A Regularity Result for the Stokes Problem
in a Convex Polygon*

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In this paper we show that the solution of the Stokes problem has square integrable second derivatives provided that the domain is a convex polygon in the plane and the nonhomogeneous term is square integrable.

1. Introduction

It is well known [10] that the solution \( u \) of the problem

\[
\Delta u = f \quad \text{in } D,
\]

\[
u = 0 \quad \text{on } \partial D
\]

has square integrable second derivatives provided that \( D \subset \mathbb{R}^2 \) is a convex polygon and \( f \in L_2(D) \). It is of interest to ask whether a similar result holds for the solution of the Stokes problem (see Section 2). In fact, such a regularity result is of importance in the analysis of numerical methods for solving the Stokes equations [6, 9], but to our knowledge a proof of the result has not appeared in the literature. In this paper we prove the appropriate result for the Stokes problem. This result is used by one of us in the study of an eigenvalue problem associated with the Stokes problem [16]. In addition, such regularity results are used in analyzing the stability of stationary solutions of the Navier–Stokes equations [17].

Our proof basically consists in applying the method used by Kondrat'ëv [12] in the analysis of single 2nd order elliptic equations, to the system of equations given by the Stokes problem. (See [18] for a sketch of some related results.) We do not use the full power of

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Kondrätěv's method, but only enough to give a self-contained proof of our main result (Theorem 2). The proof requires a use of the usual regularity estimates for the Stokes problem in a domain with a smooth boundary (see, e.g., [13]), and the Paley–Wiener theorem.

In addition, the proof involves the consideration of a system of ordinary differential equations that is closely connected with the Stokes equations. This system of ordinary differential equations should be of importance in a more detailed study of the behavior of solutions of the Stokes problem near a vertex of the boundary. In the analysis of the solution operator associated with this system of ordinary differential equations certain problems are encountered which are not present in Kondrätěv's analysis of the 2mth order elliptic equation. A fundamental point in the method of Kondrätěv is the analytic continuation of the solution operator (which depends on a complex parameter $\zeta$) up to the line $\text{Im } \zeta = 1$. If for the particular equation being considered there is a singularity on this line, the results in [12] do not apply. For the solution operator for the system of ordinary differential equations given by the Stokes problem there is a pole at $\zeta = i$. Nevertheless it is shown (in Section 5) that a certain "part" of the solution operator can be analytically continued up to $\text{Im } \zeta = 1$.

Throughout the paper $H^m(D)$, $m = 0, 1, \ldots$, will denote the $m$th Sobolev space on a domain $D$ in the plane. On this space we have the usual norm given by

$$||w||_{H^m(D)}^2 = \sum_{|\xi| \leq m} \int_D |D^\xi w|^2 \, dx \, dy;$$

note that $H^0(D) = L_2(D)$. $H^0(D)$ will denote the subspace of $H^1(D)$ consisting of those functions which vanish on $\partial D$. We will also use the Sobolev spaces $H^m[0, \omega]$ on a closed interval $[0, \omega]$ in $R^1$. We shall also use the notation

$$D^m w(x) = \left[ \sum_{|\xi| = m} |D^\xi w(x)|^2 \right]^{1/2}.$$

2. Formulation of the Problem

We let $D \subseteq R^2$ be a bounded domain with boundary $\partial D$ and we consider the generalized Stokes problem

$$
\begin{align*}
-\Delta u + p_x &= f_1 & \text{in } D, \\
-\Delta v + p_y &= f_2 & \text{in } D, \\
u_x + v_y &= g & \text{in } D, \\
u - v &= 0 & \text{on } \partial D.
\end{align*}
$$

(2.1)
These equations are obtained by suppressing the nonlinear terms in the two-dimensional stationary Navier-Stokes equations. We have set the viscosity \( \nu = 1 \) for convenience. We suppose that \( f_1, f_2 \) and \( g \) lie in \( L_2(D) \). If \( g = 0 \) we refer to the Stokes problem, as opposed to the generalized Stokes problem. We will say that functions \( u \in H_0^1(D), \nu \in H_0^1(D), \rho \in L_2(D) \) give a generalized solution of (2.1) if for any smooth function \( \varphi \) with compact support in \( D \),

\[
\int_D (\varphi_x u_x + \varphi_y u_y - \varphi_x \rho) \, dx \, dy = \int_D f_1 \varphi \, dx \, dy,
\]

\[
\int_D (\varphi_x v_x + \varphi_y v_y - \varphi_y \rho) \, dx \, dy = \int_D f_2 \varphi \, dx \, dy,
\]

and if \( u_x + v_y = g \) a.e. in \( D \). Note that the latter equation plus the boundary conditions imply that \( g \) satisfies the constraint

\[
\int_D g \, dx \, dy = 0. \tag{2.2}
\]

It is known (see, e.g., [19]) that if \( g = 0 \), or more generally if \( g \) satisfies (2.2), then there is a generalized solution \( (u, v, \rho) \) of (2.1).

If the curve \( \partial D \) and the data \( f_1, f_2, g \) are smooth, the solution of (2.1) is also smooth (see, e.g., [13]). In particular we may obtain local regularity of the solution, and global regularity of the solution if the boundary is smooth. The following result may be obtained from [13, Chap. 3, Theorem 2].

**Theorem 1.** Let \( m \geq 0 \) be an integer, let \( f_1 \in H^m(D), f_2 \in H^m(D), \) \( g \in H^{m+1}(D) \), and let \( u, v, \rho \) be a generalized solution of (2.1). Then if \( D_1 \) is any domain with \( \overline{D}_1 \subset D \), we have \( u \in H^{m+2}(D_1), v \in H^{m+2}(D_1), \) and \( \rho \in H^{m+1}(D_1) \). Furthermore, if \( \partial D \) is smooth, then \( u \in H^{m+2}(D), v \in H^{m+2}(D), \) \( \rho \in H^{m+1}(D) \), and there is a constant \( c > 0 \) depending only on \( D \) such that

\[
\| u \|_{H^{m+2}(D)} + \| v \|_{H^{m+2}(D)} + \| \rho \|_{H^{m+1}(D)} \\
\leq c \left[ \| f_1 \|_{H^m(D)} + \| f_2 \|_{H^m(D)} + \| g \|_{H^{m+1}(D)} \\
+ \| \rho \|_{L_2(D)} \right]. \tag{2.3}
\]

To state the main theorem of this paper we let \( D \) be a convex polygon and we let

\[
\delta(x, y) = \min\{\text{dist}((x, y), P); P \text{ a vertex of } D\}.
\]
Thus, $\delta(x, y)$ is the minimum distance from $(x, y)$ to one of the vertices of $D$. With this notation our main result is

**Theorem 2.** Let $D$ be a convex polygon, let $f_1 \in L^2(D)$, $f_2 \in L^2(D)$, $g \in H^1(D)$, $\delta^{-1}g \in L^2(D)$, and let $u, v, p$ be a generalized solution of (2.1). Then $u \in H^2(D)$, $v \in H^2(D)$, and $p \in H^1(D)$. Furthermore, there is a constant $c > 0$ depending only on $D$ such that

$$
\|u\|_{H^2(D)} + \|v\|_{H^2(D)} + \|D^1p\|_{L^2(D)} 
\leq c \left[\|f_1\|_{L^2(D)} + \|f_2\|_{L^2(D)} + \|D^1g\|_{L^2(D)} + \|\delta^{-1}g\|_{L^2(D)}\right].
$$

(2.4)

**Remark.** It will be shown that the requirement $\delta^{-1}g \in L^2(D)$ cannot be eliminated. This requirement may be interpreted in terms of interpolation spaces as follows. Let $H_*(D)$ denote the set of functions $w$ in $H^2(D)$ which vanish at the vertices of $D$. Since functions in $H^2(D)$ are continuous on $D$, $H_*(D)$ is a closed subspace of $H^2(D)$. Letting

$$
H_*(D) = [L^2(D), H^2(D)]_{1/2}
$$

(see [14] for a discussion of interpolation spaces), it may be shown that the expression

$$
\left\{|\nabla w|_{L^2(D)}^2 + \|\delta^{-1}w\|_{L^2(D)}^{1/2}\right\}
$$

provides a norm which is equivalent to the norm of $H_*(D)$. Thus, our requirement on $g$ may be phrased as $g \in H_*(D)$. (For more on interpolation between weighted Sobolev spaces, see [7, 11].)

To prove Theorem 2 it suffices to analyze the behavior of the solution near each of the vertices of $D$. To see this, let $P$ be one of the vertices, and let $\zeta(x, y)$ be a smooth function which is identically 1 in a neighborhood of $P$ and which satisfies

$$
\zeta(x, y) = 0 \quad \text{for } \text{dist}(x, y, P) \geq r_0,
$$

where $2r_0$ is the length of the smallest side of $D$. With no loss of generality we may set $r_0 = 1$. We let $\Omega$ denote the infinite sector whose vertex is placed at $P$, and whose sides are the extension to infinity of the two sides of $\partial D$ which meet at $P$. We obtain formally from (2.1)

$$
\begin{align*}
-\Delta(\zeta u) + (\zeta p)_{xx} &= \zeta f_1 - 2 \nabla \zeta \cdot \nabla u - u \Delta \zeta + p \zeta_x \quad \text{in } \Omega, \\
-\Delta(\zeta v) + (\zeta p)_{yy} &= \zeta f_2 - 2 \nabla \zeta \cdot \nabla v - v \Delta \zeta + p \zeta_y \quad \text{in } \Omega, \\
(\zeta u)_x + (\zeta v)_y &= \zeta g + u \zeta_x + v \zeta_y \quad \text{in } \Omega, \\
\zeta u = \zeta v &= 0 \quad \text{on } \partial \Omega \text{ and on } r = 1.
\end{align*}
$$

(2.5)
It is easily verified that \( \xi u \) and \( \xi v \) are locally in \( H^2 \), and \( \xi p \) is locally in \( H^1 \), and that (2.5) is satisfied a.e. in \( \Omega \). Also, the right sides of the first three equations of (2.5) are in \( L_2(\Omega) \), \( L_2^*(\Omega) \), and \( H^4(\Omega) \), respectively, and \( \xi u, \xi v, \xi p \) is a generalized solution of the generalized Stokes equations (2.5). If this construction is made for each vertex \( P_i \) of \( D \), \( 1 \leq i \leq I \), we see that the original solution \( u, v, p \) may be written in the form

\[
\sum_{i=0}^{I} u_i, \quad \sum_{i=0}^{I} v_i, \quad \sum_{i=0}^{I} p_i.
\]

The functions \( u_i, v_i, p_i, 1 \leq i \leq I \), are the generalized solution of a problem of the form (2.5) corresponding to the vertex \( P_i \) of \( D \). The remaining triple, \( u_0, v_0, p_0 \), is a generalized solution of the generalized Stokes problem which vanishes in a neighborhood of the vertices of \( D \). Using Theorem 1, we obtain \( u_0 \in H_0^2(D), \ v_0 \in H_0^2(D), \ p_0 \in H_0^1(D) \), and furthermore, inequality (2.3) with \( m = 0 \) holds for this solution. To prove Theorem 2 we must therefore analyze the solution of (2.5) in a sector \( \Omega \).

