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# On the $\mu$-calculus over transitive and finite transitive frames

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ABSTRACT

We prove that the modal $\mu$-calculus collapses to first order logic over the class of finite transitive frames. The proof is obtained by using some byproducts of a new proof of the collapse of the $\mu$-calculus to the alternation free fragment over the class of transitive frames.

Moreover, we prove that the modal $\mu$-calculus is Büchi and co-Büchi definable over the class of all models where, in a strongly connected component, vertexes are distinguishable by means of the propositions they satisfy.



## 1. Introduction

The modal $\mu$-calculus, introduced by Kozen in [9], is a powerful logic widely used in the area of specification and verification of computer systems, be it hardware or software; see [5]. This logic is obtained from modal logic by adding two operators $\mu$ and $\nu$ for the least and greatest fixed points of monotone operators on sets. Via Kripke semantics, the $\mu$-calculus can be used to express properties of graphs. Intuitively, least fixed points correspond to inductive definitions, and greatest fixed points correspond to coinductive definitions. For instance, with least fixed points one can express global liveness properties of a graph like "$P$ is true at some reachable point", and with greatest fixed points one expresses global safety properties of the kind "$P$ is true in all reachable points". These properties are not modally expressible (at least on arbitrary graphs) due to the local character of modal logic. Next, fixed points can be nested, and by one nesting of least and greatest fixed points we capture fairness properties like "$P$ holds infinitely often". Finally, with several nestings, one can express the existence of a winning strategy in a parity game. It turns out that, on arbitrary graphs, the number of nestings between different fixed points gives a strict infinite hierarchy; see [10,4,3].

The situation may change if one considers special subclasses of frames. Probably the most studied subclass so far, from the seventies on, is the class of transitive well-founded frames (also called the Gödel–Löb class or GL), in view of its relation with Gödel theorems and the logic of provability in Peano Arithmetic. However, the $\mu$-calculus is not very expressive inside this class, since from the de Jongh–Sambin fixed point theorem it follows that the $\mu$-calculus in GL collapses to modal logic.

In this paper, we are interested in the expressiveness of the $\mu$-calculus in the class of transitive frames and in the class of finite transitive frames. These classes are important also for applications: many natural frames are transitive. For instance, in temporal reasoning, the relation "A is posterior to B" is transitive.

A first question one can ask is whether the fixed point hierarchy collapses on transitive frames. The answer is affirmative: this result was first proved in [2] where it is shown that on transitive frames, the $\mu$-calculus collapses to its alternation free

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fragment, that is, to formulas where no real nesting of different fixed points occurs. However, by a result of Visser, the $\mu$-calculus does not collapse to modal logic on transitive (and even transitive and reflexive) frames. The non-modal example of Visser is the property stating the existence of an infinite path alternately labeled with $P$, $\neg P$, $P$, $\neg P$, etc. where $P$ is an atomic proposition.

Notice that both results, the collapse to the alternation free fragment, and the non-collapse to modal logic, extend to finite transitive frames: the first follows by restriction, and the second follows by the finite model property of the $\mu$-calculus on transitive frames. However, if we consider first order logic instead of modal logic, we see that finite transitive frames and transitive frames disagree. Consider for example the formula $F = \nu x \Diamond (x \wedge P)$, which says that there is a path starting from the root where, after the first step, $P$ always holds. Finite transitive frames and transitive frames agree on the fact that $F$ is not modal; however, on finite transitive frames, $F$ is equivalent to the first order formula $\exists y(xRy \wedge yRy \wedge P(y))$, where $R$ represents the accessibility relation on Kripke models, while the formula $F$ is not equivalent to any first order formula over transitive frames. As we shall see, this is an indication of a general pattern.

Coming back to transitive frames, in this paper we propose a new proof of the collapse of the $\mu$-calculus to the alternation free language, which relies on the Alberucci–Facchini Lemma and on an equally elementary result saying that on transitive frames we also have the property: $\Diamond \mu x \phi = \Diamond \phi(\bot)$.

Moreover, the new proof allows us to recognize that formulas without least fixed points are always equivalent, over transitive frames, to formulas where we never encounter a $\Box$ in the path from a declaration to a variable. This is the heart of the proof of our second result, which says that on finite transitive frames the $\mu$-calculus is included in first order logic, a result which also follows from a characterization of the bisimulation invariant fragment of monadic second order logic proved by Otto and Dawar [12].

The results in this paper can be extended to a broader investigation of the class of transitive frames and of the class of finite transitive frames, as well as of other classes. One may ask to what extent these classes are similar to the class of all graphs, or all finite graphs, etc. In this vein, in the final section of this paper we consider the class of finite simple graphs, a class containing (modulo bisimulation) the class of finite transitive frames, and prove that over this class the $\mu$-calculus is contained in the level $\Sigma_2 \cap \Pi_2$ of the alternation hierarchy (a tentative proof of this result already appeared in [11], but the proof contained a mistake). Notice that $\Sigma_2 \cap \Pi_2$ coincides with the alternation free fragment of the $\mu$-calculus over the class of all frames, but this does not imply that the same must hold over smaller classes. In fact, in [7] it is proved that $\Sigma_2 \cap \Pi_2$ is more expressive than the alternation free fragment on simple graphs.

## 2. Preliminaries

### 2.1. Syntax

In the alphabet of $\mu$-formulas we distinguish between *propositions* $P_1, P_2, \ldots$ and *variables* $x_1, x_2, \ldots$. $\mu$-formulas are obtained from propositions, negated propositions, and variables (also called *literals*) using disjunctions, conjunctions, modalities ($\Box$ and $\Diamond$) and fixed point operators $\mu xF$, $\nu xF$, where $x$ is a *variable*.

For a finite set of formulas $\Gamma$, we also use the abbreviation

$$\Diamond(\Gamma) := \bigwedge_{G \in \Gamma} \Diamond G.$$

We write $\sigma xF$ for one of the formulas $\mu xF$, $\nu xF$, and $F \leq G$ if $F$ is a subformula of $G$ (where the subformulas of $\sigma xF$ are $\sigma xF$ and all the subformulas of $F$).

Free and bound variables are defined as usual.

If $F(x)$ and $A$ are formulas, we define $F[x|A]$ (also denoted by $F(A)$) as the formula obtained from $F$ by the simultaneous substitution of the free occurrences of the variable $x$ with $A$. Likewise one defines a simultaneous substitution of variables with formulas and denotes it by $F(A_1..A_n)$.

