrium $(0,0)$ of equation (1) is locally asymptotically stable if $a > 1$ and $b > 1$, and it is a source if $a < 1$ and $b < 1$. Finally, if $a > 1$ and $b < 1$, or $a < 1$ and $b > 1$, the equilibrium $(0,0)$ is a saddle point. The local stable manifold theorem assures that a small piece of local unstable manifold W^u_{loc} resembles a line segment. Similarly, the local stable manifold theorem guarantees that a small piece of local stable manifold W^s_{loc} resembles a line segment. In forthcoming sections, we will "globalize" some of these local results. In particular, we will find explicitly both global stable and unstable manifolds in the case of a saddle point equilibrium. The case $a = b = 1$, where the two equilibria coincide, will be treated separately.

We summarize the local stability properties of the equilibrium $(0,0)$ of equation (1) as follows.

THEOREM 2.1.

(a) *Assume that*

$$a > 1 \qquad \textit{and} \qquad b > 1.$$

Then, the equilibrium $(0,0)$ *of equation (1) is locally asymptotically stable.*

(b) *Assume that*

$$a < 1 \qquad \textit{and/or} \qquad b < 1.$$

Then the equilibrium $(0,0)$ *of equation (1) is unstable. More precisely, it is a saddle point equilibrium if* $a < 1$ *and* $b > 1$, *or* $a > 1$ *and* $b < 1$, *and it is a source if* $a < 1$ *and* $b < 1$.

The characteristic equation of the Jacobian evaluated at $E_{a,b} = (1-b, 1-a)$ is

$$\lambda^2 - 2\lambda + 1 - (a-1)(b-1) = 0, \tag{6}$$

with roots

$$\lambda_{\pm} = 1 \pm \sqrt{(a-1)(b-1)}.$$

This shows that at least one of the characteristic roots of the Jacobian computed at the equilibrium $E_{a,b}$ has modulus greater than 1. By a well-known result [6, p. 455], the equilibrium $E_{a,b}$ of equation (1) is locally unstable for all values of a and b. Obviously, the interesting cases that locate the equilibrium $E_{a,b}$ in the first quadrant are

(a) $a < 1$, $b < 1$,
(b) $a < 1$, $b = 1$,
(c) $a = 1$, $b < 1$.

In Case (a), the equilibrium $E_{a,b}$ is a saddle point. The local unstable and stable manifold theorem assures that, near equilibrium, small pieces of local unstable and stable manifolds, W^u_{loc} and W^s_{loc}, respectively, resemble line segments, see [6, p. 457] and [7, Theorem 10.1, p. 182]. We will exhibit explicitly both stable and unstable manifolds in the special case $a = b$, and we will numerically estimate segments of both stable and unstable manifolds for other values of parameters a and b.

In Cases (b) and (c), $E_{a,b} = (0, 1-a)$ and $E_{a,b} = (1-b, 0)$ are nonhyperbolic equilibrium points and so the linearization is inconclusive. Global results for these two cases will be provided in forthcoming sections.

The following statement summarizes the local stability properties of the equilibrium $E_{a,b}$ of equation (1).

THEOREM 2.2. *The equilibrium point $E_{a,b}$ is not locally asymptotically stable for any value of parameters a and b. More precisely, we have the following.*

(a) *Assume that*

$$a < 1, \qquad b < 1.$$

Then the equilibrium $(0,0)$ of equation (1) is a source, while the equilibrium $E_{a,b}$ is a saddle point.

(b) *Assume that*

$$a < 1 \text{ and } b = 1 \qquad \text{or} \qquad a = 1 \text{ and } b < 1.$$

Then the equilibrium $(0,0)$ of equation (1) is a nonhyperbolic equilibrium point.

Our linearized stability analysis indicates several cases with different asymptotic behavior depending on the values of parameters a and b.

CASE 1. $a > 1$, $b > 1$.

CASE 2. $a > 1$, $b = 1$.

CASE 3. $a > 1$, $b < 1$.

CASE 4. $a < 1$, $b = 1$.

CASE 5. $a = b = 1$.

CASE 6. $a < 1$, $b < 1$.

CASE 7. $a < 1$, $b > 1$.

CASE 8. $a = 1$, $b > 1$.

CASE 9. $a = 1$, $b < 1$.

Defining

$$f(c,x,y) = \frac{x}{c+y},$$

we can rewrite equation (1) in the form

$$x_{n+1} = f(a, x_n, y_n), \qquad y_{n+1} = f(b, y_n, x_n). \tag{7}$$

This shows that there is a linear change of variables and parameters showing that some of the cases are essentially identical to other cases. Case 7 is reducible to Case 3, Case 8 is reducible to Case 2, and Case 9 is reducible to Case 4. Accordingly, there are six cases to consider: Cases 1–6. In this paper, we treat Cases 1–5. Case 6 will be taken up in a separate paper [1].

3. GLOBAL RESULTS—COMMON FACTS

The following results are common to all cases.

LEMMA 3.1. *Every solution of equation (1) satisfies*

$$x_n y_n < 1, \qquad n = 1, 2, \dots.$$

The initial point $(0, y_0)$ generates the solution $(0, y_0/b^n)$, and the initial point $(x_0, 0)$ generates the solution $(x_0/a^n, 0)$. Consequently, if $a > 1$, then the initial point $(x_0, 0)$ generates a solution $(x_n, 0)$ such that $\lim_{n\to\infty} x_n = 0$, while if $a < 1$, the initial point $(x_0, 0)$ generates a solution $(x_n, 0)$ such that $\lim_{n\to\infty} x_n = \infty$. Finally, if $a = 1$, then the solution is simply $(x_0, 0)$.

Analogously, if $b > 1$, then the initial point $(0, y_0)$ generates a solution $(0, y_n)$ such that $\lim_{n\to\infty} y_n = 0$, while if $b < 1$, the initial point $(0, y_0)$ generates a solution $(0, y_n)$ such that $\lim_{n\to\infty} y_n = \infty$. Finally, if $b = 1$, then the solution is simply $(0, y_0)$.

PROOF. Using equation (1), we have

$$x_{n+1} y_{n+1} = \frac{x_n}{a + y_n} \frac{y_n}{b + x_n} < \frac{x_n}{y_n} \frac{y_n}{x_n} = 1, \qquad n = 1, 2, \dots.$$

The rest of the proof is obvious. ∎

In view of the linearized stability analysis from the previous section and the obvious fact that the positive initial condition $x_0 > 0$ and/or $y_0 > 0$ implies that the corresponding solution is positive, $x_n > 0$ and/or $y_n > 0$, this result has an important consequence for the dynamics of the iteration as follows.