Let the vertex of the sector \( \Omega \) be placed at the origin \( 0 \). Suppose that one of the sides of \( \Omega \) lies on the positive \( x \)-axis. Letting the angle of \( \Omega \) be \( \omega \), the sector \( \Omega \) is given, in polar coordinates, by the inequality \( 0 < \theta < \omega \). We have thus reduced the proof of Theorem 2 to proving the following

**Theorem 3.** Let \( \omega < \pi \) and let \( u, v, \) and \( p \) be functions in \( \Omega \) such that

\[
\begin{align*}
u & \in H^1(\Omega) \cap H_0^3(\Omega), \quad \nu \in H^1(\Omega) \cap H_0^3(\Omega), \quad p \in L_2(\Omega) \cap H_0^1(\Omega), \\
u & = 0, \quad v = 0, \quad p = 0 \quad \text{for} \quad r > 1,
\end{align*}
\]

and in both the generalized and pointwise sense \( u, v, \) and \( p \) satisfy

\[
\begin{align*}
-\Delta u + p_x & = f_x \in L_2(\Omega), \\
-\Delta v + p_y & = f_y \in L_2(\Omega), \\
u_x + v_y & = g \in H^1(\Omega),
\end{align*}
\]

(2.8)

Then \( u \in H_2^2(\Omega), \ v \in H_2^3(\Omega), \ p \in H_1^1(\Omega) \), and there is a constant \( c = c(\omega) \) such that

\[
\begin{align*}
\Vert u \Vert_{H^2(\Omega)} + \Vert v \Vert_{H^2(\Omega)} + \Vert D^1 p \Vert_{L_2(\Omega)} \\
\lesssim c(\Vert f_x \Vert_{L_2(\Omega)} + \Vert f_y \Vert_{L_2(\Omega)} + \Vert D^1 g \Vert_{L_2(\Omega)} + \Vert r^{-1} g \Vert_{L_2(\Omega)}).
\end{align*}
\]

(2.9)
Remarks. (1) In the application of Theorem 3 to the proof of Theorem 2 we apply (2.9) to each vertex and use the above mentioned estimate for \( u_0, v_0, p_0 \). This yields for the solution of (2.1) the estimate

\[
\| u \|_{H^2(D)} + \| v \|_{H^2(D)} + \| D^1 p \|_{L_2(D)} \\
\leq c [\| f_1 \|_{L_2(D)} + \| f_2 \|_{L_2(D)} + \| D^1 g \|_{L_2(D)} + \| \delta^{-1} g \|_{L_2(D)} \\
+ \| u \|_{H^1(D)} + \| v \|_{H^1(D)} + \| p \|_{L_2(D)}].
\]

The presence of the terms involving \( u, v, \) and \( p \) on the right-hand side of this inequality is due to the form of the right-hand sides in (2.5). The desired estimate (2.4) is achieved by a standard argument based on uniqueness of solutions to (2.1).

(2) Similar to (2.2) it follows from (2.8) that

\[
\int_\Omega g \, dx \, dy = 0. \tag{2.10}
\]

(3) The proof of Theorem 3 will be given in Section 7. We derive in that section the stronger weighted inequality (7.2), which implies (2.9). A similar improvement can be made in the inequality in Theorem 2.

(4) The condition \( r^{-1} g \in L_2(\Omega) \), or, in Theorem 2, the condition \( \delta^{-1} g \in L_2(\Omega) \), may seem restrictive. In fact, the condition is necessary. To see this suppose \( u \in H^2(\Omega) \cap H^1_0(\Omega) \), let \( x^0 \to 0 \), and let \( L(x^0) \) be the line segment joining \( (x^0, 0) \) to \( (x^0 \cos \omega, x^0 \sin \omega) \); note that \( L(x^0) \) is perpendicular to the angle bisector of \( \Omega \) (see Fig. 1).

\[
\begin{align*}
\text{Since } u,(x^0, 0) = 0, \text{ we may use Hardy's inequality [15, p. 94] to obtain} \\
\int_{L(x^0)} |u^+(x, y)|^2/\text{dist}^2((x, y), (x^0, 0))) \, ds & \leq 4 \int_{L(x^0)} |D^2 u|^2 \, ds.
\end{align*}
\]

\[\text{Figure 1}\]
Integrating this inequality in the direction perpendicular to \( L(x^0) \), we find
\[
\int_{\Omega} \left( |u_x|^2 + r^2 \right) dx \, dy < 4 \sec^2(\omega/2) \int_{\Omega} |D^2 u|^2 \, dx \, dy.
\]

A similar inequality holds for the \( y \) derivative. Hence, if \( u \) and \( v \) belong to \( H^2(\Omega) \cap H^1_0(\Omega) \), we have \( r^{-1}(u_x + v_y) \in L_2(\Omega) \), so from the third equation in (2.1) we have \( r^{-1}g \in L_2(\Omega) \), as asserted.

It has been pointed out to us by Temam that Theorem 2 implies a similar regularity result for solutions of the Navier–Stokes equations
\[
-\Delta u + p_x + uu_x + v u_y = h, \quad \text{in } L^2(\Omega),
\]
\[
-\Delta v + p_y + u v_x + v v_y = h, \quad \text{in } L^2(\Omega),
\]
\[
u_x + v_y = 0
\]
in a convex polygon \( D \). In fact we have

**Theorem.** Let \( u \in H^1_0(D) \), \( v \in H^1_0(D) \), \( p \in L_2(D) \) and suppose (2.11)–(2.13) is satisfied in a distributional sense. Then \( u \in H^2(D) \), \( v \in H^2(D) \), \( p \in H^1(D) \).

**Proof.** Since \( H^1_0(D) \subset L^q(D) \) for any \( q < \infty \), we may conclude that, e.g., \( u \in L_8(D) \), \( v \in L_8(D) \). Hence, from Hölder’s inequality,
\[
f_1 = -uu_x - v u_y + h_1 \in L_{8/3}(D)
\]
\[
f_2 = -v v_x - v v_y + h_2 \in L_{8/3}(D).
\]

Pick a \( q > 8 \). Since \( H^1_0(D) \subset L^q(D) \), we see by interpolating that
\[
H^{1/3}(D) = [H^0(D), H^1_0(D)]_{1/3,2} \subset [L_2(D), L_q(D)]_{1/3,2} = L^{p,2}(D)
\]
where \( L^{p,2}(D) \) is the Lorentz space, defined by interpolation between \( L_2(D) \) and \( L_q(D) \) [5, Section 3.3], and where
\[
p = \frac{3q}{q - 1} > \frac{8}{3}.
\]

From the properties of the Lorentz spaces, \( L^{p,2}(D) \subset L^{p,p}(D) = L^p(D) \) so
\[
H^{1/3}(D) \subset L_{8/3}(D),
\]
with a continuous imbedding.
Taking the dual of this,

\[ L_{6/5}(D) \subset H^{-1/3}(D). \]

Hence, \( u, v, p \) satisfy the Stokes equations (2.1) with \( f_j \in H^{-1/3}(D) \), \( j = 1, 2 \), and with \( g = 0 \). We deduce from (2.1) and an integration by parts,

\[
\int_D \left[ |u_x|^2 + |u_y|^2 + |v_x|^2 + |v_y|^2 \right] \, dx \, dy
= \int_D \left[ f_1 \bar{u} + f_2 \bar{v} \right] \, dx \, dy
\leq \left[ \|u\|_{H^1(D)}^2 + \|v\|_{H^1(D)}^2 \right]^{1/2} \left[ \|f_1\|_{H^{-1}(D)}^2 + \|f_2\|_{H^{-1}(D)}^2 \right]^{1/2}.
\]

Hence, since \( u, v \in H^1(D) \),

\[
\|u\|_{H^1(D)} + \|v\|_{H^1(D)} \leq \epsilon \left[ \|f_1\|_{H^{-1}(D)} + \|f_2\|_{H^{-1}(D)} \right].
\]

Interpolating between this inequality and (2.4), we obtain

\[
\|u\|_{H^{5/3}(D)} + \|v\|_{H^{5/3}(D)} \leq \epsilon \left[ \|f_1\|_{H^{-1/3}(D)} + \|f_2\|_{H^{-1/3}(D)} \right].
\]

Thus, \( u \) and \( v \) belong to \( H^{5/3}(D) \). Hence, \( u \) and \( v \) are bounded, so \( f_j \in L_2(D) \), \( j = 1, 2 \), and again using (2.4), we have \( u \in H^2(D) \), \( v \in H^2(D) \), \( p \in H^1(D) \), as asserted.

3. A Weighted Inequality

It is convenient to write (2.8) in terms of polar coordinates \( x = r \cos \theta, y = r \sin \theta \). For this purpose we introduce the functions

\[
U = u \cos \theta + v \sin \theta, \\
V = -u \sin \theta + v \cos \theta,
\]

representing the components of velocity in the radial and tangential directions, respectively, and the functions

\[
F_1 = f_1 \cos \theta + f_2 \sin \theta, \\
F_2 = -f_1 \sin \theta + f_2 \cos \theta,
\]

representing the components of force in the radial and tangential
directions, respectively. Using these functions, (2.8) may be rewritten as
\[ -(1/r)(rU_r)_r - (1/r^2) U_{r0} + (1/r^2) U + (2/r^2) V_\theta + p_r = F_1 \quad \text{in } \Omega, \]
\[ -(1/r)(rV_\theta)_r - (1/r^2) V_{\theta0} + (1/r^2) V - (2/r^2) U_\theta + (1/r) p_\theta = F_2 \quad \text{in } \Omega, \]
\[ U_r + (1/r) U + (1/r) V_\theta = g \quad \text{in } \Omega, \]
\[ \theta = 0, \quad \theta = \omega. \]

It is necessary to consider the various norms in polar coordinates. For any function \( w \) recall the notation
\[ |D^1w|^2 = |w_x|^2 + |w_y|^2, \]
\[ |D^2w|^2 = |w_{xx}|^2 + |w_{xy}|^2 + |w_{yy}|^2. \]

Evidently
\[ |D^1w|^2 = |w_r|^2 + r^{-2} |w_\theta|^2. \]

Furthermore, it is easily seen that
\[ |D^2w|^2 \leq c[ |w_{rr}|^2 + r^{-2} |w_{r\theta}|^2 + r^{-4} |w_{\theta\theta}|^2 + r^{-2} |w_r|^2 + r^{-4} |w_\theta|^2], \]
and
\[ |w_{rr}|^2 + r^{-2} |w_{r\theta}|^2 + r^{-4} |w_{\theta\theta}|^2 \leq c[ |D^2w|^2 + r^{-2} |D^4w|^2]. \]

Using these inequalities and (3.1), we also find
\[ |D^1u|^2 + |D^1v|^2 \]
\[ \leq c[ |U_r|^2 + |V_r|^2 + r^{-2} |U_\theta|^2 + r^{-2} |V_\theta|^2 + r^{-2} |U|^2 + r^{-2} |V|^2], \]
\[ |U_r|^2 + |V_r|^2 + r^{-2} |U_\theta|^2 + r^{-2} |V_\theta|^2 \]
\[ \leq c[ |D^1u|^2 + |D^1v|^2 + r^{-2} |u|^2 + r^{-2} |v|^2], \]
\[ |D^2u|^2 + |D^2v|^2 \leq c[ |U_{rr}|^2 + r^{-2} |U_{r\theta}|^2 + r^{-4} |U_{\theta\theta}|^2 + r^{-2} |U|^2 + r^{-4} |V|^2], \]
\[ + r^{-2} |U_r|^2 + r^{-4} |U_\theta|^2 + r^{-4} |U|^2 + r^{-4} |V|^2, \]
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Lemma 1. Let \( u, v \) and \( p \) satisfy the conditions of Theorem 3. Then there is a constant \( c \) depending only on \( \omega \) such that

\[
\int_{\Omega} \int_{\Omega} \left[ r^2 \left| D^2 u \right|^2 + r^2 \left| D^2 v \right|^2 + \left| D^3 u \right|^2 + \left| D^3 v \right|^2 + r^{-2} \left| u \right|^2 \\
+ r^{-2} \left| v \right|^2 + r^2 \left| D^1 p \right|^2 + \left| p \right|^2 \right] \, dx \, dy.
\]

(3.3)