**Definition 2.1.** The *fixed point alternation-depth hierarchy* of the $\mu$-calculus is the sequence $\Sigma_0 = \Pi_0, \Sigma_1, \Pi_1, \ldots$ of sets of $\mu$-formulas defined inductively as follows.

1. $\Sigma_0 = \Pi_0$ is defined as the set of all modal fixed point free formulas.
2. $\Sigma_{k+1}$ is the closure of $\Sigma_k \cup \Pi_k$ under:
   - Compositions without capture: if $F(x_1, \ldots x_n), F_1, \ldots, F_n$ are in $\Sigma_{k+1}$, and $x_1, \ldots, x_n$ are variables, then $F(F_1, \ldots, F_n)$ is in $\Sigma_{k+1}$, provided no occurrence of a variable which was free in one of the $F_i$ becomes bound in $F(F_1, \ldots, F_n)$;
   - Least fixed points: if $F$ is in $\Sigma_{k+1}$, then $\mu x.F \in \Sigma_{k+1}$.
3. Likewise, $\Pi_{k+1}$ is the closure of $\Sigma_k \cup \Pi_k$ under composition without capture and the $\nu$-operator.

In this paper we are particularly interested in formulas of classes $\Pi_1$ and $\Pi_2$, also called $\nu$-formulas and $\nu\mu$-formulas, respectively. We also call *alternation free* a formula obtained by composition without capture of formulas in $\Sigma_1 \cup \Pi_1$.

### 2.2. Semantics

Like modal logic, the $\mu$-calculus can be given a Kripke semantics. A *Kripke model* is a tuple $M = (W, R, r, L)$ where W is a set, R is a binary relation on W, r is an element of W, and

$$L : \{P_1, P_2, \ldots\} \cup \{x_1, x_2, \ldots\} \to Powerset(W)$$

interprets propositions and variables as subsets of $W$.

Truth of a formula $F$ in a model $M$ is defined by induction on $F$. The atomic, boolean and modal cases are defined as usual, and the two fixed point clauses are as follows:

- $\mu x.F(x)$ is true in $M$ if $r$ belongs to the least fixed point of the equation $x = F(x)$;
- likewise, $\nu x.F(x)$ is true in $M$ if $r$ belongs to the greatest fixed point of the equation $x = F(x)$.

In the following, we shall use the fact that positivity implies monotonicity in the $\mu$-calculus:

**Fact 2.2.** *Suppose $A(x)$ is positive in $x$ and the implication $G \to H$ holds in all points of a model $M$. Then, $A(G) \to A(H)$ holds in the root of the model.*

Sometimes one considers also *graphs*, which are pairs $(W, R)$ as above, or *frames*, which are triples $(W, R, r)$ as above. A frame or graph will be called *transitive* if $R$ is transitive, and will be called *finite* if $W$ is finite.

## 3. From $L_\mu$ to the alternation free fragment

In this section we prove:

**Theorem 3.1.** *On transitive frames, every formula of the $\mu$-calculus is equivalent to a formula of the alternation free fragment.*

### 3.1. Well named formulas

A variable $x$ is *guarded* in $F$ if every occurrence of $x$ in $F$ is under the scope of a modality. A formula $F$ is *guarded* if for every subformula of type $\mu xG$ or $\nu xG$ in $F$, $x$ is guarded in $G$. One can easily prove that every formula is equivalent to a guarded formula. A formula is said to be *well bound* if any bound variable has a unique occurrence and a unique declaration, and it is *well named* if it is both well bound and guarded. There is an easy inductive algorithm $wn$ that, given a formula $F$, calculates an equivalent well named formula $wn(F)$ using the following equivalences (under the standard conventions about free and bound variables):

$$\nu xF(x) = \nu yF(y), \qquad \mu xF(x) = \mu yF(y), \tag{1}$$

$$\nu xF(x, x) = \nu x\nu x^* F(x, x^*), \qquad \mu xF(x, x) = \mu x\mu x^* F(x, x^*). \tag{2}$$

The equations in (1), (2) are used to assure the uniqueness of declarations and the uniqueness of variable occurrences, respectively.

A property of well named formulas that we will use many times in the sequel is the following: if $\sigma yA$ is a subformula of $F$ (with $\sigma \in \{\nu, \mu\}$) and $B$ is a subformula of $F$ containing a free occurrence of the variable $y$, then $B \le A$.

Given a formula $F$, we consider the *syntactic* tree $tree(F)$, defined as follows:

1. its nodes are labeled by the subformulas of $F$;
2. the root is labeled by $F$;
3. there is an edge between a node labeled by $G$ and a node labeled by $H$ if and only if $H$ is an immediate subformula of $G$.

Suppose $F$ is well named: although different nodes may be labeled by the same formula, this cannot happen with formulas starting with a fixed point, since we suppose that bound variables have a unique declaration. Hence, given a subformula of type $\sigma xA$ we may identify this formula with a node in $tree(F)$. For example, we shall speak of *the* path between a $\nu$-declaration and a $\mu$-declaration, or of *the* path between a declaration $\nu xA$ and its variable $x$.

### 3.2. Conventions, first results, and plan of the proof of Theorem 3.1

We shall use the following facts:

**Fact 3.2.** *$F \in \nu\mu$ if and only if $F$ does not contain any pair of subformulas $\mu xA \ge \nu yB$ with $x$ free in $B$.*

**Fact 3.3.** *$F$ is alternation free if and only if $F$ does not contain any pair of subformulas $\mu xA \ge \nu yB$ or $\nu xA \ge \mu yB$ with $x$ free in $B$.*

The easy verification of the following lemma is left to the reader.

**Lemma 3.4.** *Suppose $A(x)$, $B$ are $\nu\mu$ well named formulas with the following property: there is no free variable $z$ in $B$ such that, in $A$, the path from the root to $x$ contains a node labeled by $\mu zC$, for some $C$; then $A[x|B]$ is $\nu\mu$.*

**Definition 3.5.** A bound variable $x$ having a unique occurrence and a unique declaration in a formula $F$ is said to be

- *existential* if the path between the declaration $\sigma x$ and the variable $x$ contains no nodes labeled by a formula of the form $\Box A$;
- *weakly universal*, otherwise, i.e. if the path between the declaration $\sigma x$ and the variable $x$ contains at least one node labeled by a formula of the form $\Box A$.