COROLLARY 3.1. *The coordinate axes are parts of either the global stable manifold or the global unstable manifold of the equilibrium $(0, 0)$ of equation (1).*

That is, if $a > 1$, then the x-axis is part of the global stable manifold $W^s((0, 0))$, while if $a < 1$, then the x-axis is part of the global unstable manifold $W^u((0, 0))$.

Analogously, if $b > 1$, then the y-axis is part of the global stable manifold $W^s((0, 0))$, while if $b < 1$, then the y-axis is part of the global unstable manifold $W^u((0, 0))$.

The next two lemmas provide basic information about the behavior of solutions in the cases when $a > 1$ and/or $b > 1$, and when $a = 1$ or $b = 1$.

LEMMA 3.2. *Assume that $a > 1$. Then, for every solution $\{(x_n, y_n)\}$ of equation (1),*

$$\lim_{n\to\infty} x_n = 0.$$

Assume that $b > 1$. Then, for every solution $\{(x_n, y_n)\}$ of equation (1),

$$\lim_{n\to\infty} y_n = 0.$$

PROOF. The first equation of (1) gives

$$x_{n+1} \leq \frac{x_n}{a},$$

which, by iteration, implies

$$x_{n+1} \leq \frac{1}{a^n} x_0, \qquad n = 0, 1, \dots.$$

In the same way, the second equation of (1) gives

$$y_{n+1} \leq \frac{y_n}{b},$$

completing the proof. ∎

LEMMA 3.3. *Assume that $a = 1$. Then the first component $\{x_n\}$ of every solution $\{(x_n, y_n)\}$ of equation (1) is a strictly decreasing sequence.*

Assume that $b = 1$. Then the second component $\{y_n\}$ of every solution $\{(x_n, y_n)\}$ of equation (1) is a strictly decreasing sequence.

PROOF. Taking into account the positivity of solutions and parameters, the first equation of (1) implies

$$x_{n+1} < x_n.$$

Likewise, the second equation of (1) gives

$$y_{n+1} < y_n,$$

and the proof is complete. ∎

The following provides a kind of closed-form solution to equation (1).

LEMMA 3.4. *Every solution $\{(x_n, y_n)\}$ of equation (1) satisfies*

$$\frac{y_n + a}{x_n + b} = \frac{(y_{n-1} + a)(a x_{n-1} + y_{n-1} + ab)}{(x_{n-1} + b)(x_{n-1} + b y_{n-1} + ab)} \tag{8}$$

and

$$\frac{y_n + a}{x_n + b} = \left(\prod_{k=0}^{n-1} \frac{a x_k + y_k + ab}{x_k + b y_k + ab} \right) \frac{y_0 + a}{x_0 + b}. \tag{9}$$

If the solution has the finite limit (x_∞, y_∞), then

$$\frac{y_\infty + a}{x_\infty + b} = \left(\prod_{k=0}^{\infty} \frac{a x_k + y_k + ab}{x_k + b y_k + ab} \right) \frac{y_0 + a}{x_0 + b}. \tag{10}$$

PROOF. The first two identities are obvious consequences of equation (1) and the third follows from equation (9), assuming that the limit (x_∞, y_∞) exists.

4. GLOBAL RESULTS—CASE $a > 1$, $b > 1$

The next result, based on the previous sections, describes completely the dynamics of equation (1) in Case 1.

THEOREM 4.1. *Assume that $a > 1$ and $b > 1$. Then the equilibrium point $(0, 0)$ is globally asymptotically stable, i.e., every solution $\{(x_n, y_n)\}$ of equation (1) satisfies*

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0.$$

The global stable manifold $W^s((0,0)) = \{(x, y) : x \geq 0, y \geq 0\}$.

5. GLOBAL RESULTS—CASE $a > 1$, $b = 1$

In this case, the equilibrium points consist of all points on the y-axis. In view of Lemmas 3.1–3.3, every solution $\{(x_n, y_n)\}$ of equation (1) satisfies

$$\lim_{n\to\infty} x_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \bar{y} \geq 0.$$

In addition, both sequences $\{x_n\}$ and $\{y_n\}$ are strictly decreasing. Thus, our objective is to provide additional information about $\bar{y}$.

Taking $b = 1$ in equation (9), we obtain

$$\frac{y_n + a}{x_n + 1} = \left(\prod_{k=0}^{n-1} \frac{ax_k + y_k + a}{x_k + y_k + a}\right) \frac{y_0 + a}{x_0 + 1}.$$

Taking limits produces a closed formula for the limit in terms of the iterates

$$\bar{y} = \left(\prod_{k=0}^{\infty} \frac{ax_k + y_k + a}{x_k + y_k + a}\right) \frac{y_0 + a}{x_0 + 1} - a. \tag{11}$$

Since each of the terms in the product is greater than 1, for any finite N, we have the lower bound

$$\bar{y} > \left(\prod_{k=0}^{N} \frac{ax_k + y_k + a}{x_k + y_k + a}\right) \frac{y_0 + a}{x_0 + 1} - a.$$

A simple question will serve to showcase the power of our differential equations-based method. Under what circumstances is $\bar{y} \geq 0$ equal to zero? It seems that, at best, only a partial answer to this question can be derived from (9). Since the product in (9) is greater than 1,

$$\bar{y} > \frac{y_0 + a}{x_0 + 1} - a.$$

Therefore, $\bar{y} > 0$ if the right-hand side of this inequality is positive, which is equivalent to

$$y_0 > ax_0.$$

So, the preceding inequality gives a region in the plane of initial conditions which implies $\bar{y} > 0$. The more satisfactory answer to the question is very simple, but the reasoning is quite different.

THEOREM 5.1. *Assume that $a > 1$ and $b = 1$. Then the global stable manifold $W^s((0,0))$ is precisely the x-axis. Every solution $\{(x_n, y_n)\}$ of equation (1) satisfies*

$$\lim_{n\to\infty} x_n = 0, \qquad \lim_{n\to\infty} y_n = \bar{y},$$

where $\bar{y}$ is given by equation (11). Every solution $\{(x_n, y_n)\}$ for which the initial point (x_0, y_0) satisfies $y_0 > 0$ has a positive limit $\bar{y}$.

