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# Ruijsenaars' hypergeometric function and the modular double of $\mathscr{U}_q(\mathfrak{sl}_2(\mathbb{C}))$

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**Abstract**

Simultaneous eigenfunctions of two Askey–Wilson second-order difference operators are constructed as formal matrix coefficients of the principal series representation of the modular double of the quantized universal enveloping algebra $\mathscr{U}_q(\mathfrak{sl}_2(\mathbb{C}))$. These eigenfunctions are shown to be equal to Ruijsenaars' hypergeometric function under a proper parameter correspondence.


*Keywords:* Quantum groups; Modular double; Generalized hypergeometric functions; Hyperbolic Gamma-functions

## 1. Introduction

The main goal of this article is to construct a solution of two commuting Askey–Wilson second-order difference equations using representation theory of the modular double of the quantized universal enveloping algebra $\mathcal{U}_q$ of $\mathfrak{sl}_2(\mathbb{C})$. Furthermore, we relate this solution to Ruijsenaars' hypergeometric function from [11].

By Masuda et al. [8] there exist three inequivalent $*$-structures on $\mathcal{U}_q$, one associated to the real form $\mathfrak{su}(2)$ of $\mathfrak{sl}_2(\mathbb{C})$, one associated to $\mathfrak{su}(1,1)$, and one to $\mathfrak{sl}_2(\mathbb{R})$. Koornwinder [7], Noumi and Mimachi [9], and Koelink [5] have shown that

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the Askey–Wilson polynomials arise as matrix coefficients of $*$-unitary irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$. To prove these results they used the fact that the Askey–Wilson second-order difference operator arises as the radial part of the quantum Casimir in $\mathcal{U}_q$ when calculated with respect to Koornwinder's [7] twisted primitive elements. In [6,14] Koelink and Stokman constructed the trigonometric Askey–Wilson functions as matrix coefficients of $*$-unitary irreducible representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$. In this paper we consider matrix coefficients of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$-representations.

An essential tool is the embedding of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$ in Faddeev's [2] modular double of $\mathcal{U}_q$. The modular double consists of two commuting copies of the quantized universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ with deformation parameters $q = e^{\pi i w_1/w_2}$ and $\tilde{q} = e^{\pi i w_2/w_1}$ $(w_1, w_2 \in \mathbb{R}_{>0})$, respectively. Kharchev et al. [4] made the crucial observation that the algebraic version $\pi_\lambda$ of the principal series representation of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$ on the space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$ can be extended to a representation of the modular double on the same space. In the same article they construct generalized Whittaker functions as matrix coefficients of $\pi_\lambda$.

We construct joint eigenvectors to the action under $\pi_\lambda$ of two commuting twisted primitive elements (one for each copy of the quantized universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{R})$ inside the modular double) in terms of Ruijsenaars' [10] hyperbolic gamma function. The action of the two commuting quantum Casimir elements in the modular double shows that the corresponding matrix coefficients, for which we have an explicit integral representation, satisfy Askey–Wilson second-order difference equations in step directions $iw_1$ and $iw_2$. By construction these matrix coefficients are invariant under interchanging of $w_1$ and $w_2$. We show duality of this solution $S$ in its spectral variable $\lambda$ and its geometric variable. Consequently, it satisfies another two Askey–Wilson second-order difference equations in its spectral variable.

In a series [11–13] of papers, Ruijsenaars considered a solution $R$ of the same Askey–Wilson difference equations. These equations arose in his study of relativistic quantum integrable systems. Ruijsenaars defined the hypergeometric function $R$ as a Barnes' type integral with integrand expressed in terms of the hyperbolic gamma function. Subsequently, he established for $R$ duality, $D_4$-symmetry in the parameters, asymptotic behaviour and the reduction to Askey–Wilson polynomials. We use these properties to show equality of $R$ to $S$, which is not apparent from their explicit integral representations.

The structure of this paper is as follows. In Sections 2 and 3 we recall some properties of the hyperbolic gamma function and of Ruijsenaars' hypergeometric function $R$, respectively. In Section 4 we define the modular double of $\mathcal{U}_q$ and its principal series representation on meromorphic functions. In Section 5 we consider the corresponding eigenvalue problem of two commuting twisted primitive elements. Using the matrix coefficients of the principal series representation we construct a solution $S$ to the Askey–Wilson difference equations in Section 6, and we establish the duality of $S$. In Section 7 we show by a direct calculation that $S$ reduces to the Askey–Wilson polynomials for certain discrete values of the spectral parameter. Finally, in Section 8 we show that $S$ equals Ruijsenaars' hypergeometric function $R$.

### 1.1. Notational conventions

If $\pm$ appears inside the argument of functions we mean a product, e.g.

$$f(z \pm a) = f(z+a)f(z-a). \tag{1.1}$$

Otherwise it means that all sign combinations are possible.

Whenever we use a square root, we always mean the branch which has a cut along the negative real line and maps the positive real line to itself.

## 2. The hyperbolic gamma function

Both Ruijsenaars' and our solution to the Askey–Wilson second-order difference equations are expressible in terms of the hyperbolic gamma function, which was introduced in [10]. Let us therefore recall some basic properties of this function, see [10] and the appendices of [11] for more details. Some results are stated for the parameters $w_1$ and $w_2$ in a larger set than in the corresponding results of the references, but these extensions are all obvious.

Let us first define for $w_1, w_2 \in \mathbb{C}_+ = \{z \in \mathbb{C} | \Re(z) > 0\}$,

$$g(w_1, w_2; z) = \int_0^\infty \left( \frac{\sin(2yz)}{2\sinh(w_1 y)\sinh(w_2 y)} - \frac{z}{w_1 w_2 y} \right) \frac{dy}{y}. \tag{2.1}$$

Note that the integrand has no pole at 0. To ensure convergence of the integral at infinity however, we must impose the condition $|\Im(z)| < \Re(w)$, where $w$ is defined by

$$w = \frac{w_1 + w_2}{2}.$$

The hyperbolic gamma function $G(z) = G(w_1, w_2; z)$ for $|\Im(z)| < \Re(w)$ is now defined by

$$G(w_1, w_2; z) = e^{ig(w_1, w_2; z)}. \tag{2.2}$$

The hyperbolic gamma function $G$ owes its name to the fact that it satisfies the difference equations

$$\begin{aligned} G(z + iw_1/2) &= 2\cosh(\pi z/w_2) G(z - iw_1/2), \\ G(z + iw_2/2) &= 2\cosh(\pi z/w_1) G(z - iw_2/2). \end{aligned} \tag{2.3}$$

In these equations we suppress the $w_1$ and $w_2$ dependence of $G$, which we continue to do whenever this does not cause confusion. These two difference equations allow for an analytic continuation of $G$ to a meromorphic function on $\mathbb{C}$. The hyperbolic gamma function can also be expressed in terms of Barnes' double gamma function, or Kurokawa's double sine function. Details can be found in [11, Appendix A].

Let us first note a few symmetries of the hyperbolic gamma function, which are all obvious from (2.1):

$$G(w_1, w_2; z) = G(w_2, w_1; z), \tag{2.4}$$

$$G(w_1, w_2; z) = G(w_1, w_2; -z)^{-1}, \tag{2.5}$$

$$G(w_1, w_2; z) = \overline{G(\bar{w}_1, \bar{w}_2, -\bar{z})}, \tag{2.6}$$

$$G(\mu w_1, \mu w_2; \mu z) = G(w_1, w_2; z) \quad (\mu \in \mathbb{R}_{>0}). \tag{2.7}$$

The pole and zero locations of $G$ are easily derived from the difference equations (2.3), since $G$ has no poles or zeros in the strip $z \in \mathbb{R} \times i(-\Re(w), \Re(w))$ in view of (2.2). The zeros of $G$ are contained in the set

$$\Lambda_+ = iw + iw_1\mathbb{Z}_{\geqslant 0} + iw_2\mathbb{Z}_{\geqslant 0} \tag{2.8}$$

and the poles in $-\Lambda_+$. The pole at $z = -iw$ is simple, and its residue equals

$$\frac{i}{2\pi}\sqrt{w_1 w_2}. \tag{2.9}$$

If $w_1/w_2$ is irrational all other poles are also simple and their residues can be calculated from (2.9) and the difference equations (2.3), see [10, Proposition III.3].

For later purposes it is convenient to call an infinite sequence of points in $\mathbb{C}$ increasing (respectively decreasing) if it is contained in a set of the form $a + \Lambda_+$ (respectively $a - \Lambda_+$) for some $a \in \mathbb{C}$. In this terminology, $G$ has one increasing zero sequence and one decreasing pole sequence.

We also need an estimate for $G(z)$ as $\Re(z) \to \infty$ and $\Im(z)$ stays bounded. In fact, we only need it for the quotient of two hyperbolic gamma functions, which is easily derived from the estimate of the hyperbolic gamma itself as described in [10, Proposition III.4; 12, (3.3)]. For $a, b \in \mathbb{C}$ and $w_1, w_2 \in (0, \infty)$ the resulting estimate reads

$$\frac{G(z-a)}{G(z-b)} = \exp\left(\frac{\pi}{2iw_1w_2}\left(2z(b-a) + a^2 - b^2 + f(z)\right)\right), \tag{2.10}$$

where $f(z)$ satisfies for $\Re(z) > \max(w_1, w_2) + \max(\Re(a), \Re(b))$,

$$|f(z)| < C(w_1, w_2, \Im(z), a, b)e^{-\pi\Re(z)/\max(w_1,w_2)}, \tag{2.11}$$

with $C$ depending continuously on $(0, \infty)^2 \times \mathbb{R} \times \mathbb{C}^2$.

We also use the description of $G$ as a quotient

$$G(z) = \frac{E(z)}{E(-z)}, \tag{2.12}$$

where $E$ is an entire function with zeros at $\Lambda_+$ which are all simple if $w_1/w_2$ is irrational. For a precise definition of $E$, see [11, Appendix A].

We will occasionally meet functions defined by an integral of the form

$$M(u,d) = \int_{\mathbb{R}} \prod_{j=1}^{n} \frac{G(w_1,w_2;z-u_j)}{G(w_1,w_2;z-d_j)}\,dz \tag{2.13}$$

for $w_1, w_2 > 0$ and for parameters $u_j$ and $d_j$ satisfying $|\Im(u_j)|, |\Im(d_j)| < w$ and $\Im(\sum_{j=1}^{n}(u_j - d_j)) > 0$. These conditions ensure that the integral is well defined (the contour meets no poles and it decreases exponentially at $\pm\infty$). In [11, Appendix B] it is shown that there exists a unique analytic extension of

$$M(u,d) \prod_{j,k=1}^{n} E(-iw + u_j - d_k)$$

to the set $\{(w_1, w_2, u, d) \in \mathbb{C}_+^2 \times \mathbb{C}^{2n} | \Im(\sum(u_j - d_j)/w_1w_2) > 0\}$. Hence $M(u,d)$ is a meromorphic function which can only have poles when some $E(-iw + u_j - d_k)$ is zero.

## 3. Ruijsenaars' hypergeometric function

Ruijsenaars [11] introduced a generalization $R$ of the hypergeometric function as a Barnes' type integral. We recall several properties of $R$ from [11,12] which we will need to relate $R$ to the formal matrix coefficients we are going to define in subsequent sections.

We define Ruijsenaars' hypergeometric function in terms of a parameter set $\gamma_\mu$ ($\mu = 0, 1, 2, 3$), which is related to Ruijsenaars' original $c$-parameters by [12, (1.11)]. Dual parameters $\hat{\gamma}_\mu$ are defined as

$$\begin{pmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \\ \hat{\gamma}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}. \tag{3.1}$$

We denote the set of parameters $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ by $\gamma$ and the set of dual parameters by $\hat{\gamma}$. Note that taking dual parameters is an involution, $\hat{\hat{\gamma}} = \gamma$.