A free variable $x$ having a unique guarded occurrence in a formula $A$ is said to be existential (weakly universal) if and only if $x$ is existential (weakly universal) in $\nu x A$.

Given a well named formula $F$ we assign the sign $+$ or $-$ to greatest fixed point declarations according to whether their variable is existential or weakly universal. For example,

$$\nu x^- \Box \mu z (\nu y^+ \Diamond (y \vee \Box z)) \vee \Diamond x).$$

To prove the collapse of the $\mu$-calculus over transitive frames we use the following two lemmas, which allow us to eliminate some kind of fixed points.

**Lemma 3.6.** *Let $F(x)$ be positive in $x$. Over transitive frames it holds*

$$\Diamond(\mu x F) \equiv \Diamond F(\bot), \quad \textit{and, dually,} \quad \Box(\nu x F) \equiv \Box F(\top),$$

*where $\bot$ is identically false, and $\top$ is identically true.*

**Lemma 3.7** (*Alberucci–Facchini [2]*). [1] *Suppose the formula $\nu x A$ is well named and $x$ is weakly universal in it. Then, over transitive frames, it holds*

$$\nu x^- A(x) \equiv A(A(\top)).$$

We first notice that the proofs of the two lemmas are facilitated by the observation that the $\mu$-calculus enjoys the finite model property over transitive frames: consider the formula $F^*$ obtained from $F$ by replacing all $\Box$'s and $\Diamond$'s with $\Box^*$ and $\Diamond^*$, where $\Diamond^* P = \mu x \Diamond (P \vee x)$, and $\Box^* P = \nu x \Box (P \wedge x)$: then, if $M$ is a model and $M^*$ is like $M$ except that the accessibility relation is the transitive closure of the one in $M$, we have

$$M \models F^* \Leftrightarrow M^* \models F$$

(for details, see [6]). Using the equivalence above it is not difficult to see that the finite model property for $\mu$-formulas over transitive frames follows from the finite model property for $\mu$-formulas over the class of all frames.

**Proof of Lemma 3.6.** By the finite model property of the $\mu$-calculus over transitive frames, it suffices to prove (one) of the above equivalences over finite transitive frames. We prove the first equivalence. Since over finite frames the least fixed point is always reached after a finite number of iterations, the lemma is proved if we are able to show that, for each natural number $n$, the following is a valid implication:

$$\Diamond(F^n) \to \Diamond F(\bot),$$

where $F^0 =: \bot$, and $F^{n+1} = F(F^n)$. We proceed by induction on $n$. If $n = 0$, the implication holds trivially. Suppose the implication is valid for $n$. Consider a model $M = (W, R, r, L)$, and suppose, without loss of generality, that all points in $M$ are reachable from the root $r$. We consider two cases:

1. the set $\{v \in W : rRv\} \cap \{v \in W : M, v \models F^n\} = \emptyset$;
2. the set $\{v \in W : rRv\} \cap \{v \in W : M, v \models F^n\} \neq \emptyset$.

In the first case, we first show that $M, v \models F(F^n) \to F(\bot)$, for all points $v$ with $rRv$. This is true because, since $R$ is transitive, no point reachable from $v$ verifies $F^n$; hence, from Fact 2.2 applied to the model $M, v, A(x) := F(x)$, $G := F^n$, and $H := \bot$, we obtain $M, v \models F(F^n) \to F(\bot)$. Suppose now that $M, r \models \Diamond(F^{n+1})$; then there is a $v$ with $rRv$ and $M, v \models F(F^n)$; from the above we get $M, v \models F(\bot)$ and hence $M, r \models \Diamond F(\bot)$.

If the second case applies, then let $v$ be an $R$-successor of $r$ such that $M, v \models F^n$ holds. Then $M, r \models \Diamond(F^n)$ and, by induction, $M, r \models \Diamond F(\bot)$. $\Box$

As for Lemma 3.7, we obtain it as a corollary of a stronger result:

**Lemma 3.8.** *Consider a greatest fixed point, well named formula of the form $\nu x B(\Box(F(x)))$: then, over transitive frames,*

$$\nu x B(\Box(F(x))) \equiv B(\Box(F(B(\top)))).$$

**Proof.** First we notice that if we have a fixed point formula $D$ of type $D = \nu x C(F(x))$, then $F(D)$ is equivalent to $\nu y\, F(C(y))$. This can be proved as follows. Since $D$ is equivalent to $C(F(D))$, we have that $F(D)$ is equivalent to $F(C(F(D)))$; hence, $F(D)$

[1] A similar result can be found in the proof of the de Jongh–Sambin theorem; see Corollary 1.6 (iii) of [13]. However, the context here is different because $\mu$-formulas over transitive frames are considered instead than modal formulas over transitive and well-founded frames.

is a fixed point for $F(C(y))$, from which it follows

$$F(D) \rightarrow \nu y F(C(y)).$$

To prove the inverse implication, notice that $\nu y F(C(y))$ is equivalent to $F(C(\nu y F(C(y))))$. Therefore, to show that it implies $F(D)$, it is enough (by monotonicity of $F$) to show that

$$C(\nu y F(C(y))) \rightarrow D.$$

Since $D = \nu x C(F(x))$ this has already been showed in the first part of the proof (with $F$ and $C$ interchanged).

To prove the Lemma, consider the formula $D = \nu x B(\Box(Fx))$, and write it as $D = \nu x C(F(x))$, for $C = B(\Box(y))$.

Using the equivalence between $F(D)$ and $\nu y F(C(y))$ and Lemma 3.6 we obtain:

$$\begin{aligned} D &\equiv C(F(D)) = B(\Box(F(D)) \equiv B(\Box(\nu y F(C(y)))) \\ &\equiv B(\Box(F(C(\top)))) \equiv B(\Box(F(B(\Box(\top)))) = B(\Box(F(B(\top)))). \quad \Box \end{aligned}$$

We are now able to prove Lemma 3.7:

**Proof of Lemma 3.7.** Since $x$ is weakly universal in $A$, we may write the formula $A$ as $A = B(\Box(F(x)))$. To show that $\nu x^{-} A(x) \equiv A(A(\top))$ it is then enough to prove that $A(A(\top)) \rightarrow \nu x A$ (since the other implication is always valid). The formula $A(A(\top))$ is equal to $B(\Box(F(B(\Box(F(\top))))))$, and, since $B(\Box(F(B(x))))$ is positive in $x$, we have that $B(\Box(F(B(\Box(F(\top))))))$ implies $B(\Box(F(B(\top))))$ which is equivalent to $\nu x A$ by Lemma 3.8. $\Box$

The plan of the proof of Theorem 3.1, which uses as basic ingredient Lemmas 3.6 and 3.7, is as follows.