PROOF. If $y_0 > 0$ and $x_0 = 0$, we have the trivial case of an equilibrium point where $y_n = y_0$ and $x_n = 0$, for every $n = 0, 1, \ldots$. Suppose then, that $y_0 > 0$ and $x_0 > 0$. With $b = 1$ and $a > 1$, the implicit solution (2) to our differential equation (3) becomes

$$\frac{(y + a)y^{a-1}}{(x + 1)^a} = C. \tag{12}$$

First, we want to show that solutions to the difference equation (1) which begin with positive initial values, pass through decreasing values of C in the last expression (see Figure 1). That is, we need to show that for all $n = 0, 1, \ldots,$

$$\frac{(y_{n+1} + a)\, y_{n+1}^{a-1}}{(x_{n+1} + 1)^a} < \frac{(y_n + a)\, y_n^{a-1}}{(x_n + 1)^a} \tag{13}$$

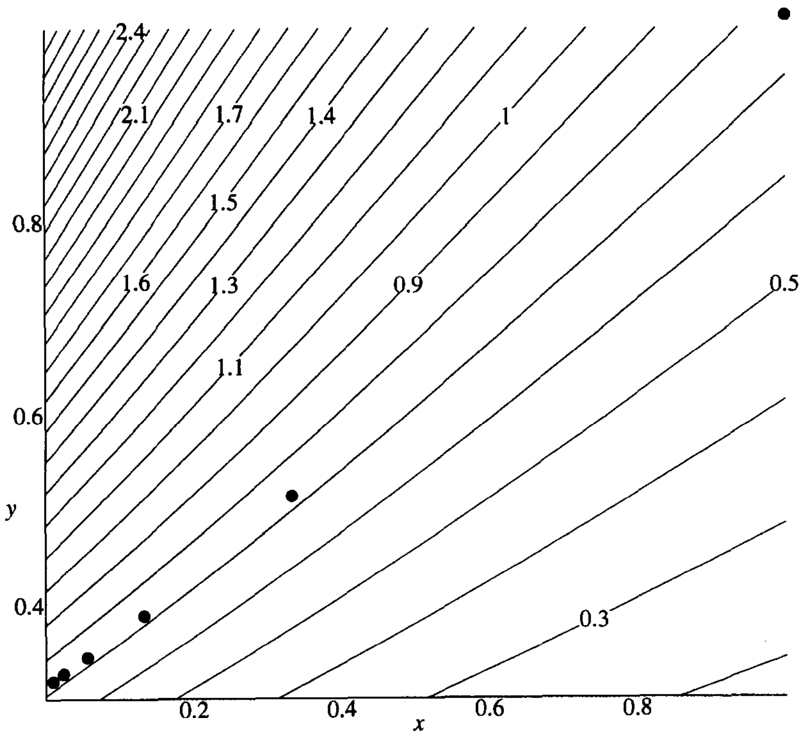

Figure 1. The iterates of the difference equation cross-solution curves of the differential equation in the direction of decreasing level numbers when $a > 1$, $b = 1$. In this illustration, $a = 2$.

or

$$\frac{(y_n + a)^{a-1}(ax_n + y_n + a)}{(x_n + y_n + a)^a} < 1.$$

Letting $y_n + a = k$, $x_n = x$, we need to show that for all $x > 0$,

$$\frac{(ax + k)k^{a-1}}{(x + k)^a} < 1.$$

Obviously, the left-hand side of this inequality equals 1 when $x = 0$. Differentiating with respect to x and using the fact that $a > 1$ and $k > 0$, we get a negative derivative

$$\frac{d}{dx}\frac{(ax + k)k^{a-1}}{(x + k)^a} = -\frac{ak^{a-1}(a - 1)x}{(x + k)^{a+1}}.$$

In view of the preceding remark, we have established inequality (13).

Set

$$c_n = \frac{(y_n + a)\, y_n^{a-1}}{(x_n + 1)^a}.$$

We have just shown that c_n is monotone decreasing. Now consider

$$\begin{aligned} c_{n+1} - c_n &= \frac{y_n^{a-1}\,(y_n + a)^a\,(y_n + a\,(x_n + 1))}{(x_n + 1)^a\,(x_n + y_n + a)^a} - \frac{(y_n + a)\, y_n^{a-1}}{(x_n + 1)^a} \\ &= c_n \frac{(y_n + a)^{a-1}\,(ax_n + y_n + a) - (x_n + y_n + a)^a}{(x_n + y_n + a)^a}. \end{aligned}$$

We want to show that $c_n \geq (1/2)c_N$. Letting $y_n + a = k_n$, we obtain

$$c_{n+1} - c_n = c_n \frac{k_n^{a-1}\,(ax_n + k_n) - (x_n + k_n)^a}{(x_n + k_n)^a}.$$

Expanding in a Taylor polynomial,

$$(x_n + k_n)^a = k_n^a + ak_n^{a-1}x_n + \frac{1}{2}(a-1)a\theta_n^{a-2}x_n^2,$$

we get, for $k_n \le \theta_n \le x_n + k_n$,

$$c_{n+1} - c_n = -c_n \frac{a(a-1)\theta_n^{a-2}x_n^2}{(x_n + k_n)^a}.$$

For any integer $N \ge 0$, $n \ge N+1$, sum both sides of the preceding relation to obtain

$$c_n - c_N = -\sum_{j=N}^{n-1} \left(c_j \frac{a(a-1)\theta_j^{a-2}x_j^2}{(x_j + k_j)^a} \right). \tag{14}$$

The sum of positive terms in the last equation has as an upper bound

$$\frac{(a-1)(x_0 + y_0 + a)^{a-1}}{2(a^a - 1)} \sum_{j=N}^{n-1} c_j x_j^2,$$

which follows because $\theta_j \le k_j + x_j$ and both x_j and y_j are monotone decreasing. However, we showed in (13) that c_n is monotone decreasing, so

$$\sum_{j=N}^{n-1} \left(c_j \frac{a(a-1)\theta_j^{a-2}x_j^2}{2(x_j + k_j)^a} \right) \le \frac{(a-1)(x_0 + y_0 + a)^{a-1} c_N}{2(a^a - 1)} \sum_{j=N}^{n-1} x_j^2.$$

Since

$$x_{j+1} = \frac{x_j}{a + y_j},$$

we have $x_j \le (1/a^j)x_0$ and

$$\begin{aligned} \sum_{j=N}^{n-1} \left(c_j \frac{a(a-1)\theta_j^{a-2}x_j^2}{2(x_j + k_j)^a} \right) &\le \frac{(a-1)(x_0 + y_0 + a)^{a-1} c_N x_0^2}{2(a^a - 1)a^{2N}} \sum_{j=0}^{n-N-1} \left(\frac{1}{a^j} \right)^2 \\ &\le \frac{(x_0 + y_0 + a)^{a-1} x_0^2 c_N}{2(a+1)a^{2N+a-3}}, \end{aligned}$$

after expanding the geometric series. Choosing N sufficiently large so that

$$\frac{(x_0 + y_0 + a)^{a-1} x_0^2}{(a+1)a^{2N+a-3}} \le 1,$$

we have, for $n \ge N+1$,

$$\sum_{j=N}^{n-1} \left(c_j \frac{a(a-1)\theta_j^{a-2}x_j^2}{2(x_j + k_j)^a} \right) \le \frac{1}{2}c_N.$$

Using this estimate and (14), we obtain $c_n - c_N \ge -(1/2)c_N$, hence, $c_n \ge (1/2)c_N$. Therefore, $y_\infty > 0$ because we assumed that $y_0 > 0$, hence, $y_N > 0$, and we have just obtained

$$(y_\infty + a)\, y_\infty^{a-1} \ge \frac{(y_N + a)\, y_N^{a-1}}{2(x_N + 1)^a}. \qquad \blacksquare$$

Our numerical simulations suggest that the limit y_∞ depends continuously on the initial values x_0, y_0 for a fixed value of a. Here is the precise statement.