Ruijsenaars' hypergeometric function $R$ is now defined for generic parameters $w_1, w_2 \in \mathbb{C}_+, \gamma \in \mathbb{C}^4$ by

$$R(\gamma; x, \lambda) = \frac{1}{\sqrt{w_1 w_2}} \int_{\mathcal{C}} \frac{G(z \pm x + i\gamma_0)G(z \pm \lambda + i\hat{\gamma}_0)}{G(\pm x + i\gamma_0)G(\pm\lambda + i\hat{\gamma}_0)G(z + iw)} \times \prod_{j=1}^{3} \frac{G(i\gamma_0 + i\gamma_j + iw)}{G(z + i\gamma_0 + i\gamma_j + iw)}\, dz. \tag{3.2}$$

Note that we use convention (1.1) in this expression. The integral is taken over a contour $\mathcal{C}$, which is a deformation of $\mathbb{R}$ seperating the poles of the numerator from the zeros of the denominator (equivalently, $\mathcal{C}$ separates the increasing pole sequences of the integrand from the downward pole sequences). $R$ has an analytic extension to a meromorphic function on $(w_1, w_2, \gamma, x, \lambda) \in \mathbb{C}_+^2 \times \mathbb{C}^6$, with possible poles for fixed values of $w_1$, $w_2$, and $\gamma$ at

$$x \in \pm(\Lambda_+ - i\gamma_j), \quad \lambda \in \pm(\Lambda_+ - i\hat{\gamma}_j) \quad (j = 0, 1, 2, 3). \tag{3.3}$$

Recall that $\Lambda_+$ is defined by (2.8).

We now look at the Askey–Wilson second-order difference equations which $R$ satisfies. The equations are obtained from [11, Theorem 3.1] by not only replacing the $c$-variables by $\gamma$, but also multiplying the equations by a constant. These descriptions of the Askey–Wilson difference equations are more convenient for the representation theoretic approach we consider in the following sections.

Let us define the function $A$ by

$$\begin{aligned} A(w_1, w_2, \gamma; x) &= -\frac{e^{\pi i w_1/w_2 + 2\pi i\hat{\gamma}_0/w_2}}{\sinh\left(\frac{2\pi x}{w_2}\right)\sinh\left(\frac{2\pi(x+iw)}{w_2}\right)} \times \prod_{j=0}^{3} \cosh\left(\frac{\pi}{w_2}\left(x + \frac{iw_1}{2} + i\gamma_j\right)\right) \\ &= \frac{1}{(1 - e^{4\pi x/w_2})(1 - e^{4\pi(x+iw)/w_2})} \prod_{j=0}^{3}(1 - e^{2\pi(iw + i\gamma_j + x)/w_2}). \end{aligned} \tag{3.4}$$

The Askey–Wilson second-order difference operator $\mathcal{L}_\gamma^x$ is defined by

$$\mathcal{L}_\gamma^x f(x) = A(w_1, w_2, \gamma; x)(f(x + iw_1) - f(x)) + A(w_1, w_2, \gamma; -x)(f(x - iw_1) - f(x)). \tag{3.5}$$

Here the superscript $x$ is added to emphasize that the operator acts on the $x$-variable (in a moment we will also consider the operator $\mathcal{L}$ acting on the spectral variable $\lambda$). We write $\widetilde{\mathcal{L}}_\gamma^x$ for the Askey–Wilson operator (3.5) with $w_1$ and $w_2$ interchanged.

Ruijsenaars' hypergeometric function $R$ is an eigenfunction of four Askey–Wilson second-order difference operators with eigenvalues expressible in terms of

$$v(w_1, w_2, \gamma; \lambda) = -2e^{\pi i w_1/w_2 + 2\pi i \hat{\gamma}_0/w_2} \times(\cosh(2\pi\lambda/w_2) + \cosh(\pi i w_1/w_2 + 2\pi i \hat{\gamma}_0/w_2)). \tag{3.6}$$

Specifically, $R$ satisfies the difference equations

$$\begin{aligned} \mathcal{L}_\gamma^x R(\gamma; x, \lambda) &= v(w_1, w_2, \gamma; \lambda) R(\gamma; x, \lambda), \\ \widetilde{\mathcal{L}}_\gamma^x R(\gamma; x, \lambda) &= v(w_2, w_1, \gamma; \lambda) R(\gamma; x, \lambda), \\ \mathcal{L}_{\hat{\gamma}}^\lambda R(\gamma; x, \lambda) &= v(w_1, w_2, \hat{\gamma}; x) R(\gamma; x, \lambda), \\ \widetilde{\mathcal{L}}_{\hat{\gamma}}^\lambda R(\gamma; x, \lambda) &= v(w_2, w_1, \hat{\gamma}; x) R(\gamma; x, \lambda). \end{aligned} \tag{3.7}$$

Actually, the last three of these equations follow from the first by various symmetries of $R$. The second difference equation can be derived from the first (and the fourth from the third) by using the fact that $R$ is invariant under the exchange of $w_1$ and $w_2$,

$$R(w_1, w_2, \gamma; x, \lambda) = R(w_2, w_1, \gamma; x, \lambda).$$

This symmetry can be directly seen from definition (3.2) of $R$ and the corresponding symmetry (2.4) of the hyperbolic gamma function. The third difference equation can be obtained from the first by using the duality of $R$ under the exchange of $x$ and $\lambda$,

$$R(\gamma; x, \lambda) = R(\hat{\gamma}; \lambda, x). \tag{3.8}$$

This duality is also a direct consequence of the definition of $R$ using the fact that $\gamma_0 + \gamma_j = \hat{\gamma}_0 + \hat{\gamma}_j$ for $j = 1, 2, 3$.

There are more symmetries of $R$ directly visible from the definition. Since the hyperbolic gamma function is scale invariant it follows that $R$ is scale invariant as well,

$$R(vw_1, vw_2, v\gamma; vx, v\lambda) = R(w_1, w_2, \gamma; x, \lambda)$$

for $v \in (0, \infty)$, where $v\gamma$ denotes the scaled parameter set $(v\gamma_0, v\gamma_1, v\gamma_2, v\gamma_3)$. Furthermore, it is immediately clear that $R$ is symmetric under permutations of $\gamma_1$, $\gamma_2$, and $\gamma_3$.

This symmetry can be extended to a $D_4$-symmetry in the four parameters $\gamma$ (where the Weyl group of type $D_4$ acts on the parameters by permutations and an even number of sign flips). To formulate this result we need the $c$-function

$$c(\gamma; y) = \frac{1}{G(2y+iw)} \prod_{j=0}^{3} G(y - i\gamma_j)$$

and the normalization constant

$$N(\gamma) = \prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + iw). \tag{3.9}$$

The $D_4$-symmetry [12, Theorem 1.1] of $R$ then reads

$$\frac{R(\gamma; x, \lambda)}{c(\gamma; x)c(\hat{\gamma}; \lambda)N(\gamma)} = \frac{R(w(\gamma); x, \lambda)}{c(w(\gamma); x)c(\widehat{w(\gamma)}; \lambda)N(w(\gamma))} \tag{3.10}$$

for all elements $w$ of the Weyl group of type $D_4$. Notice that both the $c$-function and $N$ are invariant under the action of the $S_3$-subgroup which permutes $\gamma_1$, $\gamma_2$, and $\gamma_3$.

Finally we recall the limit behaviour of $R$, cf. [12, Theorem 1.2]. Set $\alpha = 2\pi/w_1w_2$. For $w_1, w_2 \in \mathbb{R}_{>0}$, $\gamma \in \mathbb{R}^4$, and $w_1 \neq w_2$ there exists an open neighbourhood $U \subset \mathbb{C}$ of $\mathbb{R}$, such that the asymptotics of $R$ for fixed $\lambda \in U$ are given by

$$R(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)| - \hat{\gamma}_0 - w)|\Re(x)|}) \tag{3.11}$$

for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compacta. In fact, Ruijsenaars gives a precise expression for the leading term of $R$ as $\Re(x) \to \pm\infty$ when $\lambda \in \mathbb{R}$. These results easily extend to $\lambda$ in some open neighbourhood $U$ of $\mathbb{R}$.

## 4. The modular double of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$

In this section we consider a slightly extended version of Faddeev's [2] modular double of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ and define an algebraic version of its principal series representation on the space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$. We define an inner product on some suitable subspace of $\mathcal{M}$, which is compatible to the $*$-structure on $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ associated to the real form $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{sl}_2(\mathbb{C})$, cf. [8].

Throughout Sections 4–6 we assume that $w_1$ and $w_2$ are positive real numbers such that neither $w_1/w_2$ nor $w_2/w_1$ is an integer, unless specifically stated otherwise. We define

$$q = \exp(\pi i w_1/w_2), \quad \tilde{q} = \exp(\pi i w_2/w_1),$$

which both lie on the unit circle (but they are not $\pm 1$). For complex numbers $\beta$ we define

$$q^{\beta} = e^{\beta\pi i w_1/w_2}, \quad \tilde{q}^{\beta} = e^{\beta\pi i w_2/w_1}.$$

**Definition 4.1.** The quantized universal enveloping algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ of $\mathfrak{sl}_2(\mathbb{C})$ is the unital associative algebra over $\mathbb{C}$ generated by $K^{\pm 1}$, $E$, and $F$, subject to the relations

$$KK^{-1} = K^{-1}K = 1,$$

$$KE = q^2 EK,$$

$$KF = q^{-2} FK,$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

If $w_1/w_2$ is irrational, then the center of $\mathcal{U}_q$ is generated by the quantum Casimir element $\Omega$, defined as

$$\Omega = qK + q^{-1}K^{-1} + (q - q^{-1})^2 FE.$$

By simply replacing $q$ by $\tilde{q}$ (or interchanging $w_1$ and $w_2$) we obtain the quantum universal enveloping algebra $\mathcal{U}_{\tilde{q}}$. The generators of $\mathcal{U}_{\tilde{q}}$ are denoted by $\tilde{K}^{\pm 1}$, $\tilde{E}$, and $\tilde{F}$. The following concept of modular double was introduced by Faddeev [2].

**Definition 4.2.** The modular double $\mathcal{Q}$ is $\mathcal{U}_q \otimes \mathcal{U}_{\tilde{q}}$ endowed with its standard tensor product algebra structure.

For elements $X \in \mathcal{U}_q$ (respectively $\tilde{X} \in \mathcal{U}_{\tilde{q}}$) we also write $X$ (respectively $\tilde{X}$) for its image under the natural embedding of $\mathcal{U}_q$ (respectively $\mathcal{U}_{\tilde{q}}$) in $\mathcal{Q}$. In particular, $X\tilde{X} = \tilde{X}X$ in $\mathcal{Q}$ for elements $X \in \mathcal{U}_q$ and $\tilde{X} \in \mathcal{U}_{\tilde{q}}$.

We now define an extension of the modular double by formally adjoining complex powers of $K$ and $\tilde{K}$ to $\mathcal{Q}$. Let $\mathcal{A} = \bigoplus_{x\in\mathbb{C}} \mathbb{C}\hat{x}$ be the group algebra of the additive group $\hat{\mathbb{C}} = (\mathbb{C}, \oplus)$, where $\oplus$ is the translated addition $\hat{x} \oplus \hat{y} = \widehat{x + y + iw}$ (this translation in addition will make formules simpler later on). The unit of $\hat{\mathbb{C}}$ is $\widehat{-iw}$.