**Plan of the proof of Theorem 3.1.**

1. As we shall see, our final proof requires formulas where there is a limited use of conjunctions. In order to reduce the use of conjunctions, we convert a $\nu\mu$-formula into a *bidisjunctive normal form*, which is very close to the disjunctive normal form of [8].
2. We call *special* a $\nu\mu$-formula in bidisjunctive normal form such that in any path between a $\nu$-declaration $\nu x A$ and a $\mu$-declaration $\mu y B$ there is at least a node labeled by a modality. In this step we transform the formula obtained in step 1 into a special one.
3. We eliminate all weakly universal greatest fixed point declarations: in the resulting formula, which is still special, all $\nu$-declarations are existential[2]
4. Finally, we use Lemma 3.6 to eliminate all $\nu\mu$-nested declarations from the formula obtained in the previous steps. $\Box$

### *3.3. Step 1*

Here we transform the formula in bidisjunctive normal form (see Definition 3.11). First of all, we recall some definitions.

**Definition 3.9** (*[8]*)**.** The class of *disjunctive $\mu$-formulas* is the least class containing literals (i.e. propositions, negated propositions, and variables) and which is closed under:

1. disjunctions;
2. fixed point operators: if $F$ is disjunctive and the variable $x$ does not appear in a context $x \wedge G$ for some $G$ then $\mu x F$, $\nu x F$ are in the class;
3. special conjunctions: if $\Theta$ is a (possibly empty) finite set of disjunctive formulas and $\sigma$ is a conjunction of literals, then $\sigma \wedge \textsc{Cover}(\Theta)$ is in the class, where the meaning of the Cover operator is given by the definition:

$$\textsc{Cover}(\Theta) := \left( \bigwedge_{F \in \Theta} \Diamond F \right) \wedge \Box \left( \bigvee_{F \in \Theta} F \right).$$

In other words, in disjunctive formulas we may have only a restricted use of conjunctions, the one which appears in the Cover operator. Disjunctive formulas are representative of the whole $\mu$-calculus:

**Theorem 3.10** (*[8]*)**.** *Any $\mu$-calculus formula is equivalent to a disjunctive guarded formula. Moreover, any $\nu\mu$-formula is equivalent to a $\nu\mu$-disjunctive guarded formula.*

In order to apply Lemma 3.7 we shall need to work with formulas in which each bound variable has a unique occurrence and a unique declaration. This request does not fit very well with the use of the cover operators, since e.g. in $\nu x \textsc{Cover}(x) = \nu x(\Diamond(x) \wedge \Box(x))$ the bound variable $x$ appears twice. To solve this problem we introduce a new cover operator which now

2 Alberucci and Facchini already proved in [2] that any formula is equivalent to a formula in which all greatest fixed point declarations are weakly existential (that is, the path between the declaration $\sigma x$ and the variable $x$ contains at least a node labeled by a formula of the form $\Diamond A$); however, our proof needs a stronger result.

works on two sets of formulas:

$$\textsc{Cover}(\Gamma; \Delta) := \left(\bigwedge_{F\in\Gamma} \Diamond F\right) \wedge \Box \left(\bigvee_{G\in\Delta} G\right).$$

If $\Gamma = \{F_1, \dots, F_n\}$ and $\Delta = \{G_1, \dots, G_m\}$ we denote the formula $\textsc{Cover}(\Gamma; \Delta)$ also with $\textsc{Cover}(F_1, \dots, F_n; G_1, \dots, G_m)$. We then change Definition 3.9 as follows.

**Definition 3.11.** The class of *bidisjunctive $\mu$-formulas* is the least class containing literals which is closed under:

1. disjunctions;
2. fixed point operators: if $F$ is bidisjunctive and the variable $x$ does not appear in a context $x \wedge G$ for some $G$ then $\mu xF$, $\nu xF$ are in the class;
3. given two finite sets of (possibly empty) disjunctive formulas $\Gamma, \Delta$ and a conjunction $\sigma$ of literals, the formula $\sigma \wedge \textsc{Cover}(\Gamma; \Delta)$ is in the class,

**Lemma 3.12.** *Any $\nu\mu$-formula $F$ is equivalent to a well named $\nu\mu$-bidisjunctive guarded formula.*

**Proof.** This can be done by starting from a $\nu\mu$-disjunctive guarded formula equivalent to $F$ and transforming it using the equivalences (1), (2) for well naming a formula and the new cover operators: e.g. the formula $\nu x \mu y \textsc{Cover}(x, x \vee y)$ becomes

$$\nu x\ \nu z\ \nu w\ \nu u\ \mu y\ \mu s\ \textsc{Cover}(\{x, z \vee y)\}; \{w, u \vee s\}). \quad \Box$$

We say that a formula is a *cover formula* if it is of the form $\sigma \wedge \textsc{Cover}(\Gamma; \Delta)$. When we consider bidisjunctive formulas in the following, we shall use the Cover operator as a basic operator instead of the $\Box$ and $\Diamond$ operators. This implies e.g. that in the syntactic tree $t(F)$, the children of a node labeled by $\textsc{Cover}(\Gamma; \Delta)$ are labeled by the formulas in $\Gamma \cup \Delta$.

### *3.4. Step 2*

In this step we show how to transform a $\nu\mu$-formula $F$ into an equivalent formula with the following property: in the path $\pi$ between a $\nu$-declaration $\nu xA$ and a $\mu$-declaration $\mu yB$ there is at least a node labeled by a Cover; moreover, if the formula $F$ is a well named $\nu\mu$-bidisjunctive guarded formula, then the same holds for the new formula.

In the following definition we suppose that any bound variable has a unique declaration in the formula $F$ (although we do not yet insist on the fact that it has a unique occurrence). In this case we say that $F$ has *unique declarations*.

**Definition 3.13.** A pair $(\nu x, \mu y)$ is a *bad pair* in a formula $F$ if there are $F$-subformulas $\nu xA$ and $\mu yB$ such that $\mu yB$ is a subformula of $A$, and in the tree *tree*$(F)$ the path between $\nu xA$ and $\mu yB$ contains no nodes labeled by a modality (or a cover formula, if we use the operator Cover as a basic operator).

Notice that in the above definition we do not require $x$ to be free in $B$.