THEOREM 5.2. *Assume that $a > 1$ and $b = 1$. Then every solution $\{(x_n, y_n)\}$ of equation (1) which is initiated off the x-axis satisfies*

$$\lim_{n\to\infty} x_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} y_n = y_\infty > 0.$$

The limit y_∞ depends continuously on the initial values x_0, y_0 for any fixed value of $a > 1$.

PROOF. Fix $x_0, y_0 \geq 0$ and let (x_0', y_0') lie in a compact neighborhood of (x_0, y_0). Denote by y_∞ and y_∞' the respective limits of the corresponding y_n-sequences. The limit of the x_n-sequence is in every case zero. The infinite product representation (10) gives

$$y_\infty + a = \frac{y_0 + a}{x_0 + 1} \prod_{k=0}^{\infty} \left(1 + \frac{(a-1)x_k}{x_k + y_k + a}\right).$$

First, we want to show that

$$\prod_{k=0}^{\infty} \left(1 + \frac{(a-1)x_k}{x_k + y_k + a}\right) = 1 + O\left(a^{-N}\right),$$

where $O(a^{-N})$ depends only on the fixed value x_0. We will use the fact $(a > 1)$ that $x_k \leq x_0/a^k$.

Now,

$$\ln \prod_{k=N}^{\infty} \left(1 + \frac{(a-1)x_k}{x_k + y_k + a}\right) \leq \sum_{k=N}^{\infty} \frac{(a-1)x_k}{x_k + y_k + a}$$

and

$$\sum_{k=N}^{\infty} \frac{(a-1)x_k}{x_k + y_k + a} \leq (a-1)x_0 \sum_{k=N}^{\infty} \frac{1}{x_k + y_k + a}\frac{1}{a^k} \leq \frac{(a-1)x_0}{a} \sum_{k=N}^{\infty} \frac{1}{a^k} = \frac{(a-1)x_0}{a} \frac{a}{a^N(a-1)} = \frac{x_0}{a^N}.$$

Therefore,

$$\ln \prod_{k=N}^{\infty} \left(1 + \frac{(a-1)x_k}{x_k + y_k + a}\right) \leq \frac{x_0}{a^N}$$

and

$$\prod_{k=N}^{\infty} \left(1 + \frac{(a-1)x_k}{x_k + y_k + a}\right) \leq e^{x_0 a^{-N}},$$

which proves our claim that $O(a^{-N})$ depends only on the fixed value x_0. Similarly, the O' term in the following expression depends only on x_0':

$$\prod_{k=0}^{\infty} \left(1 + \frac{(a-1)x_k'}{x_k' + y_k' + a}\right) = 1 + O'\left(a^{-N}\right).$$

Fix $M > 0$ so that, for every x_0' in the aftermentioned compact neighborhood of (x_0, y_0),

$$\left|O\left(a^{-N}\right)\right| + \left|O'\left(a^{-N}\right)\right| \leq M a^{-N}.$$

For any positive integer N, these estimates result in

$$\begin{aligned} |y_\infty - y_\infty'| \leq & \left| \frac{y_0 + a}{x_0 + 1} \prod_{k=0}^{N-1} \left(1 + \frac{(a-1)x_k}{x_k + y_k + a}\right) - \frac{y_0' + a}{x_0' + 1} \prod_{k=0}^{N-1} \left(1 + \frac{(a-1)x_k'}{x_k' + y_k' + a}\right) \right| \\ & + \left| \frac{y_0 + a}{x_0 + 1} \prod_{k=0}^{N-1} \left(1 + \frac{(a-1)x_k}{x_k + y_k + a}\right) O\left(a^{-N}\right) - \frac{y_0' + a}{x_0' + 1} \prod_{k=0}^{N-1} \left(1 + \frac{(a-1)x_k'}{x_k' + y_k' + a}\right) O'\left(a^{-N}\right) \right|. \end{aligned}$$

Taking the lim sup as $(x_0', y_0') \to (x_0, y_0)$ on both sides of the preceding inequality gives

$$\limsup_{(x_0', y_0') \to (x_0, y_0)} |y_\infty - y_\infty'| \leq M a^{-N}.$$

Since N is arbitrary, we conclude that the limit of y_∞' as $(x_0', y_0') \to (x_0, y_0)$ is y_∞. ∎

6. GLOBAL RESULTS—CASE $a > 1$, $b < 1$

In this case, the only equilibrium point in first quadrant is the origin $(0,0)$. In view of Lemma 3.2, every solution $\{(x_n, y_n)\}$ of equation (1) satisfies

$$\lim_{n\to\infty} x_n = 0.$$

But then the denominator in

$$y_{n+1} = \frac{y_n}{x_n + b}$$

is, for all large n, strictly less than a constant $\beta < 1$, which in turn implies

$$y_{n+1} > \frac{1}{\beta} y_n, \qquad n \geq N.$$

Iterating this inequality, we obtain

$$y_n > \frac{1}{\beta^{n-N}} y_N, \qquad n \geq N,$$

and this forces y_n to infinity. Combined with the result of Lemma 3.2, we have the following.

THEOREM 6.1. *Assume that $a > 1$ and $b < 1$. Then the x-axis is the global stable manifold $W^s((0,0))$. Every solution $\{(x_n, y_n)\}$ of equation (1) which begins off the x-axis satisfies*

$$\lim_{n\to\infty} x_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} y_n = \infty.$$

Consequently, the global unstable manifold $W^u((0,0)) = \{(x,y) : x \geq 0, y > 0\} - W^s((0,0))$.

7. GLOBAL RESULTS—CASE $a < 1$, $b = 1$

In this case, the equilibrium points are all the points on the y-axis, and the x-axis is a part of the unstable manifold $W^u((0,0))$. In addition, all equilibria except $(0,0)$ are nonhyperbolic points.