Proof. We start with the inequality

\[
\int_{\theta=0}^{\phi} \int_{r=1/2}^{2} \left[ \left| D^2 u \right|^2 + \left| D^2 v \right|^2 + \left| D^3 u \right|^2 + \left| D^3 v \right|^2 + \left| D^1 p \right|^2 \\
+ \left| u \right|^2 + \left| v \right|^2 + \left| p \right|^2 \right] \, dx \, dy.
\]

(3.4)

This inequality follows from the inequality (2.3), applied with \( m = 0 \) on the smooth domain \( \Omega^* \) (see Fig. 2) to the functions \( \zeta u, \zeta v, \zeta p \), where \( \zeta = \zeta(r) \) is a smooth function satisfying

\[
\zeta(r) = 1, \quad \frac{1}{2} \leq r \leq 2,
\]

\[
\zeta(r) = 0, \quad r \leq \frac{1}{2} \quad \text{or} \quad r \geq 4.
\]
We next make a change of independent variables, setting \( \bar{x} = \rho x \), \( \bar{y} = \rho y \), \( \bar{r} = \rho r \). With

\[
|\mathcal{D}^1 u|^2 = |u_x|^2 + |u_y|^2,
\]
\[
|\mathcal{D}^2 u|^2 = |u_{xx}|^2 + |u_{xy}|^2 + |u_{yy}|^2,
\]

we have

\[
|\mathcal{D}^1 u| = \rho |\mathcal{D}^1 u|,
\]
\[
|\mathcal{D}^2 u| = \rho^2 |\mathcal{D}^2 u|.
\]

Setting \( \bar{\Delta} u = u_{xx} + u_{yy} \), we see that in the new variables, (2.8) becomes

\[
-\bar{\Delta} u + (\rho^{-1} p) x = \rho^{-2} f_1,
\]
\[
-\bar{\Delta} v + (\rho^{-1} p) y = \rho^{-2} f_2,
\]
\[
u_x + v_y = \rho^{-1} g.
\]

Thus, the functions \( u, v, \rho^{-1} p \) are solutions of equations of the form (2.8) with right-hand sides \( \rho^{-2} f_1, \rho^{-2} f_2, \) and \( \rho^{-1} g \). Now applying (3.4) to the transformed problem we get

\[
\int_{\theta = 0}^{\pi} \int_{r = 1/2}^{1} |\mathcal{D}^2 u|^2 + |\mathcal{D}^2 v|^2 + |\mathcal{D}^1 u|^2 + |\mathcal{D}^1 v|^2 + |u|^2 + |v|^2
\]
\[
+ \rho^{-2} |p|^2 + \rho^{-2} |\mathcal{D}^1 p|^2 \, dx \, dy
\]
\[
\leq c \int_{\theta = 0}^{\pi} \int_{r = 1/4}^{1} |\rho^{-4}(|f_1|^2 + |f_2|^2) + \rho^{-2}(|D^1 g|^2 + |g|^2)
\]
\[
+ |u|^2 + |v|^2 + \rho^{-2} |p|^2 |\, dx \, dy.
\]

Transforming back to the \( x, y \) variables, we have

\[
\int_{\theta = 0}^{\pi} \int_{r = 1/2}^{1} \rho^4 |\mathcal{D}^2 u|^2 + |\mathcal{D}^2 v|^2 + \rho^{-2}(|\mathcal{D}^1 u|^2 + |\mathcal{D}^1 v|^2)
\]
\[
+ |u|^2 + |v|^2 + \rho^{-2} |p|^2 + \rho^{-4} |D^1 p|^2 \rho^2 \, dx \, dy
\]
\[
\leq c \int_{\theta = 0}^{\pi} \int_{r = 1/4}^{1} \rho^{-4}(|f_1|^2 + |f_2|^2) + \rho^{-2} |g|^2 + \rho^{-4} |D^1 g|^2
\]
\[
+ |u|^2 + |v|^2 + \rho^{-2} |p|^2 |\, dx \, dy.
\]
Since \( \frac{1}{4} \leq \rho^{-1}/r \leq 4 \) in the region of integration we may replace \( \rho^{-1}/r \) by \( r \) in the integrands of this inequality. This gives

\[
\int_{\theta=0}^{\pi} \int_{r=1/4}^{1/\rho} \left[ r^2 \left| D^2 u \right|^2 + \left| D^2 v \right|^2 \right] + \left| D^1 u \right|^2 + \left| D^1 v \right|^2 \\
+ r^{-2} \left( \left| u \right|^2 + \left| v \right|^2 \right) + \left| p \right|^2 + r^2 \left| D^1 p \right|^2 \right] \, dx \, dy
\]

\[
\leq c \int_{\theta=0}^{\pi} \int_{r=1/4}^{1/\rho} \left[ r^2 \left( \left| f_1 \right|^2 + \left| f_2 \right|^2 \right) + \left| g \right|^2 + r^2 \left| D^1 g \right|^2 \\
+ r^{-2} \left( \left| u \right|^2 + \left| v \right|^2 \right) + \left| p \right|^2 \right] \, dx \, dy.
\]

In this inequality we set \( \rho = 2^k \) and sum over all integers \( k \). The inequality (3.3) results, proving the lemma.

With the next lemma we see that the right side of (3.3) is finite if \( u, v, \) and \( p \) satisfy the hypotheses of Theorem 3.

**Lemma 2.** For \( w \in H^1_\ominus(\Omega) \) we have

\[
\int \int r^{-2} \left| \omega \right|^2 \, dx \, dy \leq \frac{1}{2} \omega^2 \int \int \left| D^1 \omega \right|^2 \, dx \, dy.
\]

See [8, p. 138] for a proof of this well-known result.

**Lemma 3.** Let \( u, v \) and \( p \) satisfy the conditions of Theorem 3 and let \( U \) and \( V \) be given by (3.1). Then for \( 0 < \epsilon < 1 \),

\[
\int \int \left[ r^2 \left| U_{rr} \right|^2 + \left| U_{r\theta} \right|^2 + \left| U_{\theta\theta} \right|^2 + r^{-2} \left| U \right|^2 + r^2 \left| V_{rr} \right|^2 \\
+ \left| V_{r\theta} \right|^2 + \left| V_{\theta\theta} \right|^2 + \left| U_r \right|^2 + r^{-2} \left| U_\theta \right|^2 \\
+ \left| V_r \right|^2 + r^{-2} \left| V_\theta \right|^2 + \left| U \right|^2 + r^{-2} \left| V \right|^2 + r^2 \left| p_r \right|^2 \\
+ r^2 \left| p \right|^2 + \left| p_\theta \right|^2 \right] \, r \, d\theta < \infty.
\]

**Proof.** This follows directly from Lemmas 1 and 2 and the previously established relations between the \( x, y \) and \( r, \theta \) derivatives.

The only purpose of Lemma 1 is to deduce the finiteness of the integral in (3.5). A specific bound for the integral is not required.

4. Transformation of the Problem

We introduce a new variable \( \tau \) by the relation \( r = e^{-\tau} \). In the \( \tau \theta \) plane the sector \( \Omega \) becomes the strip \( S = \{(\tau, \theta): 0 < \theta < \infty, \quad \} \).
\(-\infty < \tau < \infty\) \). Setting \(q = e^{-\tau}p\), (3.2) is readily transformed into the system

\[-U_\tau - U_\theta + U + 2V_\theta - q, -q = e^{-2\tau}F_1\quad \text{in } S,\]

\[-V_\tau - V_\theta + V - 2U_\theta + q_\theta = e^{-2\tau}F_2\quad \text{in } S,\]

\[-U_\tau + U + V_\theta = e^{-\tau}g\quad \text{in } S,\]

\[U = V = 0, \quad \theta = 0, \quad \theta = \omega.\]

In the new variables, (3.5) yields

\[\int_{\theta = 0}^\infty \int_{\tau = -\infty}^\infty \left\{ \begin{array}{l}
|U_{\tau\tau}|^2 + |U_{\tau\theta}|^2 + |U_{\theta\theta}|^2 + |U_\tau|^2 + |U_\theta|^2 \\
+ |U|^2 + |V_{\tau\tau}|^2 + |V_{\tau\theta}|^2 + |V_{\theta\theta}|^2 + |V_\tau|^2 + |V_\theta|^2 \\
+ |q_\tau|^2 + |q_\theta|^2 + |q|^2 \end{array} \right\} d\tau d\theta < \infty.\]  

The purpose of introducing the dependent variable \(q\) is so that all the dependent variables appear with the same weight on the left side of (4.2). Summarizing, we have

**Lemma 4.** Let \(u, v, \) and \(p\) satisfy the conditions of Theorem 3.1. Then \(u, v,\) and \(p\) are functions in the strip \(S\) which vanish for \(\tau < 0,\) and which satisfy (4.1) and (4.2).

We now introduce the Fourier transform

\[(\hat{w}) (\zeta, \theta) = \hat{w} (\zeta, \theta) = 1/(2\pi)^{1/2} \int_{-\infty}^{\infty} e^{-it\tau} w (\tau, \theta) d\tau, \quad \zeta = \xi + i\eta.\]

Note that the combined effect of introducing \(\tau\) and taking the Fourier transform is to take the Mellin transform of \(w\) with respect to the variable \(r.\) (In [4] the Mellin transform is used directly in the study of corner singularities.). We have collected in Appendix A some facts concerning the Fourier transform. Using the formulas stated there we note that the system (4.1) is formally transformed into the system

\[-U_{\theta\theta} + (\xi^2 + 1) U + 2V_\theta - (i\xi + 1)q = \hat{F}_1 (\xi - 2i, \theta), \quad 0 \leq \theta \leq \omega,\]

\[-V_{\theta\theta} + (\xi^2 + 1) V - 2U_\theta + q_\theta = \hat{F}_2 (\xi - 2i, \theta), \quad 0 \leq \theta \leq \omega,\]

\[(1 - i\xi) \hat{U} + \hat{V}_\theta = \hat{g} (\xi - i, \theta), \quad 0 \leq \theta \leq \omega,\]

\[U = \hat{V} = 0, \quad \theta = 0, \quad \theta = \omega.\]
Regarding these transforms, we have

**Lemma 5.** Let \( u, v, \) and \( p \) satisfy the conditions of Theorem 3. Then for \( \eta = \text{Im} \, \xi < 0, \) \( \hat{U}, \hat{V}, \hat{q} \) exist and are square integrable in \((\xi, \theta)\) on \( S. \) Furthermore, the weak derivatives \( \hat{U}_\theta, \hat{U}_{\theta \theta}, \hat{V}_\theta, \hat{V}_{\theta \theta}, \hat{q}_\theta \) are square integrable on \( S. \) These functions satisfy (4.3) for \( \eta < 0. \) Finally, the mapping \( \xi \rightarrow \hat{U}(\xi, \theta) \) is an analytic mapping of the half plane \( \eta < 0 \) into the Hilbert space \( L_2[0, \omega] \) and similarly, the functions \( \hat{U}_\theta, \hat{U}_{\theta \theta}, \hat{V}_\theta, \hat{V}_{\theta \theta}, \hat{q}, \hat{q}_\theta \) give rise to analytic mappings on the half plane \( \eta < 0. \)

**Proof.** From (4.2), \( U \in L_2(S). \) Since \( U(\tau, \theta) = 0 \) for \( \tau < 0, \) we therefore see that \( \hat{U}(\xi, \theta) \) exists and is in \( L_2(S) \) as a function of \((\xi, \theta)\) if \( \eta < 0, \) and gives rise to an analytic mapping for \( \eta < 0. \) The same is true for the Fourier transforms \( \hat{V} \) and \( \hat{q}. \)

Again from (4.2) the weak derivative \( \hat{U}_\theta \in L_2(S). \) From this we see that \( \hat{U}(\xi + i\eta, \theta) \) has a weak \( \theta \)-derivative in \( L_2(S) \) if \( \eta < 0, \)

\[
(\mathcal{F} U_\theta)(\xi + i\eta, \theta) = (\mathcal{F} U)_\theta (\xi + i\eta, \theta),
\]

and \( \eta \rightarrow \hat{U}_\theta(\xi, \theta) \) is analytic.