**Example 3.14.**

$$F = \nu x.\mu y.\Box x \vee \Diamond(\mu u.\Box(u \vee y))$$

contains only one bad pair: $(\nu x, \mu y)$ with $B = \Box x \vee \Diamond(\mu u.\Box(u \vee y))$.

We want to prove that any $\nu\mu$-formula is equivalent to a $\nu\mu$-formula that does not contain any bad pair. If we let

$$BP(F) = |\{(\nu x, \mu y) :\ (\nu x, \mu y) \text{ is a bad pair in } F\}|,$$

our goal is to find a formula $G$ equivalent to $F$ with $BP(G) = 0$.

**Notation 3.15.** *Let us fix a notation which will be repeatedly used in the following: given a well named formula $A$, we denote by $A'$ the formula obtained from $A$ by renaming all bound variables $u$ and their declarations with $u'$ (which is supposed to be a fresh variable different from all variables already considered).*

**Lemma 3.16.** *Every guarded $\nu\mu$-formula $F$ with unique declarations and with $BP(F) > 0$ is equivalent to a guarded $\nu\mu$-formula $G$ with unique declarations and $BP(G) < BP(F)$. Moreover, if $F$ is a $\nu\mu$-bidisjunctive guarded formula, then the same holds for the formula $G$.*

**Proof.** Consider a bad pair $(\nu x, \mu y)$ in $F$: there is a subformula $\nu xA$ of $F$, containing a formula $\mu yB$ as a subformula and no covers in between. Without loss of generality, we may choose a bad pair in which $B$ contains no bad pairs. Let $D(z)$ be such that

$$F = D[z|\mu yB].$$

We consider the formula $G := D[z|B[y|(\mu yB)']]$ obtained from $F$ by substituting the subformula $\mu yB$ with the equivalent one $B[y|(\mu yB)']$. Notice that, thanks to the use of $(\mu yB)'$, the obtained formula has unique declarations.

In Example 3.14 we have $D(z) = \nu x.z$,

$$G = D[z|B[y|(\mu yB)']] = \nu x.\Box x \vee \Diamond(\mu u.\Box(u \vee (\mu y'.\Box x \vee \Diamond(\mu u'.\Box(u' \vee y'))))).$$

Notice that the original bad pair $(\nu x, \mu y)$ in $F$ has been replaced in $G$ by the pair $(\nu x, \mu y')$, which is not bad, because there is now a modality between $\nu x$ and $\mu y'$ in $G$.

Returning to the general case, it is clear that the formula $G$ is equivalent to $F$, since $\mu yB$ is equivalent to $B[y|(\mu yB)']$. We claim that $BP(G) < BP(F)$.

As in the example, the original bad pair has been replaced by new pairs $(\nu x, \mu y')$, which is not bad any more: since the variable $y$ was guarded in $B$, there is now a modality between copies $\nu x$ and the new declaration $\mu y'$. More precisely, we show that no new bad pairs have been created in $G$. This can be proved as follows. If $(\nu z, \mu u)$ is a bad pair in $G$ but it is not a bad pair in $F$, then, since $B$ contains no bad pairs, there must be a subformula $\nu zE$ in $B$, containing the $\mu$-variable $y$ free, and this is impossible since $F$ is a $\nu\mu$-formula.

We leave it to the reader to verify that $G$ is a bidisjunctive guarded formula, if $F$ is so. We have just to prove that $D[z|B[y|(\mu yB)']]$ is a $\nu\mu$-formula. This can be done by using Fact 3.2: suppose by contradiction that $\mu uM \geq \nu vN$ are subformulas of $D[z|B[y|(\mu yB)']]$ with $u$ free in $N$. We consider the following cases:

1. $D$ contains subformulas $\mu uC \geq \nu vE$, $\mu uC[y|\mu yB]$ and $\nu vE[y|\mu yB]$ are subformulas of $F$, $M = C[y|B[y|(\mu yB)']]$ and $N = E[y|B[y|(\mu yB)']$. Since $D$ is $\nu\mu$, the variable $u$ cannot be free in $E$. The fact that $u$ is free in $N$ implies then that $u$ must be free in $B[y|(\mu yB)']$ and hence in $\mu yB$. But then u is free in $\nu vE[y|\mu yB]$ which is impossible since $\mu uC[y|\mu yB] \geq \nu vE[y|\mu yB]$ are subformulas of $F$ which is $\nu\mu$.
2. $B$ contains a subformula $\nu vE$ and $\nu vN = \nu vE[y|(\mu yB)']$. The variable $u$ is free in $N$ but cannot be free in $E$, hence $E \neq N$ and $y$ must be free in $E$; but then we have $\mu yB \geq \nu vE$ in $F$, with $y$ free in $E$, contradicting $F \in \nu\mu$.
3. $\nu vN$ is a subformula of $(\mu yB)'$. In this case we must have $v = w'$ for some variable $w$ such that $\nu wD$ is a subformula of $B$ and $N$ is obtained from $D$ only by changing the name of bound variables and the name of $y$ to $y'$. Then either $u = y'$ and $\nu wD$ contains $y$ free, or $u$, which is a $\mu$-bound variable in $F$, is free in $\nu wD$; both cases lead to a contradiction since $F$ is a $\nu\mu$-formula. $\Box$

Starting from a well named $\nu\mu$-bidisjunctive guarded formula $F$ and applying Lemma 3.16 a finite number of times, we obtain an equivalent $\nu\mu$-bidisjunctive guarded formula with $BP(G) = 0$. Notice that the formula $G$ is not necessarily well named. However, as it is easily seen, if $G' = wn(G)$ we have $BP(G') = 0$ as well.

To summarize the results of this section, we give a definition:

**Definition 3.17.** A formula is called *special* if

1. it is a well named $\nu\mu$-bidisjunctive guarded formula;
2. in any path between a $\nu$-declaration $\nu xA$ and a $\mu$-declaration $\mu yB$ there is at least a node labeled by a Cover.

**Lemma 3.18.** *Every well named $\nu\mu$-formula is equivalent to a special formula.*

Notice that an alternative proof of Lemma 3.18 can be obtained by rewriting the formula $F$ into a Büchi automaton, transforming the transition rules of the automaton into a "system of fixed point equations" in the sense of [1], Def. 1.4.9, and then calculating the single formula $G$ equivalent to the system. This formula $G$ will be the special formula in Lemma 3.18.