**Lemma 4.3.** *There exists a unique left $\mathcal{A}$-action by algebra automorphisms on the modular double $\mathcal{Q}$ satisfying*

$$\hat{x} \cdot K^{\pm 1} = K^{\pm 1}, \quad \hat{x} \cdot \tilde{K}^{\pm 1} = \tilde{K}^{\pm 1},$$

$$\hat{x}\cdot E=-qe^{2\pi x/w_2}E,\quad \hat{x}\cdot\tilde{E}=-\tilde{q}e^{2\pi x/w_1}\tilde{E},$$

$$\hat{x}\cdot F=-qe^{-2\pi x/w_2}F,\quad \hat{x}\cdot\tilde{F}=-\tilde{q}e^{-2\pi x/w_1}\tilde{F}.$$

**Proof.** Observe that e.g. the action of $\hat{x}$ on $E$ can be rewritten as

$$\hat{x}\cdot E=e^{2\pi(x+iw)/w_2}E.$$

The lemma now follows by direct calculations. □

**Definition 4.4.** The extended modular double $\mathcal{D}=\mathcal{Q}\rtimes\mathcal{A}$ is the crossed product of the modular double $\mathcal{Q}$ and the algebra $\mathcal{A}$ under its action on $\mathcal{Q}$ as defined in Lemma 4.3.

Hence $\mathcal{D}$ is the vector space $\mathcal{Q}\otimes\mathcal{A}$ endowed with the unique algebra structure such that the natural embeddings of $\mathcal{Q}$ and $\mathcal{A}$ in $\mathcal{D}$ are algebra morphisms and such that

$$\hat{x}Q=(\hat{x}\cdot Q)\hat{x},\quad \forall x\in\mathbb{C},\ \forall Q\in\mathcal{Q},$$

where we identified $\hat{x}$ (respectively $Q$) with their images under the natural embeddings of $\mathcal{A}$ (respectively $\mathcal{Q}$) in $\mathcal{D}$.

Now we define representations $\pi_\lambda$ of the extended modular double $\mathcal{Q}$ on the space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$ depending on a complex representation label $\lambda$, cf. [4]. These representations may be viewed as algebraic versions of the principal series representations of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$. We define these representations in terms of the operators $T_y$ and $S_y$ on $\mathcal{M}$, which act by

$$T_yf(z)=f(z+y),\quad S_yf(z)=e^{2\pi iz/y}f(z)\quad (y\in\mathbb{C}).$$

**Lemma 4.5.** *For $\lambda\in\mathbb{C}$ the assignments*

$$\pi_\lambda(K)=T_{iw_1},\quad \pi_\lambda(\tilde{K})=T_{iw_2},\quad \pi_\lambda(\hat{x})=T_{x+iw},$$

$$\pi_\lambda(E)=\frac{q^{1/2}}{q-q^{-1}}S_{iw_2}\left(q^{-1/2}e^{\pi\lambda/w_2}+q^{1/2}e^{-\pi\lambda/w_2}T_{iw_1}\right),$$

$$\pi_\lambda(F)=-\frac{q^{1/2}}{q-q^{-1}}S_{-iw_2}\left(q^{-1/2}e^{\pi\lambda/w_2}+q^{1/2}e^{-\pi\lambda/w_2}T_{-iw_1}\right),$$

$$\pi_\lambda(\tilde{E}) = \frac{\tilde{q}^{1/2}}{\tilde{q}-\tilde{q}^{-1}} S_{iw_1}\left(\tilde{q}^{-1/2}e^{\pi\lambda/w_1} + \tilde{q}^{1/2}e^{-\pi\lambda/w_1}T_{iw_2}\right),$$

$$\pi_\lambda(\tilde{F}) = -\frac{\tilde{q}^{1/2}}{\tilde{q}-\tilde{q}^{-1}} S_{-iw_1}\left(\tilde{q}^{-1/2}e^{\pi\lambda/w_1} + \tilde{q}^{1/2}e^{-\pi\lambda/w_1}T_{-iw_2}\right),$$

*uniquely define a representation* $\pi_\lambda$ *of* $\mathcal{D}$ *on* $\mathcal{M}$.

Observe that the action of the generators of $\mathcal{U}_{\tilde{q}}$ are obtained from the action of the generators of $\mathcal{U}_q$ by interchanging $w_1$ and $w_2$.

**Proof.** The defining relations of $\mathcal{D}$ are easily checked using $S_xS_{-x} = 1$, $T_xT_y = T_{x+y} = T_yT_x$, and the equation

$$T_xS_y = e^{2\pi ix/y}S_yT_x. \qquad \square$$

**Remark 4.6.** Denote $v = (w_1 - w_2)/2$, then $\pi_\lambda(\widehat{iv}) = \pi_\lambda(K)$ and $\pi_\lambda(\widehat{-iv}) = \pi_\lambda(\tilde{K})$. The extension of the modular double $\mathcal{Q}$ by $\mathcal{A}$ and the extension of the representation $\pi_\lambda|_{\mathcal{Q}}$ to $\pi_\lambda$ thus have the effect of introducing non-integral powers of $T_{iw_1}$ and $T_{iw_2}$ in the image of $\pi_\lambda$. The introduction of this extension is not an essential part of the analysis later on and is only included for simplification. Using only integral powers of $K$ and $\tilde{K}$ we can simulate the action of $\hat{x}$ for $x$ in some dense subset of $\mathbb{R}$, cf. [4, Proposition 1.6].

A simple calculation shows that $\pi_\lambda(\Omega)$ acts as a scalar,

$$\pi_\lambda(\Omega)f = -2\cosh(2\pi\lambda/w_2)f, \qquad f \in \mathcal{M}. \tag{4.1}$$

Since $\pi_\lambda$ is an algebraic version of the principal series representation with representation label $\lambda \in \mathbb{C}$, this is as expected.

**Definition 4.7.** We say that $f \in \mathcal{M}$ has exponential growth with growth rate $\varepsilon \in \mathbb{R}$ if there exists a compact set $K_f \in \mathbb{R}$ such that all poles of $f$ are contained in $K_f \times i\mathbb{R} = \{x + iy | x \in K_f, y \in \mathbb{R}\}$ and if $|f(x+iy)| = \mathcal{O}(\exp(\varepsilon|x|))$ for $x \to \pm\infty$, uniformly for $y$ in compacta of $\mathbb{R}$.

On the space of meromorphic functions which have negative exponential growth and which have no poles on $\mathbb{R}$, we define a sesquilinear form by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(z)\overline{g(z)}\,dz. \tag{4.2}$$

Observe that this expression is already well defined under the milder asymptotic condition that the sum of the two exponential growths of $f$ and $g$ is negative. Note furthermore that (4.2) can be rewritten as

$$\langle f, g\rangle = \int_{\mathbb{R}} f(z)\bar{g}(z)\,dz, \tag{4.3}$$

where $\bar{g}(z) := \overline{g(\bar{z})}$ now is analytic at $z \in \mathbb{R}$.

Following [4] we define an antilinear anti-algebra involution $*$ on the extended modular double $\mathcal{D}$ by

$$K^* = K,\ E^* = -E,\ F^* = -F,\ \tilde{K}^* = \tilde{K},\ \tilde{E}^* = -\tilde{E},\ \tilde{F}^* = -\tilde{F},\ \hat{x}^* = \widehat{-\bar{x}}. \tag{4.4}$$

If we restrict this involution to $\mathcal{U}_q$ (respectively $\mathcal{U}_{\tilde{q}}$) we obtain the $*$-structure on $\mathcal{U}_q$ (respectively $\mathcal{U}_{\tilde{q}}$) corresponding to the noncompact real form $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{sl}_2(\mathbb{C})$, cf. [8].

The following lemma relates the sesquilinear form (4.2) to the $*$-structure (4.4) on $\mathcal{D}$.

**Lemma 4.8.** *Let $\lambda \in \mathbb{C}$ and $f, g \in \mathcal{M}$. If the poles of $f$ and $g$ are outside the strip $\mathbb{R} \times i[-w_1, w_1]$ and if the sum of the exponential growth rates of $f$ and $g$ is smaller than $-2\pi/w_2$, then*

$$\langle \pi_\lambda(X) f, g\rangle = \langle f, \pi_{\bar{\lambda}}(X^*) g\rangle$$

*for $X \in \mathcal{U}_{q,1} := span_{\mathbb{C}}\{1, E, F, K, K^{-1}, FK, EK^{-1}\}$.*

**Proof.** In view of (4.3) the proof follows by a change of variables and some contour shifting using Cauchy's theorem. □

A similar lemma holds for the dual algebra $\mathcal{U}_{\tilde{q}}$.

## 5. Twisted primitive elements and matrix coefficients

Koornwinder [7] introduced twisted primitive elements to obtain the Askey–Wilson polynomials as matrix coefficients of finite dimensional $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$-representations. We recall the definition of twisted primitive elements and show that they act as first-order difference operators under the representations $\pi_\lambda$. We construct eigenvectors to these operators in terms of the hyperbolic gamma function and we consider the corresponding formal matrix coefficients $\psi$ of $\pi_\lambda$. In subsequent sections we relate $\psi$ to Ruijsenaars' hypergeometric function.

Let $\rho \in \mathbb{C}$ and set

$$v_\rho = q^{2\rho/w_1} + q^{-2\rho/w_1} = 2\cos(2\pi\rho/w_2).$$

The twisted primitive element $Y_\rho \in \mathcal{U}_q \subset \mathcal{D}$ is defined as

$$Y_\rho = iq^{-1/2}E + iq^{-1/2}FK - \frac{v_\rho}{q-q^{-1}}(K-1). \tag{5.1}$$

Analogously, we define the twisted primitive element $\tilde{Y}_\rho \in \mathcal{U}_{\tilde{q}}$ by interchanging $w_1$ and $w_2$, viz.

$$\tilde{Y}_\rho = i\tilde{q}^{-1/2}\tilde{E} + i\tilde{q}^{-1/2}\tilde{F}\tilde{K} - \frac{\tilde{v}_\rho}{\tilde{q}-\tilde{q}^{-1}}(\tilde{K}-1),$$

where $\tilde{v}_\rho = 2\cos(2\pi\rho/w_1)$.

Denoting

$$\mu_\tau(\rho) = \frac{v_\rho - v_\tau}{q-q^{-1}}, \quad \tilde{\mu}_\tau(\rho) = \frac{\tilde{v}_\rho - \tilde{v}_\tau}{\tilde{q}-\tilde{q}^{-1}},$$

we now have the following lemma.

**Lemma 5.1.** *The meromorphic function*

$$H^\lambda_{\tau,\rho}(z) = \frac{G(z+\lambda/2-3iw/2\pm i\tau)}{G(z-\lambda/2-iw/2\pm i\rho)}$$

*satisfies*

$$\begin{aligned} \pi_\lambda(Y_\rho)H^\lambda_{\tau,\rho} &= \mu_\tau(\rho)H^\lambda_{\tau,\rho}, \\ \pi_\lambda(\tilde{Y}_\rho)H^\lambda_{\tau,\rho} &= \tilde{\mu}_\tau(\rho)H^\lambda_{\tau,\rho}. \end{aligned} \tag{5.2}$$

**Proof.** Since $H^\lambda_{\tau,\rho}$ is invariant under the exchange of $w_1$ and $w_2$, it is sufficient to prove only the first eigenvalue equation. A calculation shows that $\pi_\lambda(Y_\rho)f = \mu_\tau(\rho)f$ is equivalent to the first-order difference equation

$$f(z+iw_1/2) = \frac{\cosh\left(\dfrac{\pi}{w_2}(z+\lambda/2-3iw/2\pm i\tau)\right)}{\cosh\left(\dfrac{\pi}{w_2}(z-\lambda/2-iw/2\pm i\rho)\right)} f(z-iw_1/2). \tag{5.3}$$

(The exact calculation can be found in Appendix A.) Using the difference equation (2.3) for the hyperbolic gamma function it immediately follows that $H^\lambda_{\tau,\rho}$ satisfies the difference equation (5.3). □

**Remark 5.2.** For any one of the two equations (5.2) there are infinitely many solutions (we can e.g. multiply a solution to the first equation by any $iw_1$-periodic function). The crucial step in finding common solutions to both difference equations is to rewrite the first difference equation in the specific form (5.3). Indeed, the resulting solution $H^{\lambda}_{\tau,\rho}$ in terms of hyperbolic gamma functions is invariant under interchanging $w_1$ and $w_2$, hence it automatically satisfies the second difference equation. This is the main difference between our analysis and the one in [14].