The Fourier transforms \( \hat{U}_{\theta \theta}, \hat{V}_\theta, \hat{V}_{\theta \theta}, \hat{q}, \hat{q}_\theta \) can be treated similarly.

### 5. Solution of the System (4.3)

In this section we shall obtain a solution of the system (4.3) for each value of the parameter \( \xi = \xi + i\eta \) in the strip \( \mid \eta \mid < 1. \) Regarding the right-hand sides of (4.3), we shall suppose throughout this section that the mappings

\[
\zeta \rightarrow \hat{P}_j(\zeta - 2i, \theta), \quad j = 1, 2,
\]

\[
\zeta \rightarrow \hat{g}(\zeta - i, \theta),
\]

\[
\zeta \rightarrow \hat{g}_\theta(\zeta - i, \theta)
\]

are analytic functions from the strip \( \mid \eta \mid < 1 \) into \( L_2[0, \omega]. \) (This will be established in Section 6. See Appendix A for a brief discussion of analyticity.) We shall then show that the solution of (4.3) consists of analytic functions of \( \zeta \) in \( \mid \eta \mid < 1, \) and we shall deduce some estimates for the solution that will be used in later sections.

Letting \( U_j, V_j, q_j, j = 1, \ldots, 4, \) be four solutions of the related
homogeneous system we seek a particular solution of the nonhomogeneous system of the form

$$\hat{U} = \sum_j c_j U_j,$$

$$\hat{V} = \sum_j c_j V_j,$$

$$\hat{q} = \sum_j c_j q_j$$

by variation of parameters. It is easily seen that $\hat{U}$, $\hat{V}$ and $\hat{q}$ is a solution of (4.3) (neglecting the boundary conditions) if

$$\sum_j c_j U_j = 0,$$

$$\sum_j c_j U_j' = -\hat{F}_1(\zeta - 2i, \theta) + 2\hat{q}(\zeta - i, \theta),$$

$$\sum_j c_j' V_j' - q_j = -\hat{F}_2(\zeta - 2i, \theta) - \hat{g}(\zeta - i, \theta),$$

$$\sum_j c_j' V_j = \hat{q}(\zeta - i, \theta),$$

where the prime denotes differentiation with respect to $\theta$. If, for some fixed $\zeta$, the determinant of this linear algebraic system is nonzero we can solve uniquely for the $c_j' (\zeta, \theta) \in L_2[0, \omega]$. Then letting

$$c_j(\zeta, \theta) = \int_0^\theta c_j'(\zeta, t) \, dt,$$

we see that $\hat{U}$, $\hat{V}$, $\hat{q}$ is a solution of (4.3). More precisely we have

$$\hat{U} \in H^2[0, \omega], \quad \hat{V} \in H^2[0, \omega], \quad \hat{q} \in H^1[0, \omega],$$

and $\hat{U}$, $\hat{V}$, $\hat{q}$ solves (4.3).

Now we make a particular choice for $U_j$, $V_j$, $q_j$. Let $V_{1j}$, $1 \leq j \leq 4$, denote the functions

$$V_{11}(\zeta, \theta) = e^{i\theta(\zeta - 1)} \sinh \zeta \theta,$$

$$V_{12}(\zeta, \theta) = e^{i\theta} \cosh \zeta \theta,$$

$$V_{13}(\zeta, \theta) = e^{-i\theta(\zeta - 1)} \sinh \zeta \theta,$$

$$V_{14}(\zeta, \theta) = e^{-i\theta} \cosh \zeta \theta,$$
and define

\[ U_{1j} = \frac{V'_{1j}}{i\zeta - 1}, \quad q_{1j} = \frac{V''_{1j} - (\zeta + i)^2 V'_{1j}}{\zeta^2 + 1}, \quad 1 \leq j \leq 4, \quad (5.3) \]

for \( \zeta \neq \pm i \). Note that \( V'_{1j} \) is entire and \( U_{1j} \) is analytic for \( \zeta \neq -i \).

A computation shows that the point \( \zeta = i \) is a removable singularity of \( q_{1j} \), and hence, \( q_{1j} \) is analytic for \( \zeta \neq -i \). With \( U_{1j}, V_{1j}, q_{1j} \) defined in this way the determinant in (5.2) is, using (5.3), given by

\[
\kappa(\zeta) = \det \begin{bmatrix}
U_{11} & \cdots & U_{14} \\
U'_{11} & \cdots & U'_{14} \\
V'_{11} - q_{11} & \cdots & V'_{14} - q_{14} \\
V_{11} & \cdots & V_{14}
\end{bmatrix}
= \mu(\zeta)(i\zeta - 1)^{\alpha(1 + \zeta^2)}^{-1},
\]

where

\[
\mu(\zeta) = \det \begin{bmatrix}
V_{11} & \cdots & V_{14} \\
V'_{11} & \cdots & V'_{14} \\
V''_{11} & \cdots & V''_{14}
\end{bmatrix}.
\]

A computation shows that \( \mu(\zeta) = 16(1 + \zeta^2) \), and hence, \( \kappa(\zeta) = 16(i\zeta - 1)^{-2} \). For \( \zeta \neq \pm i \), \( V_{1j}, j = 1, \ldots, 4 \), are linearly independent solutions of the equation

\[
V''' + 2(1 - \zeta^2) V'' + (1 + \zeta^2)^2 V = 0 \quad (5.4)
\]

which is obtained by eliminating \( \tilde{U} \) and \( \tilde{q} \) from the homogeneous system related to (4.3). Since \( \kappa(\zeta) \neq 0 \) for \( \zeta \neq -i \), we can solve (5.2) uniquely for the \( c_j(\zeta, \theta) \). Let the particular solution defined by (5.1) (with the \( c_j \) determined as above) be denoted by \( \tilde{U}_0, \tilde{V}_0, \) and \( \tilde{q}_0 \).

It is clear that the functions \( c_j(\zeta, \theta) \) are analytic in \( \zeta \) for \( |\eta| < 1 \) since they can be written as linear combinations of \( \hat{F}_1(\zeta - 2i, \theta), \hat{F}_2(\zeta - 2i, \theta), \hat{g}(\zeta - i, \theta), \) and \( \hat{g}'(\zeta - i, \theta) \) with coefficients which are analytic for \( \zeta \neq -i \). Hence, \( c_j(\zeta, \theta) \) is analytic in the strip \( |\eta| < 1 \), and since \( U_j, V_j, \) and \( q_j \) are analytic for \( \zeta \neq -i \), it follows that \( \tilde{U}_0, \tilde{V}_0, \tilde{q}_0 \) are analytic for \( |\eta| < 1 \). Furthermore, using (5.1) we see that \( \tilde{U}_0', \tilde{U}_0'', \tilde{V}_0', \tilde{V}_0'', \) and \( \tilde{q}_0' \) are analytic for \( |\eta| < 1 \).
It will be important later to have bounds for the coefficients $c_j(\zeta, \theta)$ in sets of the form

$$\{\zeta: -\frac{1}{2} \leq \eta < 1, \quad |\zeta| < \xi_0\}.$$  

(5.5)

Since, from (5.2), each of the functions $c'_j$ is a linear combination of the functions $\hat{F}_1, \hat{F}_2, \hat{g},$ and $\hat{g}'$, with coefficients that are determinants (involving functions which are analytic at $i$) divided by $\kappa(\zeta)$, it follows from the formula for $\kappa(\zeta)$ that

$$|c_j(\zeta, \theta)| \leq c(\zeta) \left[ |\hat{F}_1(\zeta - 2i, \theta)| + |\hat{F}_2(\zeta - 2i, \theta)| \right]$$

$$+ |\hat{g}(\zeta - i, \theta)| + |\hat{g}'(\zeta - i, \theta)|, \quad 0 \leq \theta \leq \omega, \quad 1 \leq j \leq 4,$$  

(5.6)

where the constant $c(\zeta)$ is bounded in $\zeta$-sets of the form (5.5). By integrating this inequality with respect to $\theta$ and using Schwarz’s inequality we have

$$|c_j(\zeta, \theta)| \leq c(\zeta) \|\hat{F}_1(\zeta - 2i, \theta)\|_{L^2[0, \omega]}$$

$$+ \|\hat{F}_2(\zeta - 2i, \theta)\|_{L^2[0, \omega]} + \|\hat{g}(\zeta - i, \theta)\|_{H^1[0, \omega]},$$  

(5.7)

for $0 \leq \theta \leq \omega, \quad 1 \leq j \leq 4$, where $c(\zeta)$ is bounded on sets of the form (5.5).

Now we consider the boundary values in (4.3). To this end let $V_{2j}, \quad 1 \leq j \leq 4,$ be solutions of (5.4) satisfying the initial conditions $V_{2j}^{(i-1)}(\zeta, 0) = \delta_{ij}$. Then $V_{2j}(\zeta, \theta)$ are linearly independent solutions of (5.4) which together with their $\theta$-derivatives are entire in $\zeta$. Let

$$U_{2j} = \frac{V_{2j}}{i\zeta} - 1, \quad q_{2j} = \frac{V_{2j} - (\zeta + i)^2 V'_{2j}}{i^2 \zeta^2 + 1}$$

for $\zeta \neq \pm i$. Then for any constants $b_j,$

$$\hat{U} = \hat{U}_0 + \sum_j b_j U_{2j},$$

$$\hat{V} = \hat{V}_0 + \sum_j b_j V_{2j},$$

$$\hat{g} = \hat{g}_0 + \sum_j b_j q_{2j}$$

(5.8)

is a solution to the equations in (4.3). We choose the $b_j$ so as to satisfy the boundary conditions: $\hat{U}(\zeta, 0) = \hat{U}(\zeta, \omega) = \hat{V}(\zeta, 0) = \hat{V}(\zeta, \omega) = 0$. If these conditions are imposed on the functions in (5.8) we obtain a
linear system for the $b_j$. The nonhomogeneous terms in this system involve $\tilde{U}_0$, $\tilde{V}_0$ and $\tilde{Q}_0$ and are hence analytic for $|\eta| < 1$. The determinant of the system,

$$
\nu(\zeta) = \det \begin{bmatrix}
U_{21}(\zeta, 0) & \cdots & U_{24}(\zeta, 0) \\
U_{21}(\zeta, \omega) & \cdots & U_{24}(\zeta, \omega) \\
V_{21}(\zeta, 0) & \cdots & V_{24}(\zeta, 0) \\
V_{21}(\zeta, \omega) & \cdots & V_{24}(\zeta, \omega)
\end{bmatrix},
$$

is analytic for $\zeta \neq -i$. We will show that $\nu(\zeta)$ has no zeros in the strip $-1 < \eta \leq 1$.