In the following sections we shall prove that *speciality* is preserved by a number of transformations. We first state an easy lemma, whose proof is left to the reader:

**Lemma 3.19.** *If $A(z)$, $B$ do not contain bad pairs and any $\mu$-declaration $\mu x$ in $B$ is preceded (in the path from the root of $B$ to $\mu x$) by a node labeled by a modality, then $A[z|B]$ does not contain bad pairs as well.*

### 3.5. Step 3

In this step we reduce to formulas where all $\nu$-variables are existential. We first reformulate the notion of existential and weakly universal declarations in terms of the Cover operator. In a bidisjunctive formula, a declaration $\nu xA$ is existential if and only if whenever the path $\pi$ between the node $\nu xA$ and the variable $x$ crosses a node $n$ labeled by $Cover(\Gamma; \Delta)$, then the immediate successor $m$ of $n$ in $\pi$ is labeled by a formula in $\Gamma$. The declaration is weakly universal if and only if there exists at least a node $n$ labeled by $Cover(\Gamma; \Delta)$ in the path $\pi$ between the node $\nu xA$ and the variable $x$ and the immediate successor $m$ of $n$ in $\pi$ is labeled by a formula in $\Delta$.

**Definition 3.20.** A formula $F$ is in weakly universal prenex form (w.u.p. form, for short) if it is of the form

$$F = \nu x_k^- \dots \nu x_1^- A,$$

where $A$ does not contain any weakly universal $\nu$-declaration.

**Lemma 3.21.** *Suppose the well named formula $F$ is in w.u.p. form; then, over transitive frames, $F$ is equivalent to a well named formula $F^*$ having the following properties:*

1. *$F^*$ has the same free variables as $F$;*
2. *all $\nu$-variables of $F^*$ are existential;*

3. *if in $F$ all $\mu$-declarations are preceded by a cover, then the same is true in $F^*$;*
4. *if $F$ is special, then $F^*$ is special.*

**Proof.** Let

$$F = \nu x_k^- \dots \nu x_1^- A,$$

where $A$ does not contain any weakly universal $\nu$-declarations. We define $F^*$ by induction on $k$ and we use Lemmas 3.4 and 3.19 to prove that $F^*$ is special, if $F$ is so.

If $k = 1$ then $F = \nu x_1^- A$ and we define $F^* := A(A'(\top))$ (see Notation 3.15). By Lemma 3.7, $F$ is equivalent to $F^*$, which is well named and where all $\nu$-variables are existential. If all $\mu$-declarations in $A$ are preceded by a cover, then the same is obviously true for $A(A'(\top))$. Moreover, if $F$ is $\nu\mu$ then $A$ and $B = A'(\top)$ are $\nu\mu$ and the free variables in $B$ are free as well in $A$; then $F^* \in \nu\mu$ as a composition without capture of $\nu\mu$-formulas. If $F = \nu x_1^- A$ is special, then all $\mu$-declarations in $A$, and hence in $A'(\top)$, must be preceded by a cover; we may then use Lemma 3.19 to prove that $A(A'(\top))$ does not contain bad pairs. Moreover, it is clear that $A(A'(\top))$ is bidisjunctive and guarded if $F$ is so.

Suppose the result is true for $k - 1$ and let

$$F = \nu x_k^- \dots \nu x_1^- A,$$

where $A$ does not contain any weakly universal $\nu$-declarations. Since $x_k$ has a weakly universal $\nu$-declaration in $F$, from Lemma 3.7 we know that $F$ is equivalent to the formula

$$\nu x_{k-1}^- \dots \nu x_1^- A[x_k | (\nu x_{k-1}^- \dots \nu x_1^- A(\top))'].$$

By induction, the formula $H = (\nu x_{k-1}^- \dots \nu x_1^- A(\top))'$ is equivalent to a well named formula $H^*$, with the same free variables as $H$, where all $\nu$-variables are existential. Hence, $F$ is equivalent to

$$K = \nu x_{k-1}^- \dots \nu x_1^- A[x_k | (H^*)'],$$

where, as usual, the bound variables in $H^*$ have been renamed in $(H^*)'$ in such a way to be different from the bound variables in $\nu x_{k-1}^- \dots \nu x_1^- A$. Notice that $A[x_k | (H^*)']$ does not contain any weakly universal $\nu$-declarations. Another induction step applied to $K$ allows us to find the formula $F^* := K^*$ which is equivalent to $F$ and where all $\nu$-variables are existential.

To prove that $F^*$ has the same free variables as $F$, we notice that $K^*$ has the same free variables as $K$, by induction; on the other hand, as is easily verified, $K$ and $F$ have the same free variables.

To prove 3, suppose that in $F$ all $\mu$-declarations are preceded by a cover. Then, by induction, the same is true for $H^*$, and for the formula

$$K = \nu x_{k-1}^- \dots \nu x_1^- A[x_k | (H^*)']$$

as well. Another induction step proves that in $K^* = F^*$ all $\mu$-declarations are preceded by a cover.

Next, we check that if $F$ is special so is $F^*$. We first consider the formula $(H^*)'$: by induction we can suppose that this formula is special and, since all free variables in $(H^*)'$ are free in $\nu x_{k-1}^- \dots \nu x_1^- A$, we know that $K$ is $\nu\mu$ as composition without capture of $\nu\mu$-formulas; by induction we may also suppose that all $\mu$ declarations in $(H^*)'$ are preceded by a cover, and by Lemma 3.19 we know that $K$ does not contain any bad pair. It is also clear that $K$ is bidisjunctive, as a composition of bidisjunctive formulas, and guarded. Hence, we may conclude that $K$ is special. By induction again, speciality must be true for $K^* = F^*$ as well. $\square$

We want to prove that, over transitive frames, any $\nu\mu$-formula $F$ is equivalent to a formula in w.u.p. form. To prove this result, we introduce a notion of depth:

**Definition 3.22.** Given a weakly universal declaration $\nu x^- A$ in a well named formula $F$, we define its *depth* $d(\nu x A)$ as the number of nodes in the path between the root of $F$ and $\nu x$ which are *not* labeled by a greatest fixed point declaration.

Notice that a formula where all weakly universal $\nu$-declarations have depth equal to 0 is equivalent to a formula in w.u.p. form.