In view of Lemmas 3.1 and 3.3, every solution $\{(x_n, y_n)\}$ of equation (1) satisfies

$$\lim_{n\to\infty} y_n = \bar{y} \geq 0.$$

Furthermore, $\{y_n\}$ is strictly decreasing if $x_0 \neq 0$. Thus, our objective is to provide additional information about $\bar{y}$. In particular, we will find regions in the first quadrant of the plane of initial conditions $\{(x_0, y_0) : x_0 \geq 0,\ y_0 \geq 0\}$, which generate solutions with two different global asymptotic behaviors:

$$\lim_{n\to\infty} y_n = \bar{y} \geq 1 - a \qquad \text{and} \qquad \lim_{n\to\infty} x_n = 0,$$

and

$$\lim_{n\to\infty} y_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} x_n = \infty.$$

In the language of dynamical systems, we are estimating the basins of attractions of the equilibria $(0, \bar{y})$ and $(\infty, 0)$. Our first result provides information about the basin of attraction for $(0, \bar{y})$.

THEOREM 7.1. *Assume that a is fixed and $0 < a < 1$ and $b = 1$. Then there is a $\bar{y} = \bar{y}(x_0, y_0)$ such that every solution $\{(x_n, y_n)\}$ of equation (1) whose initial conditions (x_0, y_0) satisfy*

$$y_0 \geq 1 - a \qquad \text{and} \qquad \frac{(x_0 + 1)^a\, y_0^{1-a}}{y_0 + a} \leq (1 - a)^{1-a} \tag{15}$$

converges to the limit $(0, \bar{y})$, where $\bar{y} \geq 1 - a$.

PROOF. If we write the differential equation (3), with $b = 1$, in the form

$$\frac{dx}{dy} = \frac{(x+1)(y+a-1)}{(y+a)y},$$

x becomes visible as a well-defined function of y which is strictly decreasing to the left of $1-a$ and strictly increasing to the right of $1-a$. Therefore, $1-a$ is the location of a global minimum for x as a function of y along any one of the level curves. We want to know when the value of this minimum is zero. When $b = 1$, solution (2) of equation (3) is

$$\frac{y+a}{(x+1)^a y^{1-a}} = \frac{y_0+a}{(x_0+1)^a y_0^{1-a}}.$$

Substituting $y = 1-a$ and $x = 0$, we get

$$\frac{y_0+a}{(x_0+1)^a y_0^{1-a}} = (1-a)^{a-1},$$

and the corresponding particular solution of our differential equation is

$$\frac{y+a}{(x+1)^a y^{1-a}} = (1-a)^{a-1}. \tag{16}$$

If we can show that each succeeding pair of iterates in the difference equation lies on a level curve (12) with a higher index number C, then initial values on or above the critical level curve (16) with $y_0 \geq 1-a$ generate solutions which remain trapped in that region. Consequently, $\{y_n\}$ must converge to a point $\bar{y} \geq 1-a$ on the y-axis, while $\{x_n\}$ converges to zero. Therefore, we need to prove that for $n = 0, 1, \dots,$

$$\frac{y_{n+1}+a}{(x_{n+1}+1)^a\, y_{n+1}^{1-a}} > \frac{y_n+a}{(x_n+1)^a\, y_n^{1-a}}.$$

This behavior is general for Case 4 and is illustrated in Figure 2. Using equation (1) in the last inequality, we obtain

$$\frac{y_n/(x_n+1)+a}{(y_n/(x_n+a)+1)^a\,(y_n/(x_n+1))^{1-a}} > \frac{y_n+a}{(x_n+1)^a y_n^{1-a}}$$

or

$$\frac{(y_n+a)^a\,(y_n+a\,(x_n+1))}{y_n^{1-a}\,(x_n+1)^a\,(x_n+y_n+a)^a} > \frac{y_n+a}{(x_n+1)^a\, y_n^{1-a}}$$

or

$$\frac{(y_n+a)^{a-1}\,(y_n+ax_n+a)}{(x_n+y_n+a)^a} > 1.$$

Let c and r be defined as follows:

$$r(x) = \frac{ax+c}{(x+c)^a c^{1-a}},$$

where $c = y_n + a$. Obviously, $r(0) = 1$, so we only need to show that r is an increasing function of x,

$$\frac{dr}{dx} = -\frac{ac^{a-1}}{(x+c)^a} > 0.$$

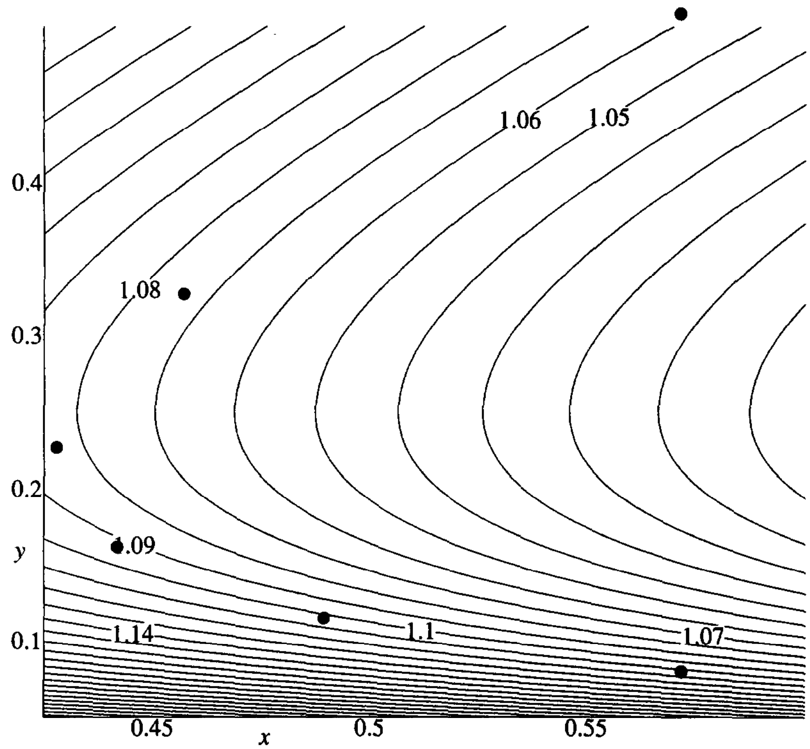

Figure 2. The iterates of the difference equation cross-solution curves of the differential equation in the direction of increasing level numbers when $a < 1$, $b = 1$. In this illustration, $a = 3/4$.

So, to get the convergence of y_n to a positive limit by means of this "trapping" argument, the initial conditions must satisfy $y_0 \geq 1 - a$ and

$$\frac{y_0 + a}{{y_0}^{1-a}(x_0+1)^a} \geq \frac{1}{(1-a)^{1-a}},$$

which is equivalent to the second condition in (15).