Suppose for some $\zeta$ that $V_j^*, 1 \leq j \leq 4$ is an independent set of solutions of (5.4). Then we can find $c_{ij}(\zeta)$ such that

$$
V_{2i}(\zeta, \theta) = \sum_j c_{ij}(\zeta) V_j^*(\zeta, \theta), \quad \det(c_{ij}(\zeta)) \neq 0.
$$

Now let

$$
U_{i^*} = V_{i'}^*/(i \zeta - 1).
$$

Then

$$
U_{2, i}(\zeta, \theta) = \sum_j c_{ij}(\zeta) U_{j^*}(\zeta, \theta).
$$

From this we see that

$$
\nu(\zeta) = \det(c_{ij}(\zeta)) \cdot \det \begin{bmatrix}
U_{1^*}(\zeta, 0) & \cdots & U_{4^*}(\zeta, 0) \\
U_{1^*}(\zeta, \omega) & \cdots & U_{4^*}(\zeta, \omega) \\
V_{1^*}(\zeta, 0) & \cdots & V_{4^*}(\zeta, 0) \\
V_{1^*}(\zeta, \omega) & \cdots & V_{4^*}(\zeta, \omega)
\end{bmatrix} = \frac{\det(c_{ij}(\zeta))}{(i \zeta - 1)^2} \cdot \det \begin{bmatrix}
V_{1^*}(\zeta, 0) & \cdots & V_{4^*}(\zeta, 0) \\
V_{1^*}(\zeta, \omega) & \cdots & V_{4^*}(\zeta, \omega) \\
V_{1^*}(\zeta, 0) & \cdots & V_{4^*}(\zeta, 0) \\
V_{1^*}(\zeta, \omega) & \cdots & V_{4^*}(\zeta, \omega)
\end{bmatrix}.
$$

Thus, to verify that $\nu(\zeta) \neq 0$, we may pick any set $V_{1^*}, \ldots, V_{4^*}$ of linearly independent solutions of (5.5), define $U_j^*$, $1 \leq j \leq 4$, as above, and calculate the determinant

$$
\nu^*(\zeta) = \det \begin{bmatrix}
V_{1^*}(\zeta, 0) & \cdots & V_{4^*}(\zeta, 0) \\
V_{1^*}(\zeta, \omega) & \cdots & V_{4^*}(\zeta, \omega) \\
V_{1^*}(\zeta, 0) & \cdots & V_{4^*}(\zeta, 0) \\
V_{1^*}(\zeta, \omega) & \cdots & V_{4^*}(\zeta, \omega)
\end{bmatrix}.
$$

We have $\nu(\zeta) \neq 0$ if and only if $\nu^*(\zeta) \neq 0$. 

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For $\zeta = 0$ let $V_j^*$ be given by
\[
\sin \theta, \quad \theta \sin \theta, \quad \cos \theta, \quad \theta \cos \theta.
\]

Then the determinant in question is
\[
\nu^*(0) = \det \begin{bmatrix}
1 & 0 & 0 & 1 \\
\cos \omega & \omega \cos \omega + \sin \omega & -\sin \omega & \omega \cos \omega - \omega \sin \omega \\
0 & 0 & 1 & 0 \\
\sin \omega & \omega \sin \omega & \cos \omega & \omega \cos \omega
\end{bmatrix}
= \omega^2 - \sin^2 \omega \neq 0,
\]
since $\omega > 0$.

For $\zeta = i$ we let $V_j^*$ be given by
\[
\sin 2\theta, \quad \cos 2\theta, \quad \theta, \quad 1.
\]

The determinant in question is
\[
\nu^*(i) = \det \begin{bmatrix}
2 & 0 & 0 & 1 \\
2 \cos 2\omega & -2 \sin 2\omega & 1 & 0 \\
0 & 1 & 0 & 1 \\
\sin 2\omega & \cos 2\omega & \omega & 1
\end{bmatrix}
= 8 \sin \omega [\omega \cos \omega - \sin \omega] \neq 0,
\]
provided $0 < \omega < \pi$. (Note the use of convexity here.)

Finally, for $-1 < \eta \leq 1$, $\zeta \neq 0, i$ we let $V_j^*$ be given by
\[
\cos(1 - i\zeta)\theta, \quad \sin(1 - i\zeta)\theta, \quad \cos(1 + i\zeta)\theta, \quad \sin(1 + i\zeta)\theta.
\]

In this case a computation shows that
\[
\nu^*(\zeta) = 4(\sinh^2 \zeta \omega - \zeta^2 \sin^2 \omega).
\]

In [12, pp. 307, 308] it is shown that this expression has no zeros for $-1 < \eta \leq 1$, $\zeta \neq 0, i$.

Thus we can solve uniquely for the $b_j$ in the strip $-1 < \eta < 1$. We see that the $b_j(\zeta)$ are analytic in $\zeta$ in this strip. Combining these results we see that the functions given by (5.8) (with the $b_j$ determined as above) provide a solution to (4.3). Furthermore $U, U', U'', \dot{V}, \dot{V}', \dot{V}''$, $\dot{q}$, and $\ddot{q}$ are analytic in $\zeta$ in the strip $|\eta| < 1$; i.e., the mappings
\[
\zeta \mapsto U(\zeta, \theta), \ldots, \quad \zeta \mapsto \ddot{q}(\zeta, \theta).
\]
from this strip to \( L_2[0, \omega] \) are analytic. This solution is the unique solution to (4.3) for \( \zeta \) in this strip. We summarize all of this in the following

**Theorem 4.** For every \( \zeta \) in the strip \(-1 < \eta < 1\), (4.3) has a unique solution \( \hat{U}, \hat{V}, \hat{q} \). Furthermore, \( \hat{U}, \hat{U}', \hat{U}'', \hat{V}, \hat{V}', \hat{V}'', \hat{q}, \) and \( \hat{q}' \) depend analytically on \( \zeta \).

**Remark.** Even if the right sides were analytic for \( \eta > 1 \), the solution would not necessarily be analytic at \( \zeta = i \). In other words, the solution operator to (4.3) has a pole at \( \zeta = i \).

We turn now to the derivation of an estimate for the solution. First we estimate \( \hat{U}_0, \hat{V}_0, \) and \( \hat{q}_0 \). It will be convenient to write

\[
N(\zeta) = \| \hat{F}_1(\zeta - 2i, \theta) \|_{L_2[0, \omega]} + \| \hat{F}_2(\zeta - 2i, \theta) \|_{L_2[0, \omega]} + \| \hat{g}(\zeta - i, \theta) \|_{H^1[0, \omega]}.
\]

Now \( \hat{U}_0 = \sum_j c_j U_{1j} \) and since the \( U_{1j} \) are analytic for \( \zeta \neq -i \), we have from (5.7)

\[
| \hat{U}_0(\zeta, \theta) | \leq c(\zeta) N(\zeta)
\]

for \( 0 \leq \theta \leq \omega \), where \( c(\zeta) \) is bounded on sets of the form (5.5). From this and Schwarz's inequality we get

\[
\| \hat{U}_0(\zeta, \theta) \|_{L_2[0, \omega]} \leq c(\zeta) N(\zeta).
\]

Using (5.2) we see that

\[
\hat{C}_0(\zeta, \theta) = \sum_j c_j U'_{1j}
\]

and again from (5.7), we have

\[
\| \hat{U}'_0(\zeta, \theta) \|_{L_2[0, \omega]} \leq c(\zeta) N(\zeta).
\]

Since

\[
\hat{U}'_0(\zeta, \theta) = \sum_j (c_j U'_{1j} + c_j U''_{1j})
\]

we have, using (5.6) and (5.7),

\[
| \hat{U}'_0(\zeta, \theta) | \leq c(\zeta) \| \hat{F}_1(\zeta - 2i, \theta) \| + \| \hat{F}_2(\zeta - 2i, \theta) \| + | \hat{g}(\zeta - i, \theta) | + | \hat{g}'(\zeta - i, \theta) | + N(\zeta),
\]
and hence,

$$\| \hat{U}_0(\zeta, \theta) \|_{L_2[0, \infty]} \leq c(\zeta) N(\zeta).$$

Combining these results we have

$$\| \hat{U}_0(\zeta, \theta) \|_{H^2[0, \infty]} \leq c(\zeta) N(\zeta)$$

(5.10)

for $-1 < \eta < 1$, where $c(\zeta)$ is bounded on the sets (5.5).

In the same way we get

$$\| \hat{V}_0(\zeta, \theta) \|_{L_2[0, \infty]} \leq c(\zeta) N(\zeta), \quad 0 \leq \theta \leq \omega$$

(5.11)

and

$$\| \hat{V}_0(\zeta, \theta) \|_{L_2[0, \infty]} \leq c(\zeta) N(\zeta).$$

Using

$$\hat{V}_0'(\zeta, \theta) = \hat{g}(\zeta - i, \theta) + \sum_j c_j V_j'$$

we get

$$\| \hat{V}_0'(\zeta, \theta) \|_{L_2[0, \infty]} \leq c(\zeta) N(\zeta).$$

and, using

$$\hat{V}_0''(\zeta, \theta) = \hat{g}'(\zeta - i, \theta) + \sum_j (c_j V_j' + c_j V_j')$$

we get

$$\| \hat{V}_0''(\zeta, \theta) \|_{L_2[0, \infty]} \leq c(\zeta) N(\zeta).$$

Thus

$$\| \hat{V}_0(\zeta, \theta) \|_{H^2[0, \infty]} \leq c(\zeta) N(\zeta).$$

(5.12)

Next we estimate $\hat{U}$ and $\hat{V}$. From (5.9), (5.11), and the fact that $\nu(\zeta)$ is never 0 in the strip $-1 < \eta \leq 1$ we see that

$$| b_j(\zeta) | \leq c(\zeta) N(\zeta).$$

Thus, since $U_{2j}$ and $V_{2j}$ are analytic for $\zeta \neq -i$, we see from (5.8), (5.10), and (5.12) that

$$\| \hat{U}(\zeta, \theta) \|_{L_2[0, \infty]} \leq c(\zeta) N(\zeta),$$

$$\| \hat{V}(\zeta, \theta) \|_{L_2[0, \infty]} \leq c(\zeta) N(\zeta).$$
Moreover, by differentiating in (5.8) we obtain

\[ \| U(\xi, \theta) \|_{H^p_{1,0}} \leq c(\xi) N(\xi), \]  
(5.13)

\[ \| \hat{V}(\xi, \theta) \|_{H^p_{1,0}} \leq c(\xi) N(\xi). \]  
(5.14)

Finally we estimate \( \hat{q}'(\xi, \theta) \) and \( (i\xi + 1) \hat{q}(\xi, \theta) \) directly from the equations (4.3). From the second equation in (4.3) and from (5.13) and (5.14) we have

\[ \| \hat{q}'(\xi, \theta) \|_{L^2_{1,0}} \leq c(\xi) N(\xi). \]

From the first equation in (4.3) and from (5.13) and (5.14) we have

\[ \| (i\xi + 1) \hat{q}(\xi, \theta) \|_{L^2_{1,0}} \leq c(\xi) N(\xi). \]

We summarize these results in the following

**Theorem 5.** For \( \xi \) in the strip \(-1 < \text{Im} \, \xi < 1\) there is a constant \( c(\xi) \) depending only on \( \xi \) and \( \omega \) such that

\[
\| U(\xi, \theta) \|_{H^{p}_{1,0}} + \| \hat{V}(\xi, \theta) \|_{H^{p}_{1,0}} + \| \hat{q}(\xi, \theta) \|_{L^2_{1,0}} \\
+ \| (i\xi + 1) \hat{q}(\xi, \theta) \|_{L^2_{1,0}} \leq c(\xi)\left\{ \| \hat{F}_1(\xi - 2i, \theta) \|_{L^2_{1,0}} \\
+ \| \hat{F}_2(\xi - 2i, \theta) \|_{L^2_{1,0}} + \| \hat{g}(\xi - i, \theta) \|_{H^{1}_{1,0}} \right\},
\]

and \( c(\xi) \) is bounded on sets of the form

\[ \{ \xi: -\frac{1}{2} \leq \text{Im} \, \xi < 1, \quad \text{Re} \, \xi \leq \xi_0 \}. \]

**Remark.** It may be noted that in the estimate of Theorem 5, one cannot include the quantity \( \| \hat{q} \|_{L^2_{1,0}} \) on the left side. This is reflected in the fact that the solution operator \( \mathcal{A}(\xi) \) associated with the system (4.3) has a pole at \( \xi = i \).