Now let us consider the adjoint $Y^*_\sigma$, which is

$$Y^*_\sigma = iq^{1/2}E + iq^{-3/2}FK + \frac{v_{\bar\sigma}}{q-q^{-1}}(K-1).$$

Since $\mu_\upsilon(\sigma)^* = -\mu_{\bar\upsilon}(\bar\sigma)$, we are interested in solutions to the equation $\pi_{\bar\lambda}(Y^*_\sigma)f = -\mu_{\bar\upsilon}(\bar\sigma)f$ and the corresponding equation $\pi_{\bar\lambda}(\tilde{Y}^*_\sigma)f = -\tilde\mu_{\bar\upsilon}(\bar\sigma)f$ for the second component of the modular double.

**Lemma 5.3.** *The function*

$$F^{\lambda}_{\upsilon,\sigma}(z) = \frac{G(z+\bar\lambda - iw/2 \pm i\bar\upsilon)}{G(z-\bar\lambda + iw/2 \pm i\bar\sigma)}$$

*satisfies*

$$\pi_{\bar\lambda}(Y^*_\sigma)F^{\lambda}_{\upsilon,\sigma} = -\mu_{\bar\upsilon}(\bar\sigma)F^{\lambda}_{\upsilon,\sigma},$$

$$\pi_{\bar\lambda}(\tilde{Y}^*_\sigma)F^{\lambda}_{\upsilon,\sigma} = -\tilde\mu_{\bar\upsilon}(\bar\sigma)F^{\lambda}_{\upsilon,\sigma}.$$

**Proof.** The proof is similar to the proof of the previous lemma. □

We will need a few results on the analytic properties of the two functions $H^{\lambda}_{\tau,\rho}$ and $F^{\lambda}_{\upsilon,\sigma}$.

**Lemma 5.4.** *The possible pole locations of $H^{\lambda}_{\tau,\rho}$ and $F^{\lambda}_{\upsilon,\sigma}$ are at*

$$-\lambda/2 \pm i\tau + iw - \Lambda_+, \qquad \lambda/2 \pm i\rho + iw + \Lambda_+$$

*and*

$$-\bar\lambda/2 \pm i\bar\upsilon - \Lambda_+, \qquad \bar\lambda/2 \pm i\bar\sigma + \Lambda_+,$$

*respectively. Furthermore, $H^{\lambda}_{\tau,\rho}$ and $F^{\lambda}_{\upsilon,\sigma}$ have exponential growth with growth rates $\pi(2\Im(\lambda)-2w)/w_1w_2$ and $\pi(-2\Im(\lambda)-2w)/w_1w_2$, respectively.*

**Proof.** The proof follows directly from the zero/pole locations and asymptotics of the hyperbolic gamma function (see Section 2). □

Define

$$\xi = \max(|\Re(\rho)|, |\Re(\sigma)|, |\Re(\tau)|, |\Re(\upsilon)|) \tag{5.4}$$

and

$$\zeta = w/2 - \xi - |\Im(\lambda/2)|. \tag{5.5}$$

We assume that the parameters $\rho, \sigma, \upsilon, \tau$ and the variable $\lambda$ are such that $\zeta > 0$. For $|\Im(x)| < \zeta$ define

$$\psi(\rho, \sigma, \tau, \upsilon; x, \lambda) = \langle \pi_\lambda(\hat{x}) H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma} \rangle, \tag{5.6}$$

which is well defined since the exponential growth $-2\pi(w_1+w_2)/w_1 w_2$ of the integrand is negative and the pole sequences of $\pi_\lambda(\hat{x}) H^\lambda_{\tau,\rho}$ and $F^\lambda_{\upsilon,\sigma}$ all stay away from the real line due to the condition $|\Im(x)| < \zeta$. Note that the increasing pole sequences of $\pi_\lambda(\hat{x}) H^\lambda_{\tau,\rho}$ and $F^\lambda_{\upsilon,\sigma}$ are all located above the real line and the decreasing pole sequences are all located below the real line due to the shifted addition in $\hat{\mathbb{C}}$. Observe furthermore that the matrix coefficient $\psi$ is invariant under the exchange of $w_1$ and $w_2$, cf. Remark 5.2. The function $\psi$ will be related to Ruijsenaars' hypergeometric function $R$ in Section 8.

Using (2.5), (2.6), and (4.3) we can write $\psi$ as

$$\begin{aligned}&\psi(\rho, \sigma, \tau, \upsilon; x, \lambda)\\ &\quad = \int_{\mathbb{R}} \frac{G(z + x + \lambda/2 - iw/2 \pm i\tau) G(z - \lambda/2 - iw/2 \pm i\sigma)}{G(z + x - \lambda/2 + iw/2 \pm i\rho) G(z + \lambda/2 + iw/2 \pm i\upsilon)}\, dz,\end{aligned} \tag{5.7}$$

which is of the form (2.13). It follows from the discussion at the end of Section 2 that

$$\begin{aligned}\Psi(\gamma; x, \lambda) = \; & E(x \pm i\gamma_0) E(x \pm i\gamma_1) E(-x \pm i\gamma_2) E(-x \pm i\gamma_3)\\ & \times E(\lambda \pm i\hat{\gamma}_0) E(\lambda \pm i\hat{\gamma}_1) E(-\lambda \pm i\hat{\gamma}_2) E(-\lambda \pm i\hat{\gamma}_3) \psi(\gamma; x, \lambda)\end{aligned} \tag{5.8}$$

has an entire extension to

$$\mathcal{O} = \{(w_1, w_2, \rho, \sigma, \tau, \upsilon, x, \lambda) \in \mathbb{C}^2_+ \times \mathbb{C}^6\}. \tag{5.9}$$

Hence $\psi$ is meromorphic on the same domain $\mathcal{O}$.

## 6. The Askey–Wilson difference equations

We show that the formal matrix coefficient $\psi$ (see Section 5) satisfies a second-order difference equation with step size $iw_1$ using a radial part calculation of the Casimir $\Omega$ with respect to twisted primitive elements. As a consequence a renormalization $S$ (6.10) of $\psi$ satisfies an Askey–Wilson second order difference equation. Since $S$, like $\psi$, is invariant under exchanging $w_1$ and $w_2$, we obtain a second difference equation with step size $iw_2$. We furthermore show that $S$ satisfies a duality in the geometric and spectral variables, and we derive various obvious symmetries of $S$.

Let us start by establishing a correspondence between the set of parameters $\rho$, $\sigma$, $\tau$, and $\upsilon$ and Ruijsenaars' parameter set $\gamma$ by

$$\gamma_0 = -\rho + \sigma, \quad \gamma_1 = \rho + \sigma, \quad \gamma_2 = -\tau - \upsilon, \quad \gamma_3 = \tau - \upsilon. \tag{6.1}$$

Observe that $\hat{\gamma}_0$ (see (3.1)) becomes

$$\hat{\gamma}_0 = \tfrac{1}{2}(\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3) = \sigma - \upsilon.$$

We will also use the abbreviation $\gamma$ for the parameters $(\rho, \sigma, \tau, \upsilon)$. In particular, we write $\psi(\gamma; x, \lambda)$ for (5.6). Later we show that (6.1) is the parameter correspondence which relates $\psi$ to Ruijsenaars' hypergeometric function.

Now we perform a radial part calculation of the Casimir element $\Omega$ with respect to the twisted primitive elements (see (5.1)). The result is stated in terms of the function $A$, see (3.4).

**Lemma 6.1.** *We have*

$$\hat{x}\Omega = \hat{x}\Omega(x) \bmod \hat{x}\mathcal{U}_{q,2}(Y_\rho - \mu_\tau(\rho)) + (Y_\sigma - \mu_\upsilon(\sigma))\hat{x}\mathcal{U}_{q,2},$$

*where* $\mathcal{U}_{q,2} := span_{\mathbb{C}}\{1, K^{-1}, F\}$ *and* $\Omega(x)$ *is defined as the element*

$$\Omega(x) = B(x)K + C(x) + D(x)K^{-1}$$

*with coefficients*

$$B(x) = q^{-1}A(\gamma_0, -\gamma_0, \gamma_1, -\gamma_1; x),$$

$$C(x) = q^{-1+2\hat{\gamma}_0/w_1}\left[-A(\gamma; x) - A(\gamma; -x) + 1 + q^{2-4\hat{\gamma}_0/w_1}\right],$$

$$D(x) = q^{-1}A(\gamma_2, -\gamma_2, \gamma_3, -\gamma_3; -x).$$

**Proof.** The proof involves a radial part calculation similar to the one performed in [14, Proposition 3.3]. In fact, we can use the calculation in [14] using an embedding $\phi$ of the extended quantum universal enveloping algebra $\mathcal{U}_q \rtimes \hat{\mathbb{C}}$ into the one in [14], given by

$$\phi(K) = K^2, \quad \phi(\hat{x}) = \left(\widehat{\frac{2x + iw_2}{iw_1}}\right) K,$$

$$\phi(E) = -iKX^+, \quad \phi(F) = iX^-K^{-1}.$$

A direct calculation gives $\phi(\Omega) = (q - q^{-1})^2\Omega + 2$ and $\phi(Y_\rho) = Y_{2\rho/w_1}$ (on the right-hand side we use the $\Omega$ and $Y$ from [14], which have a slightly different definition). Note that in [14] the radial part is calculated modulo a larger vector space. However, it is easily verified that the present smaller space suffices for the proof. □

Using this radial part calculation we can prove that $\psi$ (5.6) satisfies a gauge transformed Askey–Wilson second-order difference equation.

**Lemma 6.2.** *The function* $\psi(x) = \psi(\gamma; x, \lambda)$ *satisfies the difference equation*

$$-2\cosh(2\pi\lambda/w_2)\psi(x) = B(x)\psi(x + iw_1) + C(x)\psi(x) + D(x)\psi(x - iw_1), \tag{6.2}$$

*and a similar equation with* $w_1$ *and* $w_2$ *interchanged. These equations hold as identities between meromorphic functions on the domain* $\mathcal{O}$ (*see* (5.9)).

**Proof.** Observe that by the symmetry of $\psi$ in $w_1$ and $w_2$ we only have to prove the difference equation (6.2).

We first prove the lemma under restricted parameter conditions, which allow us to use expression (5.6) of $\psi$ as a matrix coefficient of the $\mathcal{D}$-representation $\pi_\lambda$. Using analytic continuation we can subsequently remove these parameter constraints, cf. the discussion at the end of Section 5.