**Lemma 3.23.** *For any $\nu\mu$, well named (special) formula $F$, there exists a well named (special) formula $F^\circ$ which is in w.u.p. form and is equivalent to $F$ on transitive frames.*

**Proof.** Without loss of generality, we suppose that on consecutive sequences of greatest fixed point declarations, the weakly universal precede the existential ones. Suppose $F$ is not in w.u.p. form. Then there exists a weakly universal declaration $\nu x_1^- A$ in $F$ such that

1. $A$ does not contain weakly universal $\nu$-declarations;
2. $d(\nu x_1^- A) > 0$, i.e. the path between the root of $F$ and $\nu x_1$ contains a node which is not labeled by a greatest fixed point declaration.

Consider now the *cluster* $H$ of $\nu x_1^- A$, that is, the longest subformula of $F$ of the form

$$\nu x_k^- \ldots \nu x_1^- A,$$

and let $y_1, \ldots, y_m$ be the free variables of $H = \nu x_k^- \ldots \nu x_1^- A$ which become bound in $F$. Notice that, since $F$ is $\nu\mu$, these variables must be declared as greatest fixed points in $F$. From Lemma 3.21 we know that $H$ is equivalent to a formula $H^*$, having the same free variables as $H$, but without weakly universal $\nu$-declarations. If $C(v)$ is such that $F = C[v|H]$, we let

$$K = wn(C[v|(H^*)']).$$

Notice that although the variables $y_1, \ldots, y_m$ have a unique occurrence in $H$, they may have several occurrences in $H^*$. Hence, to well name $C[v|(H^*)']$ we need to rename the different occurrences of $y_i$, and to introduce new greatest fixed point declarations binding these new variables next to the original declaration $\nu y_i$ of $y_i$ in $F$.

Let us make an example. Let

$$F = \nu y^- \nu u^+ . \Diamond u \vee \Box \nu x^- \Box (x \vee y).$$

We consider the subformula $H = \nu x \Box (x \vee y)$, which is in w.u.p. form, and find its equivalent $H^*$ which is well named and without weakly universal declarations:

$$H^* = \Box(\Box(\top \vee y) \vee y).$$

Notice that the free variable $y$ occurs twice in $H^*$. Then $F = C[v|H]$, for $C(v) = \nu y \nu u . \Diamond(u) \vee \Box v$, and we get

$$C[v|(H^*)'] = \nu y \nu u . \Diamond(u) \vee \Box(\Box(\Box(\top \vee y) \vee y)).$$

We have

$$K = wn(C[v|(H^*)'] = \nu y'^- \nu y^- \nu u^+ . \Diamond u \vee \Box(\Box(\Box(\top \vee y') \vee y)),$$

in which we still have two weakly universal declarations, but of smaller depth than the one of $\nu x$ in $F$.

Returning to the general case, we see that, by going from $F$ to $K$, we suppress $k$ declarations $\nu x_1, \ldots, \nu x_k$ of depth equal to $d(\nu x_1)$ which were present in $H$, and in the same time we add new weakly universal declarations (the ones relative to the new copies of the $y_i$) but of smaller depth. If we perform this transformation on all clusters $\nu x_k^- \ldots \nu x_1^- A$ of maximal depth, we obtain an equivalent formula where this maximal depth is lowered. It is then clear that after a finite number of steps of this type we get a formula where all weakly universal $\nu$-declarations have depth equal to 0, which is then equivalent to a formula $F^\circ$ in w.u.p. form.

To finish the proof, we are just left to prove that speciality is preserved from $F$ to $K = wn(C[v|(H^*)']$.

Suppose $F$ is special. In particular, $F$ is a $\nu\mu$-formula. We prove that $C[v|(H^*)']$ is $\nu\mu$, from which it easily follows that $K = wn(C[v|(H^*)']$ is $\nu\mu$. We use Lemma 3.4. Let $z$ be a free variable in $(H^*)'$; then $z$ is free in $H = \nu x_k^- \ldots \nu x_1^- A$ as well. Since $F = C[v|H]$ is $\nu\mu$, the variable $z$ cannot be declared as a $\mu$-variable in $C$ in the path from the root of $C$ to $v$; hence $C[v|(H^*)']$ is $\nu\mu$.

To prove that $C[x|(H^*)']$ (and hence $K = wn(C[v|(H^*)'])$ does not contain any bad pair, we simply apply Lemma 3.19, since any $\mu$-variable in $H^*$ must be preceded by a cover. Finally, it is clear that $C[v|(H^*)']$ is bidisjunctive and guarded if $F$ is so, allowing us to conclude the proof of the lemma. $\Box$

**Corollary 3.24.** *Over transitive frames, any well named $\nu\mu$-formula is equivalent to a formula $G$ without weakly universal $\nu$-declarations. Moreover, if $F$ is special, the same holds for $G$.*

**Proof.** We use Lemma 3.23 to obtain a (special) formula in weakly universal prenex form, and Lemma 3.21 to eliminate all weakly universal $\nu$-declarations. $\Box$

### *3.6. Step 4*

We finally want to apply Lemma 3.6 to convert any $\nu\mu$-formula into an alternation free one. Remember that in previous steps we have converted any $\nu\mu$-formula into a special formula (that is, a well named $\nu\mu$-bidisjunctive guarded formula in which between every $\nu$ declaration and every $\mu$-declaration below it there is a Cover), in which all $\nu$-declarations are existential.

If $F$ is a special formula we let

$$ALT(F) = |\{\mu y B \le F : \exists\, \nu x A \ge \mu y B,\ x \text{ is free in } \mu y B\}|.$$

Notice that if $ALT(F) = 0$, then $F$ is alternation free.

**Lemma 3.25.** *If $F$ is a special formula in which all $\nu$-declarations are existential and $ALT(F) > 0$, then we can find an equivalent special formula $G$, in which all $\nu$-declarations are existential, and such that $ALT(G) < ALT(F)$.*

**Proof.** Let $F$ be a special formula with $ALT(F) > 0$. Consider two subformulas $\nu x A \ge \mu y B$ with $x$ free in $\mu y B$, and (without loss of generality) such that no node in the path from $\nu x A$ to $\mu y B$ is labeled by a least fixed point. Suppose $C$ is such that

$F = C[v|\mu yB]$, and define $G$ as follows:

$$G := C[v|B(\bot)].$$

It is clear that $ALT(G) < ALT(F)$. To finish the proof, we have to prove that $G$ is special and equivalent to $F$, and that if all $\nu$-declarations are existential in $F$, the same holds in $G$. We only prove that $G$ is equivalent to $F$, leaving the other verifications to the reader. Since $F$ is special, we know that there exists a node $n$ between $\nu xA$ and $\mu yB$ which is labeled by a formula $H := \sigma \wedge \text{Cover}(\Gamma; \Delta)$. We may suppose without loss of generality that no node between $n$ and $\mu yB$ is labeled by a cover formula. It follows that no node between $n$ and $\mu yB$ can be labeled by a greatest fixed point formula $\nu uC$, otherwise the pair $(\nu u, \mu y)$ would be a bad pair. Since all $\nu$-variables in $F$ are existential, we know that $\mu yB$ must be a subformula of a formula in $\Gamma$. We then consider the sequence of nodes $n_1, \ldots, n_k$ between the node $n = n_1$, which is labeled by $H$ and the node $n_k$, labeled by $\mu yB$. We consider two cases.