### 6. Analytic Continuation of \( \hat{U}, \hat{V}, \) and \( \hat{q} \)

Our object in this section is to use (4.3) to continue \( \hat{U}, \hat{V}, \) and \( \hat{q} \) analytically up to the line \( \eta = 1 \), and to obtain \( L_2 \) estimates for these functions and their first and second \( \theta \)-derivatives. We start by establishing the region of analyticity of the functions \( \hat{F}_1, \hat{F}_2, \) and \( \hat{g} \) which appear on the right-hand sides of (4.3).

**Lemma 6.** Let \( F_1, F_2, r^{-1}g, \) and \( |D|^lg \) lie in \( L_2(\Omega) \) and vanish for
$r \geq 1$. Then the functions $\bar{F}_j(\xi, \eta), j = 1, 2,$ are analytic for $\eta < -1$, and $\bar{g}(\xi, \theta)$ is analytic for $\eta < 0$. Furthermore,

$$\int_{\theta = 0}^{\infty} \int_{\xi = -\infty}^{\infty} \left[ |\bar{F}_1(\xi + (\eta - 2)i, \theta)|^2 + |\bar{F}_2(\xi + (\eta - 2)i, \theta)|^2 ight] \, d\xi \, d\theta$$

$$+ (1 + \xi^2 + (\eta - 1)^2) |\bar{g}(\xi + (\eta - 1)i, \theta)|^2$$

$$+ |\bar{g}_0(\xi + (\eta - 1)i, \theta)|^2 \right] \, d\xi \, d\theta$$

$$\leq \int_{\Omega} \left[ |f_1|^2 + |f_2|^2 + r^{-2} |g|^2 + |D^1g|^2 \right] \, dx \, dy, \quad \eta < 1.$$  \hspace{1cm} (6.1)

**Proof.** Since

$$\int_{\tau = -\infty}^{\infty} \int_{\theta = -\infty}^{\infty} e^{-2\tau} |F_j(\tau, \theta)|^2 \, d\tau \, d\theta = \int_{\Omega} |F_j|^2 \, dx \, dy < \infty,$$

we see that $e^{-\tau}F_j \in L_2(S)$. Hence, since $F_j = 0$ for $\tau < 0$, $\bar{F}_j$ is analytic for $\eta < -1$ and

$$\int_{\theta = 0}^{\infty} \int_{\xi = -\infty}^{\infty} |\bar{F}_j(\xi - i\eta, \theta)|^2 \, d\xi \, d\theta$$

$$= \int_{\theta = 0}^{\infty} \int_{\tau = -\infty}^{\infty} e^{2\eta\tau} |F_j|^2 \, d\tau \, d\theta$$

$$\leq \int_{\Omega} |F_j|^2 \, dx \, dy, \quad \eta < -1.$$  

Since

$$\int_{\theta = 0}^{\infty} \int_{\tau = -\infty}^{\infty} |g(\tau, \theta)|^2 \, d\tau \, d\theta = \int_{\Omega} r^{-2} |g|^2 \, dx \, dy < \infty,$$

and $g = 0$ for $\tau < 0$ we see that $\bar{g}(\zeta, \theta)$ is analytic for $\eta < 0$. Furthermore,

$$\int_{\theta = 0}^{\infty} \int_{\xi = -\infty}^{\infty} \left[ (1 + \xi^2 + \eta^2)|\bar{g}(\xi + \eta i, \theta)|^2 + |\bar{g}_0(\xi + \eta i, \theta)|^2 \right] \, d\xi \, d\theta$$

$$= \int_{\theta = 0}^{\infty} \int_{\tau = -\infty}^{\infty} e^{2\eta\tau}[|g_\tau|^2 + |g_\theta|^2 + |g|^2] \, d\tau \, d\theta$$

$$\leq \int_{\Omega} \left[ |D^1g|^2 + r^{-2} |g|^2 \right] \, dx \, dy$$

for $\eta < 0$. This completes the proof.
Lemma 7. The functions \( \hat{U}, \hat{V}, \) and \( \hat{q} \) may be extended analytically to the region \( \eta < 1. \) Furthermore, for each \( \zeta \) with \( -\frac{1}{2} \leq \eta < 1, \)

\[
\hat{U}(\zeta, \theta) \in H^2[0, \omega], \quad \hat{V}(\zeta, \theta) \in H^2[0, \omega], \quad \hat{q}(\zeta, \theta) \in H^1[0, \omega],
\]

and there is a constant \( c \) depending only on \( \omega \) such that

\[
(6.2) \quad \int_{0}^{\omega} \frac{1}{2} \left( 1 + |\zeta|^2 \right) \left( |\hat{U}(\zeta, \theta)|^2 + |\hat{V}(\zeta, \theta)|^2 \right) + \left( 1 + |\zeta|^2 \right) \left( |\hat{U}_{\theta}(\zeta, \theta)|^2 + |\hat{V}_{\theta}(\zeta, \theta)|^2 \right) + |\hat{g}(\zeta, \theta)|^2 + |\hat{q}(\zeta, \theta)|^2 + |\hat{q}_{\theta}(\zeta, \theta)|^2 \, d\theta 
\]

for \( -\frac{1}{2} \leq \eta = 1 \omega \zeta < 1. \)

Proof. From Lemma 5 we know that \( \hat{U}(\zeta, \theta), \hat{V}(\zeta, \theta), \) and \( \hat{q}(\zeta, \theta) \)
are analytic for \( \eta < 0. \) Using Lemma 6 we see that \( \hat{F}_{\zeta}(\zeta - 2i, \theta), \)
\( \hat{F}_{\theta}(\zeta - 2i, \theta), \) and \( \hat{g}(\zeta - i, \theta) \)
are analytic for \( \eta < 1. \) We also know that for each such \( \zeta, \) \( \hat{F}_{\zeta}(\zeta - 2i, \theta) \in L^2[0, \omega] \) and \( \hat{g}(\zeta - i, \theta) \in H^1[0, \omega]. \)
Hence, from Theorem 4, the system (4.3) can be solved for \( -1 < \eta < 1. \) For \( -1 < \eta < 0, \) \( \hat{U}(\zeta, \theta), \hat{V}(\zeta, \theta), \hat{q}(\zeta, \theta) \)
is the solution of (4.3). Thus, the solution constructed in Section 5 provides an analytic extension of \( \hat{U}(\zeta, \theta), \hat{V}(\zeta, \theta), \) and \( \hat{q}(\zeta, \theta) \) up to the line \( \eta = 1. \)

It remains to prove (6.2). From the Agmon–Nirenberg type estimate given in Appendix B we see that there is a \( \xi_0 > 0 \) such that (6.2) holds for \( \xi \) which satisfy \( -\frac{1}{2} \leq \eta < 1, \) \( |\xi| > \xi_0. \) For \( \xi \) such that \( -\frac{1}{2} \leq \eta < 1, \) \( |\xi| \leq \xi_0, \) (6.2) follows directly from Theorem 5 in Section 5.

7. Proof of Theorem 3

Let \( u, v \) and \( p \) satisfy the conditions of Theorem 3 and let \( U, V, \) and \( q \) be the transformed functions introduced above. We shall first establish the inequality

\[
\int_{\eta = 0}^{\omega} \int_{r = -\infty}^{\infty} \left( |U_{\tau \tau}|^2 + |U_{\tau \theta}|^2 + |U_{\theta \theta}|^2 + |U_{\tau}|^2 + |U_{\theta}|^2 + |V_{\tau \tau}|^2 + |V_{\tau \theta}|^2 + |V_{\theta \theta}|^2 + |V_{\tau}|^2 + |V_{\theta}|^2 + |q_{\tau}|^2 + |q_{\theta}|^2 \right) e^{2\tau} \, d\tau \, d\theta \leq cI
\]

(7.1)
where

\[ I = \int_{\Omega} \int \left[ |f_1|^2 + |f_2|^2 + r^{-2} |g|^2 + |D^1g|^2 \right] \, dx \, dy. \]

Toward this end we define a function \( \hat{\mathcal{z}}(\zeta, \theta) \) for \( \eta < 0 \) by

\[ \hat{\mathcal{z}}(\zeta, \theta) = (\zeta + i)^2 \hat{U}(\zeta + i, \theta). \]

From Lemma 7, \( \hat{\mathcal{z}} \) is an analytic function from the half plane \( \eta < 0 \) into \( L_2[0, \omega] \). For \( \eta \leq -1 \), \( -(\zeta + i)^2 \hat{U}(\zeta + i, \theta) \) is the Fourier transform of \( U_{\tau}(\tau, \theta) \) at \( \zeta + i \) and hence

\[
\begin{align*}
\int_{\theta=0}^{\infty} \int_{\eta=-\infty}^{\infty} |\xi + i(\eta + 1)|^4 |\hat{U}(\xi + i(\eta + 1), \theta)|^2 \, d\xi \, d\theta \\
= \int_{\theta=0}^{\infty} \int_{\tau=0}^{\infty} |U_{\tau}(\tau, \theta)|^2 e^{2(\eta+1)\tau} \, d\tau \, d\theta.
\end{align*}
\]

This shows that

\[
\int_{\theta=0}^{\infty} \int_{\eta=-\infty}^{\infty} |\xi + i(\eta + 1)|^4 |\hat{U}(\xi + i(\eta + 1), \theta)|^2 \, d\xi \, d\theta
\]

is increasing in \( \eta \) for \( \eta < -1 \). Using this fact we see that

\[
E = \sup_{\eta < 0} \int_{\tau=0}^{\infty} \int_{\eta=-\infty}^{\infty} |\hat{\mathcal{z}}(\xi + i\eta, \theta)|^2 \, d\xi \, d\theta
\]

\[
= \sup_{-3/2 < \eta < 0} \int_{\tau=0}^{\infty} \int_{\eta=-\infty}^{\infty} |\hat{\mathcal{z}}(\xi + i\eta, \theta)|^2 \, d\xi \, d\theta.
\]

Now, integrating (6.2) with respect to \( \xi \) from \( \xi = -\infty \) to \( \xi = \infty \) and using (6.1), we see that \( E \leq cI < \infty \). Thus we can apply the Paley–Wiener theorem (Appendix A) and conclude that there is a function \( \mathcal{z}(\tau, \theta) \in L_2(S) \) such that \( \mathcal{z}(\tau, \theta) = 0 \) for \( \tau < 0 \), \( ||z||_{L_2(S)} \leq E \), and \( \hat{\mathcal{z}} \) is the Fourier transform of \( \mathcal{z} \):

\[ \hat{\mathcal{z}}(\zeta, \theta) = \mathcal{F}(\mathcal{z})(\zeta, \theta), \quad \eta < 0. \]

In particular,

\[ \xi^2 \hat{U}(\zeta, \theta) = \hat{\mathcal{z}}(\zeta - i, \theta) = \mathcal{F}(e^{-\tau z})(\zeta, \theta), \quad \eta < 1. \]

On the other hand we have

\[ \xi^2 \hat{U}(\zeta, \theta) = \mathcal{F}(-U_{\tau})(\zeta, \theta), \quad \eta < 0. \]
Hence, we may conclude that

$$-U_{\tau\tau}(\tau, \theta) = e^{-\tau z}(\tau, \theta),$$

so $z = -e^\tau U_{\tau\tau}$ is square-integrable. Also

$$\int_{\tau=0}^{\infty} \int_{\theta=0}^{\infty} e^{2\pi |U_{\tau\tau}|^2} d\tau d\theta = \int_{\theta=0}^{\infty} \int_{\tau=0}^{\infty} |z(\tau, \theta)|^2 d\tau d\theta \leq E \leq cI.$$

This is the required bound for the first term in (7.1).