Let us assume that $w_1, w_2 > 0$ and that

$$w_2 > 7w_1 + 4\xi + 2|\Im(\lambda)| + 4|\Im(x)| \tag{6.3}$$

holds. Then $|\Im(x)| < \zeta$, so $\psi$ is defined by (5.6) (recall that $\xi$ and $\zeta$ are defined by (5.4) and (5.5), respectively). By (4.1),

$$-2\cosh(2\pi\lambda/w_2)\psi(x) = \langle \pi_\lambda(\hat{x}\Omega)H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle \tag{6.4}$$

holds. By Lemma 6.1 there exist $X, Z \in \mathcal{U}_{q,2}$ such that

$$\hat{x}\Omega = \hat{x}\Omega(x) + \hat{x}X(Y_\rho - \mu_\tau(\rho)) + (Y_\sigma - \mu_\upsilon(\sigma))\hat{x}Z. \tag{6.5}$$

Since $\pi_\lambda(Y_\rho - \mu_\tau(\rho))H^\lambda_{\tau,\rho} = 0$, we have

$$\langle \pi_\lambda(\hat{x}X(Y_\rho - \mu_\tau(\rho)))H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle = 0. \tag{6.6}$$

The exponential growth of $\pi_\lambda(\hat{x}Z)H^\lambda_{\tau,\rho}$ is at most the exponential growth of $H^\lambda_{\tau,\rho}$ plus $2\pi/w_2$ (due to the possible occurrence of an $S_{iw_2}$ factor in $\pi_\lambda(Z)$). The sum of the exponential growths of $\pi_\lambda(\hat{x}Z)H^\lambda_{\tau,\rho}$ and $F^\lambda_{\upsilon,\sigma}$ is at most $-2\pi/w_1$, hence strictly smaller than $-2\pi/w_2$, since the restrictions on the parameters imply that $w_2 > w_1$. Moreover condition (6.3) implies that neither $\pi_\lambda(\hat{x}Z)H^\lambda_{\tau,\rho}$ nor $F^\lambda_{\upsilon,\sigma}$ has any poles in the strip $\mathbb{R} \times i[-w_1, w_1]$. Using Lemma 4.8 and the fact that $Y_\sigma \in \mathcal{U}_{q,1}$, we thus obtain

$$\langle \pi_\lambda((Y_\sigma - \mu_\upsilon(\sigma))\hat{x}Z)H_{\tau,\rho}, F_{\upsilon,\sigma}\rangle = \langle \pi_\lambda(\hat{x}Z)H_{\tau,\rho}, \pi_{\bar{\lambda}}(Y^*_\sigma + \mu_{\bar{\upsilon}}(\bar{\sigma}))F_{\upsilon,\sigma}\rangle = 0. \tag{6.7}$$

Combining (6.4)–(6.7) now yields

$$-2\cosh(2\pi\lambda/w_2)\psi(x) = \langle \pi_\lambda(\hat{x}\Omega)H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle = \langle \pi_\lambda(\hat{x}\Omega(x))H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle. \tag{6.8}$$

Furthermore, by Lemma 6.1 (remember that $\widehat{x+iw_1}$ and $\hat{x}K$ act in the same way under $\pi_\lambda$) we have

$$\langle \pi_\lambda(\hat{x}\Omega(x))H_{\tau,\rho}, F_{\upsilon,\sigma}\rangle = B(x)\psi(x+iw_1) + C(x)\psi(x) + D(x)\psi(x-iw_1). \tag{6.9}$$

The lemma for the restricted parameter conditions follows now directly from (6.8) and (6.9). □

Using the function

$$\Delta(\gamma; x) = \frac{G(x+i\gamma_2)G(x+i\gamma_3)}{G(x-i\gamma_0)G(x-i\gamma_1)},$$

we can define a renormalization $S$ of $\psi$ as

$$S(\gamma; x, \lambda) = \frac{N(\gamma)\psi(\gamma; x, \lambda)}{\sqrt{w_1 w_2}\Delta(\gamma; x)\Delta(\hat{\gamma}; \lambda)}. \tag{6.10}$$

The function $N$ (3.9) is a convenient normalization factor when matching $S$ to $R$ in Section 8.

**Lemma 6.3.** *$S(\gamma; x, \lambda)$ is meromorphic on $\mathcal{O}$ with possible poles at*

$$\lambda = \pm(v - i\hat{\gamma}_k), \qquad x = \pm(v - i\gamma_k), \qquad i\gamma_0 + i\gamma_l = -v - iw$$

*for $v \in \Lambda_+$, $k = 0, 1, 2, 3,$ and $l = 1, 2, 3$.*

**Proof.** Using (5.8) and (2.12) we can express $S$ as

$$S(\gamma; x, \lambda) = \frac{\Psi(\gamma; x, \lambda)N(\gamma)}{\prod_{k=0}^{3} E(\pm x + i\gamma_k)E(\pm\lambda + i\hat{\gamma}_k)}.$$

From this expression we can easily read off that the possible pole hyperplanes are as stated in the lemma (they have to be either poles of $N(\gamma)$ or zeros of one of the $E$-functions in the denominator). □

**Theorem 6.4.** *The function $S(\gamma; x, \lambda)$ is a simultaneous eigenfunction of the two Askey–Wilson type second-order difference operators $\mathcal{L}_\gamma^x$ and $\widetilde{\mathcal{L}}_\gamma^x$ (see* (3.5)) *with eigenvalues $v(\lambda; w_1, w_2, \gamma)$ and $v(\lambda; w_2, w_1, \gamma)$ respectively, where $v$ is defined by* (3.6).

**Proof.** Note that $\Delta$ satisfies the first-order difference equation

$$\Delta(x + iw_1/2) = \frac{\cosh\left(\frac{\pi}{w_2}(x + i\gamma_2)\right)\cosh\left(\frac{\pi}{w_2}(x + i\gamma_3)\right)}{\cosh\left(\frac{\pi}{w_2}(x - i\gamma_0)\right)\cosh\left(\frac{\pi}{w_2}(x - i\gamma_1)\right)}\Delta(x - iw_1/2).$$

The desired eigenvalue equation (3.5) for $\mathcal{L}_\gamma^x$ now follows immediately from Lemma 6.2.

To prove the result for the operator $\widetilde{\mathcal{L}}_\gamma^x$ we note that $S$ is symmetric in $w_1$ and $w_2$, while interchanging $w_1$ and $w_2$ transforms $\mathcal{L}$ to $\tilde{\mathcal{L}}$. We could also prove the second difference equation by repeating the argument for the first difference equation using the component $\mathcal{U}_{\tilde{q}}$ of the modular double. □

We continue the analysis of the eigenfunction $S$ by proving its duality in the geometric variable $x$ and the spectral variable $\lambda$, similar to duality (3.8) for Ruijsenaars' hypergeometric function $R$. The duality transformation $\gamma \to \hat{\gamma}$ of the parameters (see (3.1)) is equivalent to interchanging $\rho$ and $\upsilon$ under the parameter correspondence (6.1): $(\rho, \sigma, \tau, \upsilon) \to (\upsilon, \sigma, \tau, \rho)$.

**Theorem 6.5** (*Duality*). *We have*

$$S(\gamma; x, \lambda) = S(\hat{\gamma}; \lambda, x)$$

*as meromorphic functions on $\mathcal{O}$.*

**Proof.** Assume that $w_1, w_2 > 0$ and $w/2 > \xi + |\Im(x)| + |\Im(\lambda)|$, where $\xi$ is as in (5.4). Note that these restrictions on the parameters are invariant under the exchange $(x, \gamma) \leftrightarrow (\lambda, \hat{\gamma})$. Then we can use the integral representation (5.7) for both $\psi(\gamma; x, \lambda)$

and $\psi(\hat{\gamma};\lambda,x)$ to compute

$$\begin{aligned}\psi(\gamma;x,\lambda) &= \int_{\mathbb{R}} \frac{G(z+x+\lambda/2-iw/2\pm i\tau)G(z-\lambda/2-iw/2\pm i\sigma)}{G(z+x-\lambda/2+iw/2\pm i\rho)G(z+\lambda/2+iw/2\pm i\upsilon)}\,dz\\ &= \int_{\mathbb{R}} \frac{G(z+x/2+\lambda-iw/2\pm i\tau)G(z-x/2-iw/2\pm i\sigma)}{G(z+x/2+iw/2\pm i\rho)G(z-x/2+\lambda+iw/2\pm i\upsilon)}\,dz\\ &= \psi(\hat{\gamma};\lambda,x),\end{aligned}$$

where we used the change of integration variable $z \to z+(\lambda-x)/2$ and a contour shift in the second equality. This contour shift is allowed since the integrand converges to zero exponentially at $\pm\infty$, and the conditions on the parameters ensure that there are no poles picked up by shifting the contour back to $\mathbb{R}$.

Since $\Psi$ (see (5.8)) is entire on $\mathcal{O}$, it follows that $\psi(\gamma;x,\lambda)=\psi(\hat{\gamma};\lambda,x)$ holds as identity between meromorphic functions on $\mathcal{O}$. The desired duality for $S$ now follows from $N(\gamma)=N(\hat{\gamma})$ and $\hat{\hat{\gamma}}=\gamma$. □

**Corollary 6.6.** *The function $S(\gamma;x,\lambda)$ is a simultaneous eigenfunction of the Askey–Wilson second-order difference operators $\mathcal{L}^x_\gamma$, $\widetilde{\mathcal{L}}^x_\gamma$, $\mathcal{L}^\lambda_{\hat{\gamma}}$, and $\widetilde{\mathcal{L}}^\lambda_{\hat{\gamma}}$ with eigenvalues $v(\lambda;w_1,w_2,\gamma)$, $v(\lambda;w_2,w_1,\gamma)$, $v(x;w_1,w_2,\hat{\gamma})$, and $v(x;w_2,w_1,\hat{\gamma})$, respectively.*

**Proof.** The fact that $S$ is an eigenfunction of $\mathcal{L}^x_\gamma$ and $\widetilde{\mathcal{L}}^x_\gamma$ was proved in Theorem 6.4. The proof for the other two difference operators follows from this fact and duality (Theorem 6.5). □

It is immediately clear from the integral representation (5.7) that $\psi$ is invariant under sign flips of the parameters $\rho$, $\sigma$, $\tau$, and $\upsilon$. This leads to the following symmetries for $S$.

**Lemma 6.7.** *Let $W_n$ be the Weyl group of type $D_n$, which acts on n-tuples by permutations and even numbers of sign changes. Let $V=W_2\times W_2\subset W_4$ be the Weyl group of type $D_2\times D_2$, where the first (respectively second) component acts on the parameters $(\gamma_0,\gamma_1)$ (respectively $(\gamma_2,\gamma_3)$) of the four-tuple $(\gamma_0,\gamma_1,\gamma_2,\gamma_3)$. For an element $v\in V$ we have*

$$\frac{S(\gamma;x,\lambda)}{c(\gamma;x)c(\hat{\gamma};\lambda)N(\gamma)} = \frac{S(v(\gamma);x,\lambda)}{c(v(\gamma);x)c(\widehat{v(\gamma)};\lambda)N(v(\gamma))}$$

*as meromorphic functions on $\mathcal{O}$.*

**Proof.** Note that the action of $V \simeq \mathbb{Z}_2^{\times 4}$ on the parameters $(\rho,\sigma,\tau,\upsilon)$ is by sign flips of $\rho$, $\sigma$, $\tau$, and $\upsilon$. Under the conditions $\zeta>0$ and $|\Im(x)|<\zeta$ it follows from the integral representation (5.7) of $\psi$ that $\psi$ is invariant under the action of $V$ on $\gamma$ (note that the parameter restrictions are $V$-invariant).

Observe that

$$c(\gamma; x)\Delta(\gamma; x) = \frac{G(x \pm i\gamma_2)G(x \pm i\gamma_3)}{G(2x + iw)}$$

is also $V$-invariant. Since the action of $V$ commutes with taking dual parameters (which is obvious in the parameters $\rho, \sigma, \tau, \upsilon$, since $V$ acts by flipping signs while taking dual parameters amounts to interchanging $\rho$ and $\upsilon$) we have a similar result for $c(\hat{\gamma}; \lambda)\Delta(\hat{\gamma}; \lambda)$. Combining these results and using (6.10) now yields the desired symmetry of $S$ for the restricted parameter set. These extra conditions on the parameters can be removed by analytic continuation (compare with the proof of Theorem 6.5). □

**Remark 6.8.** The symmetries described in Lemma 6.7 should be compared to the $D_4$ symmetry (3.10) of $R$. Note that for $R$ only an $S_3 \subset W_4$ symmetry holds trivially from its integral representation (3.2), where $S_3$ acts by permuting $\gamma_1$, $\gamma_2$, and $\gamma_3$.