Furthermore, we can estimate a lower bound for this limit as the larger of the two y-roots in the following implicit equation:

$$\frac{y+a}{(x+1)^a y^{1-a}} - \frac{y_0 + a}{(x_0+1)^a y_0^{1-a}} = 0,$$

so long as convergence condition (15) is satisfied for the initial values. ∎

Our next sufficient condition provides a clearer indication of how large y_0 must be relative to x_0 to ensure the asymptotic behavior described in the previous theorem.

COROLLARY 7.1. *Assume that a is fixed and $0 < a < 1$ and $b = 1$. Then there is a $\bar{y} = \bar{y}(x_0, y_0)$ such that every solution $\{(x_n, y_n)\}$ of equation (1) whose initial conditions (x_0, y_0) satisfy*

$$y_0 \geq (1-a)^{1-1/a} x_0 + (1-a)^{1-1/a} - a \tag{17}$$

converges to the limit $(0, \bar{y})$, with $\bar{y} \geq 1 - a$.

PROOF. First, we show that the second part of (15) is equivalent to

$$x_0 \leq (1-a)^{1/a-1}\left(1 + \frac{a}{y_0}\right)^{1/a-1} y_0 + a(1-a)^{1/a-1}\left(1+\frac{a}{y_0}\right)^{1/a-1} - 1.$$

The preceding inequality is equivalent to

$$(x_0+1)^a \le (1-a)^{1-a}\left(\frac{y_0+a}{y_0}\right)^{1-a}(y_0+a)^a,$$

which is equivalent to the second part of (15). Now, the right-hand side of the last inequality is greater than

$$(1-a)^{1/a-1}y_0 + a(1-a)^{1/a-1} - 1,$$

so if we take

$$(1-a)^{1/a-1}y_0 + a(1-a)^{1/a-1} - 1 \ge x_0,$$

we will satisfy condition (15). In other words, if the initial pair lies on or above the straight line

$$y = (1-a)^{1-1/a}x + (1-a)^{1-1/a} - a, \tag{18}$$

it will satisfy the second part of (15).

With respect to the first part of condition (15), note that when $x = 0$,

$$y = (1-a)^{1-1/a} - a = 1 - a + (1-a)^{1-1/a} - 1 > 1 - a.$$

The conclusion is that any initial pair on or above the line (18) satisfies both parts of condition (15).

Interestingly, it turns out that there is a universal condition, i.e., independent of a, which implies (15).

COROLLARY 7.2. *Assume that a is fixed and $0 < a < 1$ and $b = 1$. Then there is a $\bar{y} = \bar{y}(x_0, y_0)$ such that every solution $\{(x_n, y_n)\}$ of equation (1) whose initial conditions (x_0, y_0) satisfy*

$$y_0 \ge 1 \qquad \text{and} \qquad y_0 e^{1/y_0} \ge e\,(x_0 - 1) \tag{19}$$

converges to the limit $(0, \bar{y})$, where $\bar{y} \ge 1 - a$.

PROOF. If we rewrite (15) as

$$x_0 + 1 \le (1-a)^{(1-a)/a}\left(1 + \frac{a}{y_0}\right)^{1/a} y_0,$$

and let a go to zero, elementary arguments show that the curve determined by (19) is an upper bound for all the curves determined by (15). Therefore, an initial pair satisfying (19) will result in convergence for any a, $0 < a < 1$. See Figure 3.

We follow with a universal sufficient condition for the other type of asymptotic behavior in this case: $x_n \to \infty$ and $y_n \to 0$. For any $a \in (0,1)$, this sufficient condition on the initial pair is simply

$$y_0 \le x_0. \tag{20}$$

This condition is obtained from (18) by taking $a \to 1$. As $a \to 1$, line (18) squeezes the curve (16) onto the bisector $y_0 = x_0$, squeezing out of existence the region where both types of asymptotic behavior are possible. See the discussion immediately following the proof of the next theorem.

THEOREM 7.2. *Assume that $0 < a < 1$ and $b = 1$. Every solution $\{(x_n, y_n)\}$ of equation (1) for which condition (20) is satisfied converges to $(\infty, 0)$.*

PROOF. Since $a < 1$ and $b = 1$, identity (8) implies

$$\frac{y_n + a}{x_n + 1} = \frac{y_{n-1} + a}{x_{n-1} + 1}\,\frac{a x_{n-1} + y_{n-1} + a}{x_{n-1} + y_{n-1} + a}.$$

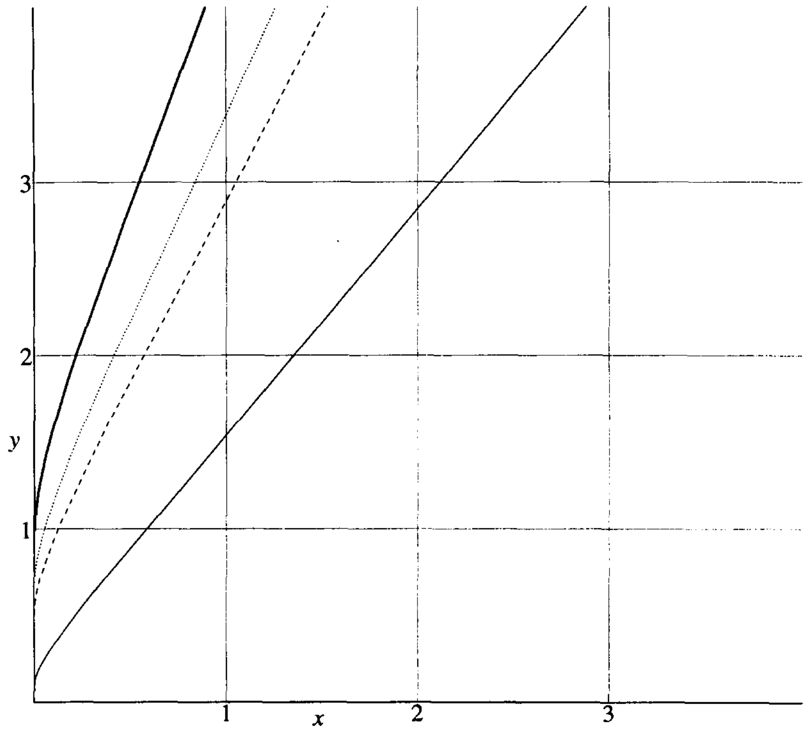

Figure 3. $0 < a < 1$, $b = 1$. The lowest line is the graph of (16) with $a = 9/10$. In the dashed line, $a = 1/2$. In the dotted line, $a = 1/3$. The heavy line shows the curve defined by (19).