The remaining terms can all be treated in the same way. For the sake of clarity we shall consider the term involving $q_\tau + q$ in detail. Let

$$z(\zeta, \theta) = (i(\zeta + i) + 1) q(\zeta + i, \theta) = i\zeta q(\zeta + i, \theta), \quad \eta < 0.$$  

By Lemma 7, $z(\zeta, \theta)$ is analytic in $\eta < 0$. For $\eta < -1$, $i\zeta q(\zeta + i, \theta)$ is the Fourier transform of $q_\tau + q$ at $\zeta + i$. Thus

$$\int_{\theta=0}^{\infty} \int_{\xi=-\infty}^{\infty} |\xi + i\eta|^2 |\hat{q}(\xi + i(\eta + 1), \theta)|^2 d\xi d\theta$$

$$= \int_{\theta=0}^{\infty} \int_{\tau=0}^{\infty} |(q_\tau + q)(\tau, \theta)|^2 e^{2(\eta + 1)\tau} d\tau d\theta, \quad \eta \leq -1$$

which shows that

$$\int_{\theta=0}^{\infty} \int_{\xi=-\infty}^{\infty} |\xi + i\eta|^2 |\hat{q}(\xi + i(\eta + 1), \theta)|^2 d\xi d\theta$$

is increasing in $\eta$ for $\eta < -1$. Using this, (6.2), and (6.1), we find

$$\sup_{\eta < -1} \int_{\theta=0}^{\infty} \int_{\xi=-\infty}^{\infty} |\hat{z}(\xi + i\eta, \theta)|^2 d\xi d\theta \leq cI.$$  

Hence, from the Paley–Wiener theorem,

$$\hat{z}(\zeta, \theta) = \mathcal{F}(z)(\zeta, \theta), \quad \eta < 0,$$

where $z \in L_2(S)$ and $z(\tau, \theta) = 0$ for $\tau < 0$. Thus,

$$(i\zeta + 1) q(\zeta, \theta) = \hat{z}(\zeta - i, \theta) = \mathcal{F}(e^{-\tau z})(\zeta, \theta), \quad \eta < 1.$$  

Also

$$(i\zeta + 1) q(\zeta, \theta) = \mathcal{F}(q_\tau + q)(\zeta, \theta), \quad \eta < 0.$$
Together these relations imply
\[ (q_\tau + q) e^\tau = z \in L_2(S). \]

Furthermore,
\[
\int_{\theta=0}^{\infty} \int_{r=-\infty}^{\infty} \frac{1}{r^2} \left| q_\tau + q \right|^2 \, d\tau \, d\theta = \int_{\theta=0}^{\infty} \int_{r=-\infty}^{\infty} \left| z(\tau, \theta) \right|^2 \, d\tau \, d\theta
\leq \sup_{\eta < 0} \int_{\theta=0}^{\infty} \int_{\xi=\eta}^{\infty} \left| \hat{z}(\xi + i\eta, \theta) \right|^2 \, d\xi \, d\theta
\leq cI.
\]

This completes the proof of (7.1).

To complete the proof of Theorem 3 we transform (7.1) into the \( r, \theta \) variables. We obtain
\[
\int_{\Omega} \int \left[ \left| U_{rt} \right|^2 + r^{-2} \left| U_{r\theta} \right|^2 + r^{-4} \left| U_{\theta\theta} \right|^2 + r^{-2} \left| U_r \right|^2 + r^{-4} \left| U_\theta \right|^2 + r^{-4} \left| U \right|^2 
+ \left| V_{rt} \right|^2 + r^{-2} \left| V_{r\theta} \right|^2 + r^{-4} \left| V_{\theta\theta} \right|^2 + r^{-2} \left| V_r \right|^2 + r^{-4} \left| V_\theta \right|^2 + r^{-4} \left| V \right|^2 
+ \left| p_r \right|^2 + r^{-2} \left| p_\theta \right|^2 \right] \, rdr \, d\theta \leq cI.
\]

Now, using the inequalities of Section 3 for the change between polar coordinates and cartesian coordinates, we obtain
\[
\int_{\Omega} \int \left[ \left| D^2 u \right|^2 + r^{-2} \left| D^1 u \right|^2 + r^{-4} \left| u \right|^2 + \left| D^2 v \right|^2 + r^{-2} \left| D^1 v \right|^2 
+ r^{-4} \left| v \right|^2 + \left| D^1 p \right|^2 \right] \, dx \, dy
\leq c \int_{\Omega} \int \left[ \left| f_1 \right|^2 + \left| f_2 \right|^2 + \left| D^1 g \right|^2 + r^{-2} \left| g \right|^2 \right] \, dx \, dy. \quad (7.2)
\]

This last estimate is a sharper version of (2.9). The proof of Theorem 3 is now complete.

**Remark.** The analysis presented here differs from that of Kondrat'ev in one important detail. The solution operator of the system (4.3) has a pole on the line \( \eta = \text{Im} \xi = 1 \), which is the line to which analytic continuation is desired. (See the remark in Section 5, following Theorem 4.) Our analysis suggests the possibility of extending, in modified form, the estimates of Kondrat'ev to lines containing singularities of the solution operator.
In this appendix we discuss briefly the Fourier transform

\[(\mathcal{F}z)(\xi, \theta) = \hat{z}(\xi, \theta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} z(\tau, \theta) e^{i\xi\tau} d\tau.\]

We pattern the discussion after the development of the Fourier transform given in [20, Chap. 6]. We define \(\mathcal{F}(S)\) to be the set of rapidly decreasing functions in the strip \(S\); that is, functions \(z(\tau, \theta)\) which are infinitely differentiable in \(S\) and such that

\[\sup_{(\tau, \theta) \in S} |\tau|^{\beta} \left| \frac{\partial^{\alpha_1 + \alpha_2} z(\tau, \theta)}{\partial \tau^{\alpha_1} \partial \theta^{\alpha_2}} \right| < \infty\]

for all nonnegative integers \(\beta, \alpha_1, \alpha_2\). It may be verified by classical arguments that for \(z \in \mathcal{F}, \mathcal{F}z\) is well defined and \(\mathcal{F}z \in \mathcal{F}\). Furthermore, one has the usual formulas for \(z, w \in \mathcal{F}\):

\[(\mathcal{F}^{-1}\hat{z})(\xi, \theta) = z(\tau, \theta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{z}(\xi, \theta) e^{i\xi\tau} d\xi.\]

Using (A.1) and (A.2) we see that \(\mathcal{F}\) may be extended to an isomorphism of \(L_2(S)\) onto \(L_2(S)\). Similarly, if \(z \in H^1(S)\), then \(\hat{z}\) and \(\hat{z}\) belong to \(L_2(S)\) and \(\mathcal{F}\) gives an isomorphism of \(H^1(S)\) onto the Hilbert space of function \(\hat{z} \in L_2(S)\) such that \(\hat{z}\) has a weak derivative \(\hat{z} \in L_2(S)\) and

\[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ |(1 + |\xi|^2)| \hat{z}(\xi, \theta)|^2 + |\hat{z}\theta(\xi, \theta)|^2 \right] d\xi d\theta < \infty.\]

A similar characterization may be given for the Fourier transform of \(H^m(S), m > 0\).

We require the extension of the Fourier transform to the complex plane. Let \(z \in L_2(S)\) and suppose that \(z(\tau, \theta) = 0\) for \(\tau < 0\). Then with \(z = \xi + i\eta\), we may define the Fourier transform \(\hat{z}(\xi, \theta)\) for \(\eta \leq 0\) by

\[\hat{z}(\xi, \theta) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} z(\tau, \theta) e^{-i\xi\tau} d\tau = \mathcal{F}(e^{i\xi}z)(\xi, \theta).\]
Since $e^{\gamma z} \in L_2(S)$, $\hat{z}$ is well defined and, for each $\gamma \leq 0$, $\hat{z} \in L_2(S)$ as a function of $(\xi, \theta)$. Also, using (A.2),

$$
\int_0^\infty \int_{-\infty}^{\infty} |\hat{z}(\xi, \theta)|^2 d\xi d\theta = \int_0^\infty \int_{-\infty}^{\infty} e^{2\tau \gamma} |z(\tau, \theta)|^2 d\tau d\theta 
\leq \int_0^\infty \int_{-\infty}^{\infty} |z(\tau, \theta)|^2 d\tau d\theta,
$$

so

$$
\sup_{\gamma < 0} \int_0^\infty \int_{-\infty}^{\infty} |\hat{z}(\xi, \theta)|^2 d\xi d\theta < \infty. \quad (A.3)
$$

We remark here that $\int_0^\infty \int_{-\infty}^{\infty} |\hat{z}(\xi + i\eta, \theta)|^2 d\xi d\theta$ is increasing in $\eta$.

For $\zeta$ fixed in the lower half plane we can alternatively view $\hat{z}(\zeta, \theta)$ as a function of $\theta$ which belongs to $L_2[0, \omega]$. We now wish to consider the mapping $\zeta \mapsto \hat{z}(\zeta, \theta)$ from the lower half plane into $L_2[0, \omega]$ and note that it is analytic in $\zeta$. Analyticity for such a mapping can be characterized by requiring that

$$
\int_0^\infty \hat{z}(\zeta, \theta) g(\theta) d\theta = \int_0^\infty \left( \int_0^\infty z(\tau, \theta) e^{-i\xi \tau} d\tau \right) g(\theta) d\theta \quad (A.4)
$$

is analytic in $\zeta$ for each $g \in L_2[0, \omega]$; see, e.g., [20, p. 128]. To prove this, observe first that $e^{-i\xi \tau}z(\tau, \theta) g(\theta)$ is integrable in $\tau$, $\theta$ and $\zeta$, $\zeta$ varying over a compact subset of the lower half plane. It is easily seen that the expression in (A.4) is continuous in $\zeta$, and if $C$ is any closed curve in the lower half plane we have

$$
\int_C \left( \int_0^\infty \hat{z}(\zeta, \theta) g(\theta) d\theta \right) d\zeta 
\quad = \int_0^\infty \int_C e^{-i\xi \tau} z(\tau, \theta) g(\theta) d\tau d\theta 
\quad = 0
$$

since $e^{-i\xi \tau}$ is analytic in $\zeta$ for $\tau$ fixed. It now follows from Morera's theorem that the mapping $\zeta \mapsto \hat{z}(\zeta, \theta)$ is analytic in $\zeta$.

The converse of these statements is given by the Paley–Wiener theorem, extended to functions in a strip.

**Theorem.** Let $\hat{z}(\zeta + i\eta, \theta)$, for each fixed $\eta < 0$, belong to $L_2(S)$, and satisfy (A.3). Furthermore, let $\zeta \mapsto \hat{z}(\zeta, \theta)$ be, for $\eta < 0$, an analytic
mapping of ζ → L_2[0, ω]. Then there is a function z(τ, θ) ∈ L_2(S) with
z(τ, θ) = 0 for τ < 0, and such that ẑ(ζ, θ) = (Fz)(ζ, θ). Furthermore,

\[ \|z\|_{L^2(S)}^2 \leq \sup_{\tau < 0} \int_0^\omega \int_{-\infty}^{\infty} |ẑ(ζ, θ)|^2 dζ dθ. \]  \hspace{1cm} (A.5)

Proof. Let \( \eta_n \uparrow 0 \), \( \{ẑ(ζ + i\eta_n, θ)\} \) is a sequence in \( L_2(S) \) and by
(A.3) the sequence of norms is bounded. Hence, by the local weak
compactness of \( L_2(S) \) there is a subsequence of \( \eta_n \) (again denoted by
\( \eta_n \)) such that

\[ ẑ(ζ + i\eta_n, θ) \rightharpoonup ẑ(ζ, θ) ∈ L_2(S). \] \hspace{1cm} (A.6)

For any \( δ < 0 \),

\[ \int_{-N}^N \int_0^\omega \int_{-\infty}^0 |ẑ(ζ + i\eta, θ)|^2 dη dθ dζ \]

\[ \leq \int_0^\omega \int_{-\infty}^\infty \int_{-N}^N |ẑ(ζ + i\eta, θ)|^2 dθ dζ dη < \infty \]

for all \( N > 0 \). This implies there is a sequence \( N_k \) such that

\[ \lim_{k \to \infty} N_k = \infty, \quad \lim_{k \to \infty} \int_0^\omega \int_{-\infty}^0 |ẑ(±N_k + i\eta, θ)|^2 dη dθ = 0. \] \hspace{1cm} (A.7)

By Cauchy’s integral formula we have

\[ ẑ(ζ_0, θ) = \frac{1}{2\pi i} \oint_{C_k} \frac{ẑ(λ, θ)}{λ - ζ_0} dλ \] \hspace{1cm} (A.8)

for \( \text{Im} \ ζ_0 < 0 \), where the contour \( C_k \) is indicated in Fig. 3.