Let us now consider asymptotics of $S$, compare with asymptotics (3.11) of $R$.

**Lemma 6.9.** *Let $w_1, w_2 \in \mathbb{R}_{>0}$, $\gamma \in \mathbb{C}^4$, and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $\zeta > 0$, where $\zeta$ is given by* (5.5). *Then*

$$S(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)| - \Re(\hat{\gamma}_0) - w)|\Re(x)|})$$

*for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compact subsets of $(-\zeta, \zeta)$, where $\alpha = 2\pi/w_1 w_2$.*

**Proof.** Under the parameter restrictions as stated in the lemma, $S$ does not have $x$-independent poles (see Lemma 6.3) and the integral representation (5.7) for $\psi$ holds.

In view of (6.10) and the asymptotics

$$\frac{1}{\Delta(\gamma; x)} = \mathcal{O}(e^{\mp\alpha\hat{\gamma}_0 x}) \tag{6.11}$$

for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compacta, it suffices to prove

$$\psi(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)| - w)|\Re(x)|}) \tag{6.12}$$

for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. The asymptotic formula (6.11) follows directly from estimates (2.10) and (2.11) for the hyperbolic gamma function.

Note that it suffices to prove (6.12) for $\Re(x) \to \infty$ since

$$\psi(\gamma; x, \lambda) = \psi(\check{\gamma}; -x, -\lambda), \tag{6.13}$$

where $\check{\gamma} = (\sigma, \rho, \upsilon, \tau)$ (in the $\gamma_\mu$ notation, $\check{\gamma} = (-\gamma_0, \gamma_1, \gamma_2, -\gamma_3)$). Eq. (6.13) follows by the change of integration variable $z \to -z$ in (5.7) and a subsequent contour shift.

To prove (6.12) for $\Re(x) \to \infty$ we consider the integral representation (5.7) of $\psi$. We define

$$\varepsilon = \max(w_1, w_2) + \tfrac{1}{2}|\Re(\lambda)| + \max(|\Im(\rho)|, |\Im(\sigma)|, |\Im(\tau)|, |\Im(\upsilon)|) \tag{6.14}$$

and we consider the division of $\mathbb{R}$ in five intervals

$$I_1 = (-\infty, -\Re(x) - \varepsilon), \qquad I_2 = (-\Re(x) - \varepsilon, -\Re(x) + \varepsilon),$$

$$I_3 = (-\Re(x) + \varepsilon, -\varepsilon), \qquad I_4 = (-\varepsilon, \varepsilon), \qquad I_5 = (\varepsilon, \infty), \tag{6.15}$$

for $\Re(x) > 2\varepsilon$. We write integral (5.7) defining $\psi$ as the sum of five integrals over $I_j$ $(j = 1, 2, \ldots, 5)$ and we bound the integral over each $I_j$ seperately. The intervals are chosen in such a way that estimates (2.10) and (2.11) for the hyperbolic gamma function can be used to bound the integrand over the intervals $I_1$, $I_3$, and $I_5$. To estimate the integrals over the remaining intervals $I_2$ and $I_4$ we use the fact that their lenghts are finite and independent of $\Re(x)$. For each interval $I_j$ we show that the integral over $I_j$ is $\mathcal{O}(e^{\alpha(|\Im(\lambda)|-w)|\Re(x)|})$ as $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compact subsets of $(-\zeta, \zeta)$. As a consequence $\psi$ is also of this order. Details are given in Appendix A. $\square$

## 7. Reduction to Askey–Wilson polynomials

Using an indirect method, Ruijsenaars [11, Theorem 3.2] proved that $R$ reduces to the Askey–Wilson polynomials [1] when the spectral parameter is specialized to certain specific discrete values. We now show by a direct calculation that $S$ (6.10) reduces to the Askey–Wilson polynomials for the same discrete spectral values.

Let us first introduce some standard notations for basic hypergeometric series, see [3]. For $q \in \mathbb{C}$ we write

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k),$$

$$(a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n.$$

The $q$-hypergeometric series is defined by

$$ {}_{s+1}\phi_s\left[\begin{matrix} a_1,\dots,a_{s+1} \\ b_1,\dots,b_s \end{matrix} ;q,z\right] = \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_{s+1};q)_k}{(b_1,\dots,b_s,q;q)_k} z^k $$

provided that either $|q| < 1$ or that the series terminates. The Askey–Wilson polynomials [1] are defined as

$$ r_n(x;a,b,c,d|q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{2\pi x/w_2}, ae^{-2\pi x/w_2} \\ ab, ac, ad \end{matrix} ;q,q\right]. $$

Note that the $q^{-n}$ term in the above expression causes the series to terminate. This implies that $r_n$ is a polynomial of degree $n$ in $\cosh(2\pi x/w_2)$. Finally, if we use the parameter correspondence

$$ a = -e^{2\pi i\gamma_0/w_2}q, \quad b = -e^{2\pi i\gamma_1/w_2}q, \quad c = -e^{2\pi i\gamma_2/w_2}q, \quad d = -e^{2\pi i\gamma_3/w_2}q, \tag{7.1} $$

and if we define

$$ \lambda_n = iw + i\hat{\gamma}_0 + inw_1, \tag{7.2} $$

then the Askey–Wilson polynomials satisfy the Askey–Wilson second-order difference equation

$$ \mathcal{L}_\gamma^x r_n(x;a,b,c,d|q^2) = v(w_1,w_2,\gamma;\lambda_n) r_n(x;a,b,c,d|q^2). $$

Here we use the Askey–Wilson operator $\mathcal{L}_\gamma^x$ (3.5) and eigenvalue $v$ (3.6).

Ruijsenaars has shown in [11] by an indirect method that

$$ R(w_1,w_2,\gamma;x,\lambda_n) = r_n(x;a,b,c,d|q^2) \tag{7.3} $$

for $n \in \mathbb{Z}_{\geqslant 0}$, under the parameter correspondence (7.1). Similarly we have

**Theorem 7.1.** *Under the parameter correspondence* (7.1) *we have*

$$ S(w_1,w_2,\gamma;x,\lambda_n) = r_n(x;a,b,c,d|q^2) $$

*for* $n \in \mathbb{Z}_{\geqslant 0}$.

**Proof.** Without loss of generality, we assume that the parameters $w_1, w_2, \gamma, x$ are generic.

For generic $\lambda$ we can express $\psi$ as an integral

$$\psi(\gamma; x, \lambda) = \int_{\mathcal{C}} I(\gamma; x, \lambda, z)\, dz \tag{7.4}$$

with $I(z) = I(\gamma; x, \lambda, z)$ given by

$$I(z) = \frac{G(z + x + \lambda/2 - iw/2 \pm i\tau)G(z - \lambda/2 - iw/2 \pm i\sigma)}{G(z + x - \lambda/2 + iw/2 \pm i\rho)G(z + \lambda/2 + iw/2 \pm i\upsilon)}$$

and with contour $\mathcal{C}$ a deformation of $\mathbb{R}$ seperating the upward pole sequences of $I$ from the downward pole sequences of $I$. When $\lambda \to \lambda_n$, the pole $z_k := \lambda/2 - iw/2 - i\sigma - ikw_1$ from a downward pole sequence of $I$ will collide with the pole $-\lambda/2 + iw/2 - i\upsilon + i(n-k)w_1$ from an upward pole sequence of $I$ for $0 \leqslant k \leqslant n$. In order to compute the limit $\lambda \to \lambda_n$ in (7.4), we therefore first shift the contour $\mathcal{C}$ over the poles at $z_k$ $(0 \leqslant k \leqslant n)$ while picking up poles. In the resulting integral the colliding poles are on the same side of the integration contour, hence the limit $\lambda \to \lambda_n$ can be taken.

To calculate the residues of $I$ at $z_k$ we first remark that $k$ consecutive applications of the difference equation (2.3) yield

$$\frac{G(z)}{G(z - ikw_1)} = e^{k\pi z/w_2} q^{-k^2/2} (-e^{-2\pi z/w_2} q; q^2)_k.$$

Using this equation we can write

$$\begin{aligned} I(z) = {} & \frac{G(z + ikw_1 + x + \lambda/2 - iw/2 \pm i\tau)G(z + ikw_1 - \lambda/2 - iw/2 \pm i\sigma)}{G(z + ikw_1 + x - \lambda/2 + iw/2 \pm i\rho)G(z + ikw_1 + \lambda/2 + iw/2 \pm i\upsilon)} \\ & \times q^{2k} \frac{(-e^{-\frac{2\pi}{w_2}(z + ikw_1 + x - \lambda/2 + iw/2 \pm i\rho)} q, -e^{-\frac{2\pi}{w_2}(z + ikw_1 + \lambda/2 + iw/2 \pm i\upsilon)} q; q^2)_k}{(-e^{-\frac{2\pi}{w_2}(z + ikw_1 + x + \lambda/2 - iw/2 \pm i\tau)} q, -e^{-\frac{2\pi}{w_2}(z + ikw_1 - \lambda/2 - iw/2 \pm i\sigma)} q, q^2)_k}. \end{aligned}$$

Using the fact that the residue of the hyperbolic gamma function at $z = -iw$ equals (2.9), we obtain that the residue $Res_k$ of $I$ at $z_k$ equals

$$\begin{aligned} Res_k = {} & \frac{i\sqrt{w_1 w_2}}{2\pi} \frac{G(x + \lambda - iw - i\sigma \pm i\tau)G(-iw - 2i\sigma)}{G(x - i\sigma \pm i\rho)G(\lambda - i\sigma \pm i\upsilon)} \\ & \times q^{2k} \frac{(-e^{-\frac{2\pi}{w_2}(x - i\sigma \pm i\rho)} q, -e^{-\frac{2\pi}{w_2}(\lambda - i\sigma \pm i\upsilon)} q; q^2)_k}{(e^{-\frac{2\pi}{w_2}(x + \lambda - i\sigma \pm i\tau)} q^2, e^{\frac{2\pi}{w_2} 2i\sigma} q^2, q^2; q^2)_k}. \end{aligned}$$