Iterating the relation backwards terminates with

$$\frac{y_n + a}{x_n + 1} < \frac{y_0 + a}{x_0 + 1}\frac{ax_0 + y_0 + a}{x_0 + y_0 + a}$$

and

$$y_n + a < \frac{y_0 + a}{x_0 + 1}\frac{ax_0 + y_0 + a}{x_0 + y_0 + a}(x_n + 1).$$

On the right-hand side above, substitute

$$x_n = \frac{y_n}{y_{n+1}} - 1$$

to get

$$y_n + a < \frac{y_0 + a}{x_0 + 1}\frac{ax_0 + y_0 + a}{x_0 + y_0 + a}\frac{y_n}{y_{n+1}}$$

and

$$y_{n+1} < \frac{y_0 + a}{x_0 + 1}\frac{ax_0 + y_0 + a}{x_0 + y_0 + a}\frac{y_n}{y_n + a} = K\frac{y_n}{y_n + a}.$$

In view of the comparison result for difference inequalities, (see [8, pp. 14–21]), $y_n \le z_n$ for every $n = 1, 2, \ldots$, if $y_0 \le z_0$, where z_n is a solution of the corresponding difference equation

$$z_{n+1} = K\frac{z_n}{z_n + a}.$$

This is Riccati's difference equation, with explicit solution of the form

$$z_n = \frac{1}{1/(K - a) + (1/z_0 - 1/(K - a))(a/K)^n}.$$

Clearly, $z_n \to 0$ as $n \to \infty$ if and only if $a > K$. The last requirement is equivalent to

$$y_0 (y_0 + a) < a x_0 (x_0 + 1) . \tag{21}$$

When satisfied, equation (21) implies that $y_n \to 0$ as $n \to \infty$. The denominator in the first equation of (1) is eventually strictly less than 1 and this forces $x_n \to \infty$.

Now, it is easy to see that $y_0 \le a x_0$ implies (21), above. So, to prove the original claim, we consider the inverse image of the line $y = ax$ under the map

$$\mathbf{f} : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \dfrac{x}{a+y} \\ \dfrac{y}{b+x} \end{pmatrix}$$

of system (1). The inverse map of $\mathbf{f}$ is

$$\mathbf{f}^{-1}(x,y) = S(x,y) = \left(\frac{x(a+by)}{1-xy}, \frac{y(b+ax)}{1-xy} \right).$$

Set $b = 1$ and consider the image of the line in question

$$S(t, at) = \left(\frac{(at+a)t}{1-at^2}, \frac{a(at+1)t}{1-at^2} \right).$$

Restricting t to $0 < t < 1/\sqrt{a}$, we have

$$x = \frac{(at+a)t}{1-at^2}, \qquad y = \frac{a(at+1)t}{1-at^2}, \tag{22}$$

and

$$\frac{dy}{dx} = \frac{at^2 + 2at + 1}{at^2 + 2t + 1}.$$

When $t = 0$, $\frac{dy}{dx} = 1$. Also, as t approaches $1/\sqrt{a}$, $\frac{dy}{dx}$ approaches $\sqrt{a}$. In fact, the line $y = \sqrt{a}x$ is a lower bound for the parametric curve given by equation (22). It is clear that any point on or underneath the line $y = \sqrt{a}x$ will be mapped by $\mathbf{f}$ to a point underneath the line $y = ax$. Now we repeat the argument, replacing the line $y = ax$ with the line $y = \sqrt{a}x$ to get a new line $y = a^{1/4}x$ bounded above by $S(S(t, at))$. Any point on this line will, in two iterations of $\mathbf{f}$, be underneath the line $y = ax$ and, hence, be part of sequence in which $x_n \to \infty$ and $y_n \to 0$.

Finally, it becomes clear how the proof terminates. We iterate the preceding argument forever and get as our ultimate line $y = x$. Any solution which starts on or below this line (i.e., our asserted universal condition $y_0 \le x_0$) will eventually be underneath the line $y = ax$ and be part of a sequence with the desired asymptotic behavior. Note that this asymptotic result is independent of $a \in (0,1)$.

Our development thus far shows that the positive plane is separated into three regions. We have just treated the region which lies below the lines $y = 1 - a$, $0 < x \le 1 - a$ and $y = x$, $x \ge 1 - a$, where $x_n \to \infty$ and $y_n \to 0$, i.e., the region on or below the dashed line in Figure 4. Previously, we showed that all initial pairs on or above the curve (16) (the heavy line in Figure 4) result in convergence of y_n to a finite limit $\bar{y} \ge 1 - a$, while x_n goes to zero. Based on our numerical simulations, points slightly to the right of the latter curve can show the same convergent behavior because, as illustrated in Figure 2, solutions might be able to "jump over" the heavy line in Figure 4 and enter the region of convergence. However, a precise meaning to "slightly" remains elusive at this time. For every $a \in (0,1)$, these two regions separate the two fundamentally

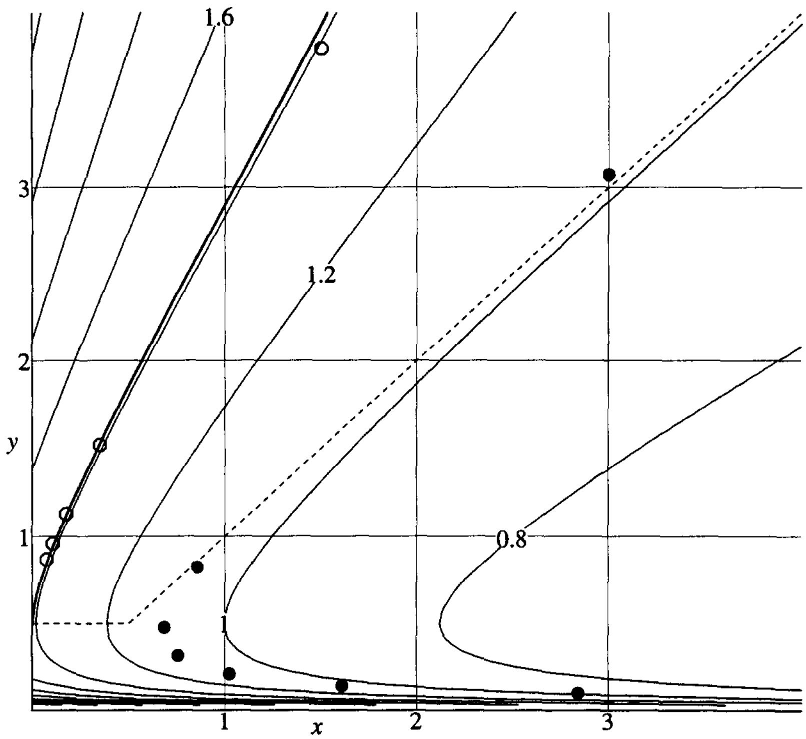

Figure 4. $a = 1/2$, $b = 1$. A divergent solution is initiated at the point (3,3). Most solutions in the region between the dashed and heavy lines also diverge, yet it is possible for solutions close enough to the heavy line to cross into the region of convergence, e.g., the sequence of open circles initiated at (1.5,3.8).

different types of asymptotic behavior. In the middle region, the behavior is predominantly divergent, however, both types are possible in view of the preceding remark.