Figure 3
Letting $k \to \infty$ and using (A.7) yields

$$
\hat{z}(\zeta_0, \theta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{z}(t + i\eta_1, \theta)}{t + i\eta_1 - \zeta_0} \, dt - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{z}(t + i\eta_0, \theta)}{t + i\eta_0 - \zeta_0} \, dt. \tag{A.9}
$$

Note that here we are considering $\zeta \to \hat{z}(\zeta, \theta)$ to be an analytic mapping with values in $L_2[0, \omega]$. Thus to pass from (A.8) to (A.9) we must show the integrals

$$
\int_{\eta_1}^{\eta_0} \frac{\hat{z}(\pm N_k + it, \theta)}{\pm N_k + it - \zeta_0} \, dt
$$

converge to 0 in the norm of $L_2[0, \omega]$. This follows directly from (A.7). The first term on the right in (A.9) tends to 0 as $\eta_1 \to -\infty$ as a result of (A.3) and Schwarz's inequality. Setting $\eta_0 = \eta_n$, we thus have

$$
\hat{z}(\zeta_0, \theta) = \frac{1}{2\pi i} \int_{-\infty}^{\omega} \frac{\hat{z}(t + i\eta_n, \theta)}{\zeta_0 - (t + i\eta_n)} \, dt \, d\theta, \quad n = 1, 2, \ldots.
$$

Now since $\hat{z}(t + i\eta_n, \theta) \to \hat{z}(t, \theta)$ in $L_2(S)$, we have for each $g \in L_2[0, \omega]$,

$$
\int_0^{\omega} \hat{z}(\zeta_0, \theta) g(\theta) \, d\theta = \frac{1}{2\pi i} \int_{-\infty}^{\omega} \int_{-\infty}^{\infty} \frac{\hat{z}(t + i\eta_n, \theta) g(\theta)}{\zeta_0 - (t + i\eta_n)} \, dt \, d\theta
$$

and thus,

$$
\int_0^{\omega} \hat{z}(\zeta_0, \theta) g(\theta) \, d\theta = \frac{1}{2\pi i} \int_0^{\omega} \left( \int_{-\infty}^{\infty} \frac{\hat{z}(t, \theta)}{\zeta_0 - t} \, dt \right) g(\theta) \, d\theta.
$$

Hence,

$$
\hat{z}(\zeta_0, \theta) = \frac{1}{2\pi i} \int_{-\infty}^{\omega} \frac{\hat{z}(\xi, \theta)}{\zeta_0 - \xi} \, d\xi, \quad \text{Im} \, \zeta_0 < 0. \tag{A.10}
$$

We now introduce the inverse Fourier transform of $(\zeta_0 - \xi)^{-1}$:

$$
\frac{1}{2\pi^{1/2}} \int_{-\infty}^{\infty} e^{i\xi \tau} \frac{1}{\zeta_0 - \xi} \, d\xi = 0, \quad \tau > 0
$$

$$
= i(2\pi)^{1/2} e^{i\zeta_0 \tau}, \quad \tau < 0, \quad \text{Im} \, \zeta_0 < 0. \tag{A.11}
$$
Using (A.10), (A.2), and (A.11), we then have
\[
\hat{z}(\xi_0, \theta) = \frac{1}{2\pi^{1/2}} \int_{-\infty}^{\infty} z(\tau, \theta) e^{-i\xi_0 \tau} d\tau, \quad \text{Im} \xi_0 < 0,
\]
where \( z(\tau, \theta) \in L_2(S) \) is the inverse Fourier transform of \( \hat{z}(\xi, \theta) \). Hence, \( \hat{z} = \mathcal{F}(z\psi) \), where \( \psi \) is the characteristic function of the interval \((0, \infty)\). By the uniqueness of the Fourier transform, \( z = z\psi \), so \( z = 0 \) for \( \tau < 0 \).

To prove the inequality asserted in the theorem, we deduce from Parseval's equation and the weak convergence in (A.6),
\[
\| z \|^2_{L_2(S)} = \| \hat{z} \|^2_{L_2(S)} \\
\leq \liminf_{n \to \infty} \int_{-\infty}^{\infty} |\hat{z}(\xi + i\eta_n, \theta)|^2 d\xi d\theta \\
\leq \sup_{\eta < 0} \int_{-\infty}^{\infty} |\hat{z}(\xi, \theta)|^2 d\xi d\theta.
\]

**APPENDIX B. AN AGMON–NIRENBERG TYPE INEQUALITY**

In this section we shall establish bounds for the solution of (4.3) that are uniform in \( \xi = \xi + iq \), provided \( \eta \) is bounded and \( |\xi| \) is sufficiently large. Such bounds are established in [2] and [3] for 2mth order elliptic problems. The method of proof given here is the same as in [2], and is presented for the sake of completeness.

Specifically we shall prove the

**Lemma.** Let \( \eta_0 > 0 \) be given. Then there are constants \( c > 0 \) and \( \xi_0 > 0 \), depending on \( \eta_0 \) and \( \omega \), such that if \( \xi = \xi + iq \) with \( |\eta| < \eta_0 \), \( |\xi| > \xi_0 \), and if \( \hat{U}, \hat{V} \in H^2[0, \omega] \), \( \hat{U} = \hat{V} = 0 \) at \( \theta = 0, \theta = \omega \), \( \hat{q}(\theta) \in H^1[0, \omega] \), then, setting
\[
-\hat{U}'' + (\xi^2 + 1) \hat{U} + 2\hat{V}' - (i\xi + 1)\hat{q} = h_1(\theta), \quad 0 < \theta < \omega,
-\hat{V}'' + (\xi^2 + 1) \hat{V} - 2\hat{U}' - \hat{q}' = h_2(\theta), \quad 0 < \theta < \omega,
(1 - i\xi) \hat{U} + \hat{V}' = h_3(\theta), \quad 0 < \theta < \omega, \quad (B.1)
\]
we have
\[
\int_{0}^{\omega} \left[ |\hat{U}|^2 + |\xi|^2 |\hat{U}'|^2 + |\xi|^4 |\hat{U}|^2 + |\hat{V}|^2 + |\xi|^2 |\hat{V}'|^2 \right] d\theta \\
+ \left[ |\xi|^4 |\hat{V}|^2 + |\hat{q}'|^2 + |\xi|^2 |\hat{q}''|^2 \right] d\theta \\
\leq c \int_{0}^{\omega} \left[ |h_1|^2 + |h_2|^2 + |h_3|^2 + |\xi|^2 |h_3|^2 \right] d\theta. \quad (B.2)
\]
Proof. Let $\mu(t)$ be a smooth function, identically 1 for $|t| < 1$, and identically zero for $|t| > 2$. Using (B.1), the functions

\begin{align*}
U^*(\theta, t) &= \tilde{U}(\theta) \mu(t) e^{it\theta}, \\
V^*(\theta, t) &= \tilde{V}(\theta) \mu(t) e^{it\theta}, \\
q^*(\theta, t) &= \tilde{q}(\theta) \mu(t) e^{it\theta}
\end{align*}

are easily seen to satisfy the system of partial differential equations

\begin{align}
\begin{cases}
-U_{tt}^* - U_{\theta\theta}^* + U^* + 2V^* - q_t^* - q^* = h_1^*, \\
-V_{tt}^* - V_{\theta\theta}^* + V^* - 2U^* + q_\theta^* = h_2^*, \\
-U_t^* + U^* - V_\theta^* = h_3^*,
\end{cases}
\end{align}

where

\begin{align}
\begin{align*}
h_1^* &= \mu e^{it\theta} h_1 - 2i\xi\mu' \tilde{U} e^{it\theta} - \mu'' \tilde{U} e^{it\theta} - \mu' \tilde{q} e^{it\theta}, \\
h_2^* &= \mu h_2 e^{it\theta} - \mu'' \tilde{V} e^{it\theta} - 2i\xi\mu' \tilde{V} e^{it\theta}, \\
h_3^* &= \mu h_3 e^{it\theta} - \mu' \tilde{U} e^{it\theta}.
\end{align*}
\end{align}

Since $U^* = V^* = 0$ for $\theta = 0, \theta = \omega$, and for $|t| > 2$, we may pick a bounded domain $D^*$ with smooth boundary $\partial D^*$ such that $D^*$ contains the rectangle $Q^*$ defined by the inequalities $|t| < 1, 0 < \theta < \omega$, and such that $U^* = V^* = 0$ on $\partial D^*$. (B.5)

It may be verified that the system (B.3) is elliptic in the sense of Agmon–Douglas–Nirenberg [1], and that the boundary conditions (B.5) are complementing [1]. (In fact, this boundary value problem, which is identical with the problem (4.1), may be obtained from the generalized Stokes problem by a change of variables. The ellipticity of the Stokes problem then gives the ellipticity of (B.3).) Hence, from the a priori estimate for elliptic systems [1], there is a constant $c > 0$ depending only on $\omega$ such that

\begin{align}
\| U^* \|_{L^2(D^*)} + \| V^* \|_{L^2(D^*)} + \| q^* \|_{H^1(D^*)} 
\leq c \left( \| h_1^* \|_{L^2(D^*)} + \| h_2^* \|_{L^2(D^*)} + \| h_3^* \|_{H^1(D^*)} \right. \\
\left. + \| U^* \|_{L^2(D^*)} + \| V^* \|_{L^2(D^*)} + \| q^* \|_{L^2(D^*)} \right].
\end{align}

Taking the integrals in the norms on the left over the rectangle $Q^*$, and
using the boundedness of $\eta$, it is easily seen that the left side of (B.6) is bounded from below by
\[
\begin{align*}
c_1(\eta_0) & \left( \| \hat{U}'' \|_{L_2(0,\omega)} + | \zeta | \| \hat{V}' \|_{L_2(0,\omega)} + | \zeta |^2 \| \hat{V} \|_{L_2(0,\omega)} \\
& \quad + \| \hat{V}'' \|_{L_2(0,\omega)} + | \zeta | \| \hat{V}' \|_{L_2(0,\omega)} + | \zeta |^2 \| \hat{V} \|_{L_2(0,\omega)} \right)
\end{align*}
\]
where $c_1(\eta_0) > 0$. Using (B.4) and the boundedness of $\eta$, we see that the right side of (B.6) is bounded from above by
\[
\begin{align*}
c_2(\eta_0) & \left( \| \hat{U} \|_{L_2(0,\omega)} + | \zeta | \| \hat{V} \|_{L_2(0,\omega)} \right.
\end{align*}
\]
Combining these inequalities, it is easily seen that if $| \xi |$ is large enough, we obtain (B.2).

**References**