Now we can rewrite $S$ as

$$S(\gamma; x, \lambda) = \frac{N(\gamma)}{\sqrt{w_1 w_2}\Delta(\gamma; x)\Delta(\hat{\gamma}; \lambda)} \left( -2\pi i \sum_{k=0}^{n} Res_k + \int_{\mathcal{C}'} I(z)\, dz \right),$$

where the contour $\mathcal{C}'$ is chosen in such a way that all upward pole sequences and the poles $z_k$ $(0\leqslant k\leqslant n)$ are above $\mathcal{C}'$, while all poles in downward pole sequences except $z_k$ $(0\leqslant k\leqslant n)$ are below $\mathcal{C}'$. In this expression the integral $\int_{\mathcal{C}'} I(z)\,dz$ has an analytic extension to $\lambda=\lambda_n$. Furthermore $S(\gamma;x,\lambda)$ is analytic at $\lambda=\lambda_n$, while $\Delta(\hat{\gamma};\lambda)$ and $Res_k$ $(0\leqslant k\leqslant n)$ have simple poles at $\lambda=\lambda_n$. Hence we obtain

$$\begin{aligned}S(\gamma;x,\lambda_n) &= \lim_{\lambda\to\lambda_n} -\frac{2\pi i N(\gamma)}{\sqrt{w_1w_2}\Delta(\gamma;x)\Delta(\hat{\gamma};\lambda)}\sum_{k=0}^{n} Res_k \\ &= e^{\frac{2n\pi}{w_2}(x-iw-i\gamma_0)}\frac{(e^{-\frac{2\pi}{w_2}(x-iw+\gamma_{2/3})}q^{-2n};q^2)_n}{(e^{-\frac{2\pi}{w_2}(i\gamma_0+i\gamma_{2/3})}q^{-2n};q^2)_n} \\ &\quad\times {}_4\phi_3\left[\begin{matrix} q^{-2n}, e^{-\frac{2\pi}{w_2}(x-iw-i\gamma_{0/1})}, e^{-\frac{2\pi i}{w_2}(\gamma_2+\gamma_3)}q^{-2n} \\ e^{-\frac{2\pi}{w_2}(x-iw+i\gamma_{2/3})}q^{-2n}, e^{\frac{2\pi i}{w_2}(\gamma_0+\gamma_1)}q^2\end{matrix};q^2,q^2\right],\end{aligned}$$

where the notation $\gamma_{0/1}$ (respectively $\gamma_{2/3}$) means that there are two terms, one with $\gamma_0$ and another with $\gamma_1$ (respectively, $\gamma_2$ and $\gamma_3$). Inserting the parameter correspondence (7.1) we obtain

$$\begin{aligned}S(\gamma;x,\lambda_n) &= e^{2\pi nx/w_2}a^{-n}\frac{(e^{-2\pi x/w_2}c^{-1}q^{-2n+2}, e^{-2\pi x/w_2}d^{-1}q^{-2n+2};q^2)_n}{(a^{-1}c^{-1}q^{-2n+2}, a^{-1}d^{-1}q^{-2n+2};q^2)_n} \\ &\quad\times {}_4\phi_3\left[\begin{matrix} q^{-2n}, e^{-2\pi x/w_2}a, e^{-2\pi x/w_2}b, c^{-1}d^{-1}q^{-2n+2} \\ e^{-2\pi x/w_2}c^{-1}q^{-2n+2}, e^{-2\pi x/w_2}d^{-1}q^{-2n+2}, ab\end{matrix};q^2,q^2\right].\end{aligned}$$

Using Sears' transformation [3, (III.15)] of a terminating balanced ${}_4\phi_3$ series with parameters specialized to $a=ae^{-2\pi x/w_2}$, $b=be^{-2\pi x/w_2}$, $c=c^{-1}d^{-1}q^{-2n+2}$, $d=ab$, $e=e^{-2\pi x/w_2}c^{-1}q^{-2n+2}$, and $f=e^{-2\pi x/w_2}d^{-1}q^{-2n+2}$ now yields the desired result. □

## 8. Equality to Ruijsenaars' hypergeometric function

We have already seen in previous sections that Ruijsenaars' hypergeometric function $R$ and the renormalized formal matrix coefficient $S$ have several properties in common. They satisfy the same Askey–Wilson second-order difference equations, they have the same duality property, they specialize in the same way to the Askey–Wilson polynomials and their possible pole locations coincide. These common properties suffice to show that $R$ and $S$ are equal.

**Theorem 8.1.** *We have*

$$R(w_1,w_2,\gamma;x,\lambda)=S(w_1,w_2,\gamma;x,\lambda). \tag{8.1}$$

This theorem is equivalent to the following identity between hyperbolic integrals.

**Corollary 8.2.** *For* $w_1, w_2, \Re(\gamma_j) > 0$ *and* $|x|, |\lambda|, |\gamma_j| < w/6$ *we have*

$$\int_{\mathbb{R}} \frac{G(z+x+\lambda/2-iw/2 \pm i(\gamma_3-\gamma_2)/2)G(z-\lambda/2-iw/2 \pm i(\gamma_0+\gamma_1)/2)}{G(z+x-\lambda/2+iw/2 \pm i(\gamma_0-\gamma_1)/2)G(z+\lambda/2+iw/2 \pm i(\gamma_2+\gamma_3)/2)}\, dz$$
$$= \frac{G(x+i\gamma_2)G(x+i\gamma_3)G(\lambda+i\hat{\gamma}_2)G(\lambda+i\hat{\gamma}_3)}{G(x+i\gamma_0)G(x-i\gamma_1)G(\lambda+i\hat{\gamma}_0)G(\lambda-i\hat{\gamma}_1)}$$
$$\times \int_{\mathcal{C}} \frac{G(z \pm x + i\gamma_0)G(z \pm \lambda + i\hat{\gamma}_0)}{G(z+iw)\prod_{j=1}^{3} G(z+i\gamma_0+i\gamma_j+iw)}\, dz,$$

*where the contour* $\mathcal{C}$ *is the real line with a downward indentation at the origin.*

**Proof.** The proof consists of inserting the integral representations of $R$ and $S$ in (8.1). See (3.2) for the integral representation of $R$, and (5.7), (6.10) for the integral representation of $S$. □

In order to prove Theorem 8.1 we first consider the Casorati-determinant of $S$ and $R$ in the $iw_1$ direction.

**Lemma 8.3.** *The Casorati-determinant*

$$\begin{aligned}\delta(\gamma; z, \lambda) = \; & S(\gamma; z+iw_1/2, \lambda)R(\gamma; z-iw_1/2, \lambda) \\ & -S(\gamma; z-iw_1/2, \lambda)R(\gamma; z+iw_1/2, \lambda)\end{aligned}$$

*of* $S$ *and* $R$ *in the* $iw_1$ *direction is identically zero.*

**Proof.** We suppress the $\lambda$ and $\gamma$ dependence of $\delta(z)$ whenever this does not cause confusion. We prove the lemma for generic parameters $w_1, w_2 \in \mathbb{R}_{>0}$, $\gamma \in \mathbb{R}^4$, and $\lambda \in U \setminus \mathbb{R}$, under the condition $w_2 > 2\xi + 2|\Im(\lambda)| + 3w_1$, where $U$ is an open subset such that asymptotics (3.11) of $R$ hold for $\lambda \in U$.

A simple calculation involving the Askey–Wilson difference equations satisfied by $R$ and $S$ (see (3.7) and Theorem 6.6, respectively) shows that

$$\delta(z+iw_1/2) = \frac{A(\gamma; -z)}{A(\gamma; z)}\, \delta(z-iw_1/2),$$

where $A$ is defined by (3.4). Since the function

$$T(z) = \sinh(2\pi z/w_2) \prod_{j=0}^{3} \frac{G(z-i\gamma_j-iw_1/2)}{G(z+i\gamma_j+iw_1/2)}$$

satisfies the same difference equation, we conclude that

$$m(z) = \frac{\delta(z)}{T(z)}$$

is an $iw_1$-periodic function.

We now show that $m(z)$ is an entire function in $z$. Let us look at the possible poles of the Casorati-determinant $\delta(z)$. By Lemma 6.9 the possible poles of $S$ are located at

$$\pm(\Lambda_+ - i\gamma_j), \quad (j = 0, 1, 2, 3).$$

From (3.3) the possible poles of $R$ are located at the same points. Hence $\delta(z)$ can only have poles at

$$\pm(\Lambda_+ - i\gamma_j) \pm iw_1/2 \quad (j = 0, 1, 2, 3)$$

Here all sign combinations are possible. Furthermore, using the pole and zero locations (2.8) of the hyperbolic gamma function, we can easily see that the possible zeros of $T(z)$ are located at

$$\pm(\Lambda_+ + i\gamma_j + iw_1/2), \qquad riw_2, \qquad (j = 0, 1, 2, 3;\ r \in \mathbb{Z}).$$

By the assumption that the parameters are generic, we conclude that $m$ has no pole sequences of the form $p + ikw_1$ ($k \in \mathbb{Z}$). By the $iw_1$-periodicity of $m$ it now follows that $m$ cannot have any poles.

In the limit $\Re(z) \to \infty$ we have

$$\frac{1}{T(z)} = \mathcal{O}(e^{\alpha(\hat{\gamma}_0 + w_1)z})$$

uniformly for $\Im(z)$ in compacta, in view of estimates (2.10) and (2.11) for the hyperbolic gamma function. Here $\alpha = 2\pi/w_1w_2$ as before.

Furthermore, using the asymptotics for $S$ (see Lemma 6.9) and for $R$ (see (3.11)) we have for $\Re(z) \to \infty$

$$\delta(z) = \mathcal{O}(e^{2\alpha(|\Im(\lambda)| + |\hat{\gamma}_0| - w)|\Re(z)|})$$

uniformly for $\Im(z)$ in compact subsets of $(-\zeta + w_1/2, \zeta - w_1/2)$. Observe that the interval $(-\zeta + w_1/2, \zeta - w_1/2)$ is nonempty due to the conditions on the parameters.

Combining these two asymptotic estimates we obtain

$$m(z) = \frac{\delta(z)}{T(z)} = \mathcal{O}(e^{\alpha(2|\Im(\lambda)| - w_2)|\Re(z)|}) \to 0 \tag{8.2}$$

for $\Re(z) \to \infty$, uniformly for $\Im(z)$ in compacta of $(-\zeta + w_1/2, \zeta - w_1/2)$.

The asymptotics of $m(z)$ for $\Re(z) \to -\infty$ can be obtained in a similar way and is also given by (8.2). Combining the asymptotics with the fact that $m(z)$ is analytic and $iw_1$-periodic we conclude that $m(z)$ is bounded on $\mathbb{C}$ since $\zeta - w_1/2 > w_1/2$.

For these parameters we conclude by Liouville's theorem that $m(z)$ is constant. In fact, by the asymptotic expansion (8.2), $m$ is identically zero. We can now extend this result to all values of the parameters by analytic continuation, which proves the lemma. □

**Proof of Theorem 8.1.** Consider the quotient

$$Q(\gamma; x, \lambda) = \frac{R(\gamma; x, \lambda)}{S(\gamma; x, \lambda)}.$$

By Lemma 8.3, $Q$ is an $iw_1$-periodic meromorphic function in $x$. Since $Q$ is symmetric in $w_1$ and $w_2$ (for both $R$ and $S$ are invariant under interchanging $w_1$ and $w_2$), $Q$ is also $iw_2$-periodic. If we choose $w_1, w_2 > 0$ such that $w_1/w_2 \notin \mathbb{Q}$, then the set $\{kw_1 + lw_2 | k, l \in \mathbb{Z}\}$ is dense on the real line, hence $Q(\gamma; x, \lambda)$ is constant as meromorphic function in $x$. Analytic continuation (in $w_1$, $w_2$, and $\gamma$) allows us to extend this result to all possible values of $w_1$ and $w_2$ in $\mathbb{C}_+$ and $\gamma \in \mathbb{C}^4$.

By the duality properties of $R$ and $S$ (see (3.8) and Theorem 6.5, respectively), we have

$$Q(\gamma; x, \lambda) = Q(\hat{\gamma}; \lambda, x).$$

This implies that $Q$ is also constant as function in $\lambda$.

In particular, we have

$$Q(w_1, w_2, \gamma; x, \lambda) = \frac{S(w_1, w_2, \gamma; x, \lambda_0)}{R(w_1, w_2, \gamma; x, \lambda_0)}$$

with $\lambda_0$ given by (7.2). By Theorem 7.1 we have $S(w_1, w_2, \gamma; x, \lambda_0) \equiv 1$, and by (7.3) we have $R(w_1, w_2, \gamma; x, \lambda_0) \equiv 1$. Hence $Q \equiv 1$, as desired. □

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## Appendix A. Eigenfunction of $\pi_\lambda(Y_\rho)$

In this appendix we give the explicit calculation to rewrite the eigenvalue equation $\pi_\lambda(Y_\rho)f = \mu_\tau(\rho)f$ as the first-order difference equation (5.3).