As in Case 2, numerical simulations indicate that the limit y_∞ depends continuously on the initial values x_0, y_0 for a fixed value of a so long as the initial point (x_0, y_0) lies in the interior of the region where the finite limiting value y_∞ exists. The continuous dependence is not true on the boundary of the region defined by (15). Every initial pair with $x_0 > 0$ and $y_0 \le 1 - a$ has x_n diverging to ∞, so there exist initial pairs arbitrarily close to $(0, 1-a)$ which then move arbitrarily far from $(0, 1-a)$. On the other hand, if $(x_0, y_0) = (0, 1-a)$, then $(x_n, y_n) = (0, 1-a)$ for every n and so $y_\infty = 1 - a$. However, it is possible to show that any limit y_∞ which results from starting in the interior B of the above-mentioned region of guaranteed convergence (determined by (15)) will exhibit continuous dependence on the initial conditions as follows.

THEOREM 7.3. *Assume that $0 < a < 1$ and $b = 1$. Then every solution $\{(x_n, y_n)\}$ of equation (1) which starts in the region B satisfies*

$$\lim_{n\to\infty} x_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} y_n = y_\infty.$$

Furthermore, y_∞ depends continuously on the initial values x_0, y_0 for any fixed value of a, $0 < a < 1$.

PROOF. The proof is similar to the proof of Theorem 5.2, so we will only provide an outline. First, displace the region B vertically by small $\epsilon > 0$ to get a new region B_ϵ (in other words, choose a slightly higher-level curve of the corresponding differential equation (3)). Because every solution of the difference equation which starts in B_ϵ remains there, we have $y_n > 1 - a + \epsilon$, and thus,

$$x_{n+1} < \frac{x_n}{1+\epsilon} \Longrightarrow x_n < \frac{x_0}{(1+\epsilon)^n}.$$

Setting $A = 1+\epsilon > 1$, the proof can proceed as in the proof of Theorem 5.2, so long as the neighborhood centered at (x_0, y_0) is contained entirely within the region B_ϵ. But now the arbitrariness of ϵ allows us to conclude that the limit y_∞ resulting from any choice of initial pair (x_0, y_0) in the original region B of the plane of initial conditions, i.e., the region above the line $y = 1-a$ and the curve

$$\frac{(x+1)^a y^{1-a}}{y+a} = (1-a)^{1-a}$$

shows continuous dependence on initial pairs taken from B. ∎

8. GLOBAL RESULTS—CASE $a = b = 1$

In this case, the equilibrium points $(0,0)$ and $E_{a,b}$ coincide and both axes consist of equilibrium points. $(0,0)$ is a nonhyperbolic equilibrium point. For $a = b = 1$, system (1) becomes

$$x_{n+1} = \frac{x_n}{1+y_n}, \quad y_{n+1} = \frac{y_n}{1+x_n}, \qquad n = 0, 1, \dots, \tag{23}$$

and expression (8) becomes

$$\frac{y_n+1}{x_n+1} = \frac{y_{n-1}+1}{x_{n-1}+1}.$$

Thus, the expression

$$I(x_n, y_n) = \frac{y_n+1}{x_n+1} \tag{24}$$

is an invariant for system (23), therefore, $I(x_n, y_n) = I(x_0, y_0)$, $n = 0, 1, \dots$. It follows from (23) that x_n and y_n are strictly decreasing sequences, hence, convergent

$$\lim_{n\to\infty} x_n = \bar{x}, \qquad \lim_{n\to\infty} y_n = \bar{y}.$$

Using the invariant I, we obtain $I(\bar{x}, \bar{y}) = I(x_0, y_0)$. System (23) implies that

$$\frac{y_{n+1}-y_n}{x_{n+1}-x_n} = \frac{y_n+1}{x_n+1} = I(x_n, y_n) = I(x_0, y_0).$$

Thus, we obtain

$$y_{n+1} - I(x_0, y_0)\, x_{n+1} = y_n - I(x_0, y_0)\, x_n,$$

which shows that the expression $J(x_n, y_n) = y_n - I(x_0, y_0)x_n$ is an additional invariant of (23). Therefore,

$$J(x_n, y_n) = J(x_0, y_0),$$

that is,

$$y_n - I(x_0, y_0)\, x_n = y_0 - I(x_0, y_0)\, x_0 = \frac{y_0 - x_0}{x_0+1}.$$

So, we have

$$y_n - y_0 = I(x_0, y_0)\,(x_n - x_0). \tag{25}$$

Thus, every solution of (23) lies on the straight line $y - y_0 = I(x_0, y_0)(x - x_0)$. Substituting

$$y_n = \frac{x_n}{x_{n+1}} - 1$$

into equation (25) yields

$$x_{n+1} = \frac{x_0+1}{y_0+1}\frac{x_n}{x_n+1} = A\frac{x_n}{x_n+1}.$$

This is Riccati's difference equation, whose general solution is

$$x_n = \frac{1}{1/(A-1) + (1/x_0 - 1/(A-1))1/A^n}, \qquad \text{if } A \neq 1, \tag{26}$$

and

$$x_n = \frac{1}{n + 1/x_0}, \qquad \text{if } A = 1. \tag{27}$$

If $A < 1$, which is equivalent to $x_0 < y_0$, then $\{(x_n, y_n)\} \to (0, (y_0 - x_0)/(y_0 + 1))$ as $n \to \infty$. If $A > 1$, which is equivalent to $x_0 > y_0$, then $\{(x_n, y_n)\} \to ((x_0 - y_0)/(y_0 + 1), 0)$ as $n \to \infty$. Finally, if $A = 1$, which is equivalent to $x_0 = y_0$, then $\{(x_n, y_n)\} \to (0, 0)$ as $n \to \infty$.

Our results for this case, which support the claims made on the very elementary web page mentioned in the Introduction, are summarized as follows.

THEOREM 8.1. *Assume that $a = b = 1$. Then, system (23) possesses an invariant (24), and all solutions belong to the lines (25). In addition, every solution is given in closed form by formulas (26), (27), and (25).*

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