Using the explicit expression (5.1) of $Y_\rho$, the eigenvalue equation becomes

$$iq^{-1/2}\pi_\lambda(E)f + iq^{-1/2}\pi_\lambda(FK)f - \frac{v_\rho}{q-q^{-1}}(\pi_\lambda(K-1))f = \frac{v_\rho - v_\tau}{q-q^{-1}}\,f.$$

By the explicit definition (Lemma 4.5) of $\pi_\lambda$ we obtain

$$\begin{aligned}
&\frac{i}{q-q^{-1}}\,e^{2\pi z/w_2}\left(q^{-1/2}e^{\pi\lambda/w_2}f(z) + q^{1/2}e^{-\pi\lambda/w_2}f(z+iw_1)\right)\\
&\quad - \frac{i}{q-q^{-1}}e^{-2\pi z/w_2}\left(q^{-1/2}e^{\pi\lambda/w_2}f(z+iw_1) + q^{1/2}e^{-\pi\lambda/w_2}f(z)\right)\\
&\quad - \frac{v_\rho}{q-q^{-1}}\,(f(z+iw_1) - f(z)) = \frac{v_\rho - v_\tau}{q-q^{-1}}f(z).
\end{aligned}$$

Multiplying by $q-q^{-1}$ and rearranging the terms yields

$$\begin{aligned}
&\left(ie^{2\pi z/w_2}q^{-1/2}e^{\pi\lambda/w_2} - ie^{-2\pi z/w_2}q^{1/2}e^{-\pi\lambda/w_2} + v_\tau\right)f(z)\\
&\quad = \left(-ie^{2\pi z/w_2}q^{1/2}e^{-\pi\lambda/w_2} + ie^{-2\pi z/w_2}q^{-1/2}e^{\pi\lambda/w_2} + v_\rho\right)f(z+iw_1),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&f(z+iw_1)\\
&\quad = \frac{\cosh(\pi i/2 + 2\pi z/w_2 - \pi i w_1/(2w_2) + \pi\lambda/w_2) + \cosh(2\pi i\tau/w_2)}{\cosh(-\pi i/2 + 2\pi z/w_2 + \pi i w_1/(2w_2) - \pi\lambda/w_2) + \cosh(2\pi i\rho/w_2)}\,f(z).
\end{aligned}$$

Replacing the variable $z$ by $z - iw_1/2$ we can now rewrite the latter equation as

$$\begin{aligned}
&\frac{f(z+iw_1/2)}{f(z-iw_1/2)}\\
&\quad = \frac{\cosh(\pi i/2 + 2\pi z/w_2 - 3\pi i w_1/(2w_2) + \pi\lambda/w_2) + \cosh(2\pi i\tau/w_2)}{\cosh(-\pi i/2 + 2\pi z/w_2 - \pi i w_1/(2w_2) - \pi\lambda/w_2) + \cosh(2\pi i\rho/w_2)}\\
&\quad = \frac{\cosh\left(\frac{\pi}{w_2}(z + \lambda/2 - 3iw_1/4 + iw_2/4 \pm i\tau)\right)}{\cosh\left(\frac{\pi}{w_2}(z - \lambda/2 - iw_1/4 - iw_2/4 \pm i\rho)\right)}\\
&\quad = \frac{\cosh\left(\frac{\pi}{w_2}(z + \lambda/2 - 3iw/2 \pm i\tau)\right)}{\cosh\left(\frac{\pi}{w_2}(z - \lambda/2 - iw/2 + i\rho)\right)},
\end{aligned}$$

where we used the $i\pi$-antiperiodicity of the hyperbolic cosine in the last equality.

## Appendix B. The limit behaviour of $\psi$

In this appendix we give the details on the calculation of the limit behaviour of $\psi$, cf. the proof of Lemma 6.9. Throughout this section we assume that $w_1, w_2 \in \mathbb{R}_{>0}$, $\lambda \notin \mathbb{R}$, $\gamma \in \mathbb{C}^4$, and that $\zeta > 0$ (with $\zeta$ given by (5.5)). We prove that

$$\psi(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)|-w)|\Re(x)|}) \tag{B.1}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. As explained in the proof of Lemma 6.9, we prove (B.1) by splitting $\mathbb{R}$ in five intervals and bounding the integral representation (5.7) of $\psi$ over each interval.

### *B.1 Preparations*

Let us first define a function $K$ by

$$K(z, \lambda, a, b) = \frac{G(z + \lambda/2 - iw/2 \pm ia)}{G(z - \lambda/2 + iw/2 \pm ib)}.$$

The integral representation (5.7) for $\psi$ can then be written as

$$\psi(\gamma; x, \lambda) = \int_{\mathbb{R}} K(z + x, \lambda, \tau, \rho) K(z, -\lambda, \sigma, \upsilon)\, dz. \tag{B.2}$$

The behaviour of $K$ in the limit $z \to \pm\infty$ is controlled by

$$K_{\pm}(z, \lambda, a, b) = e^{\mp i\alpha(z(\lambda - iw) - a^2/2 + b^2/2)}.$$

Explicitely, for fixed $a$, $b$, and $\lambda$ we have

$$K(z, \lambda, a, b) = K_{\pm}(z, \lambda, a, b) e^{g(z,\lambda,a,b)} \tag{B.3}$$

for $\pm\Re(z) > \max(w_1, w_2) + |\Re(\lambda)|/2 + \max(|\Im(a)|, |\Im(b)|)$, where

$$|g(z, \lambda, a, b)| < C(\Im(z)) e^{-\alpha \min(w_1, w_2)|\Re(z)|/2}, \tag{B.4}$$

with $C$ depending continuously on $\Im(z)$, cf. (2.10) and (2.11).

### *B.2 General estimation scheme*

Let $\varepsilon$ and the intervals $I_j$ ($j \in \{1, \ldots, 5\}$) be defined as in (6.14) and (6.15). We only consider the asymptotics for $\Re(x) \to \infty$. Assume that $\Re(x) > 2\varepsilon$, causing the intervals to form a partition of the real line. We write integral (B.2) defining $\psi$ as

$$\psi(x) = \sum_{j=1}^{5} \psi_j(x), \tag{B.5}$$

where

$$\psi_j(x) = \int_{I_j} K(z+x, \lambda, \tau, \rho) K(z, -\lambda, \sigma, \upsilon)\, dz$$

for $j \in \{1, 2, \ldots, 5\}$. We bound these integrals using (B.3) (if one of them is applicable for the interval at hand).

For $j = 1$ we have

$$\begin{aligned}\psi_1(x) &= \int_{-\infty}^{-\Re(x)-\varepsilon} K_-(z+x, \lambda, \tau, \rho) K_-(z, -\lambda, \sigma, \upsilon) e^{g_1(z+x,x)}\, dz \\ &= e^{i\alpha\lambda x} e^{-\alpha w \bar{x}} e^{i\alpha(\rho^2+\upsilon^2-\tau^2-\sigma^2)/2} \int_{-\infty}^{-\varepsilon} e^{2\alpha w z + g_1(z+i\Im(x),x)}\, dz \\ &= \mathcal{O}(e^{-\alpha(\Im(\lambda)+w)\Re(x)})\end{aligned}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. Here $g_1(z, x) = g(z, \lambda, \tau, \rho) + g(z-x, -\lambda, \sigma, \upsilon)$ which satisfies an equation like (B.4) for $z < -\varepsilon$

$$|g_1(z + i\Im(x), x)| < Ce^{-\alpha \min(w_1, w_2)|\Re(z)|/2},$$

where the constant $C$ is independent of $\Im(x)$, because $\Im(x)$ is bounded. In particular, $g_1(z + i\Im(x), x)$ is uniformly bounded for $z \in (-\infty, -\varepsilon)$ and $x \in \{z \in \mathbb{C} | \Re(z) \geqslant 2\varepsilon, |\Im(z)| < \zeta\}$.

Likewise we have for $j = 5$,

$$\begin{aligned}\psi_5(x) &= \int_{\varepsilon}^{\infty} K_+(z+x, \lambda, \tau, \rho) K_+(z, -\lambda, \sigma, \upsilon) e^{g_5(z,x)}\, dz \\ &= e^{-\alpha x(w+i\lambda)} e^{i\alpha(\sigma^2+\tau^2-\rho^2-\upsilon^2)/2} \int_{\varepsilon}^{\infty} e^{-2\alpha w z + g_5(z,x)}\, dz \\ &= \mathcal{O}(e^{\alpha(\Im(\lambda)-w)\Re(x)})\end{aligned}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. Here $g_5$ is a function which satisfies a bound like (B.4) for $z > \varepsilon$, cf. the previous paragraph.

For $j=3$ we need to be a bit more careful. First observe that

$$\begin{aligned}\psi_3(x) &= \int_{-\Re(x)+\varepsilon}^{-\varepsilon} K_+(z+x,\lambda,\tau,\rho)K_-(z,-\lambda,\sigma,\upsilon)e^{g_3(z,x)}\,dz\\ &= e^{-\alpha x(w+i\lambda)}e^{i\alpha(\tau^2+\upsilon^2-\rho^2-\sigma^2)/2}\int_{-\Re(x)+\varepsilon}^{-\varepsilon} e^{-2i\alpha\lambda z+g_3(z,x)}\,dz,\end{aligned}$$

where $g_3=g(z+x,\lambda,\tau,\rho)+g(z,-\lambda,\sigma,\upsilon)$ is bounded on $z\in(-\Re(x)+\varepsilon,-\varepsilon)$ by $Ce^{-\alpha\min(w_1,w_2)\min(-z,z-\Re(x))/2}$, and hence by the constant $C$ itself. Therefore we have

$$\begin{aligned}|\psi_3(x)| &\leqslant e^{\alpha(\Im(\lambda)-w)\Re(x)+\alpha\Im(x)\Re(\lambda)}e^{\alpha\Im(\tau^2+\upsilon^2-\rho^2-\sigma^2)/2}\int_{-\Re(x)+\varepsilon}^{-\varepsilon} e^{2\alpha\Im(\lambda)z+C}\,dz\\ &= \mathcal{O}(e^{\alpha(|\Im(\lambda)|-w)\Re(x)})\end{aligned}$$

for $\Re(x)\to\infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta,\zeta)$. Here we get the final approximation by evaluating the integral and using that $\Im(\lambda)\neq 0$.

For $j=4$ we cannot use (B.3) for the entire integrand. However we still have

$$\begin{aligned}\psi_4(x) &= \int_{-\varepsilon}^{\varepsilon} K_+(z+x,\lambda,\tau,\rho)K(z,-\lambda,\sigma,\upsilon)e^{g_4(z,x)}\,dz\\ &= e^{i\alpha x(iw-\lambda)}\int_{-\varepsilon}^{\varepsilon} K_+(z,\lambda,\tau,\rho)K(z,-\lambda,\sigma,\upsilon)e^{g_4(z,x)}\,dz\\ &= \mathcal{O}(e^{\alpha(\Im(\lambda)-w)\Re(x)})\end{aligned}$$

for $\Re(x)\to\infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta,\zeta)$. Here $g_4$ is a function satisfying the bound $g_4(z,x) < Ce^{-\alpha\min(w_1,w_2)(\Re(x)-\varepsilon)/2}\leqslant C$, for $z\in[-\varepsilon,\varepsilon]$ and $\Re(x)>2\varepsilon$.

Finally for $j=2$ we have in a similar way

$$\begin{aligned}\psi_2(x) &= \int_{-\varepsilon}^{\varepsilon} K(z+i\Im(x),\lambda,\tau,\rho)K_-(z-\Re(x),-\lambda,\sigma,\upsilon)e^{g_2(z,x)}\,dz\\ &= e^{-i\alpha\Re(x)(-\lambda-iw)}\int_{-\varepsilon}^{\varepsilon} K(z+\Im(x),\lambda,\tau,\rho)K_-(z,-\lambda,\sigma,\upsilon)e^{g_2(z,x)}\,dz\\ &= \mathcal{O}(e^{-\alpha(\Im(\lambda)+w)x})\end{aligned}$$

for $\Re(x)\to\infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta,\zeta)$, where $g_2$ is a bounded function, cf. the previous paragraph.

By (B.5) we conclude that asymptotics (B.1) for $\psi$ holds, as desired.

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