1. If $k = 2$, we have $\Gamma = \{\mu yB\} \cup \Sigma$; if we apply Lemma 3.6 we know that the subformula $H = \text{Cover}(\{\mu yB\} \cup \Sigma; \Delta)$ is equivalent to $H' := \text{Cover}(\{B(\bot)\} \cup \Sigma; \Delta)$. Hence, if $D$ is such that $F = D[u|H]$, we have $G = D[u|H'] \equiv D[u|H] = F$.
2. If $k > 2$, we know that all $n_i$ for $1 < i < k$ are labeled by a disjunction $F_1^i \vee F_2^i$, since no node $n_i$ for $1 < i < k$ can be labeled by a cover formula or by a fixed point operator; we may suppose without loss of generality that $x$ appears free in all $F_2^i$. Then $F_2^i = F_1^{i+1} \vee F_2^{i+1}$, for $2 < i + 1 < k$, and

   $$\Gamma = \{F_1^2 \vee (F_1^3 \vee \cdots (F_1^{k-1} \vee \mu yB) \ldots)\} \cup \Sigma.$$

   By Lemma 3.6 and the fact that the diamond operator commutes with disjunctions, we know that the $F$ subformula $H = \text{Cover}(\Gamma; \Delta)$ is equivalent to the formula

   $$H' := \text{Cover}(\{F_1^2 \vee (F_1^3 \vee \cdots (F_1^{k-1} \vee B(\bot)) \ldots)\} \cup \Sigma; \Delta).$$

   We then proceed as before, and substitute $H = Cover(\Gamma; \Delta)$ in $F$ with $H'$, obtaining in this way the formula $G$ which is equivalent to $F$. □

Notice that by iterating the step from $F$ to $G$ a finite number of times we arrive to an alternation free formula.

We are now able to prove:

**Theorem 3.26.** *Over transitive frames, any $\nu\mu$-formula is equivalent to an alternation free formula.*

**Proof.** This can be done using steps 1–4 in the preceding paragraph. □

**Corollary 3.27.** *Over transitive frames, the alternation hierarchy of the $\mu$-calculus collapses to the alternation free fragment.*

**Proof.** Using Theorem 3.26 we see that the class of $\nu\mu$-formulas collapses to the alternation free fragment. By induction this implies that the whole hierarchy collapses to the alternation free fragment. □

## 4. $L_\mu$ is first order definable on finite transitive frames

By using some byproducts of the proofs given in the preceding paragraph, in this section we show that on finite transitive frames the $\mu$-calculus is included in first order logic. First of all, let us fix the first order language $\mathcal{L}_F$ corresponding to a $\mu$-calculus formula $F$ containing the propositions $P_1, \ldots, P_n$ and the free variables $x_1, \ldots, x_m$: we let

$$\mathcal{L} = \{r, R, P_1, \ldots, P_n, x_1, \ldots, x_m\},$$

where $r$ is a constant representing the root, $R$ is a binary predicate representing the accessibility relation, and $P_1, \ldots, P_n, x_1, \ldots, x_m$ are unary predicates representing propositions and free variables.

As is well known and easily verified, every modal formula $F$ is equivalent to a first order formula in $\mathcal{L}_F$, without restriction on the class of frames: e.g. the formula $\Diamond(x \wedge P)$ is equivalent to $\exists v(rRv \wedge x(v) \wedge P(v))$ on all models. We are going to show that on finite transitive models this is also true for $\mu$-formulas: e.g. $\nu x.\Diamond(x \wedge P)$ is equivalent to $\exists v(rRv \wedge vRv \wedge P(v))$ on finite transitive frames.

**Theorem 4.1.** *On finite transitive frames, every formula of the $\mu$-calculus is equivalent to a formula of first order logic.*

Notice the following fact:

**Fact 4.2.** *Suppose $F(x)$, $G$ are two $\mu$-formulas such that the composition $F[x|G]$ is without capture. Then, if $F$, $G$ are equivalent in the above sense to first order formulas, the same holds for $F[x|G]$.*

**Plan of the Proof of Theorem 4.1.**

1. By Corollary 3.27 and Fact 3.3, we know that every $\mu$-formula is equivalent on transitive frames to a composition without capture of $\nu$-formulas and their negations. By Fact 4.2, to prove Theorem 4.1 it is enough to show that every $\nu$-formula is first order definable on finite transitive frames.
2. Every $\nu$-formula is equivalent to a $\nu$-formula in disjunctive normal form (this result is well known; see e.g. [1,8]); then just applying the transformations given in steps 2, 3 of the previous section we get a $\nu$-bidisjunctive formula in which all $\nu$-variables are existential. We call $\nu^+$ a formula with these properties (in particular, $\nu^+$ formulas are bidisjunctive). So, every $\nu$-formula is equivalent on transitive frames to a $\nu^+$ formula.

3. We show that every $\nu^+$ formula is equivalent to a composition without capture of modal formulas and $\Box$-free formulas of a special class $\mathcal{N}$.
4. By using some kinds of systems of equations which we call $\nu$-systems, we show that every formula in $\mathcal{N}$ is expressible in first order logic over finite transitive frames.
5. Once we have proved the preceding points, the proof of Theorem 4.1 is concluded by Fact 4.2.

Notice that to prove Theorem 4.1 we are only left to prove points 3, 4 of the above plan. We start with point 3.

If $\Gamma$ is a set of $\mu$-formulas we define $Comp(\Gamma)$ as the smallest set such that

1. $Comp(\Gamma)$ contains $\Gamma$;
2. $Comp(\Gamma)$ contains

$$A[x_1|B_1, \dots, x_n|B_n],$$

provided $A(x_1, \dots, x_n)$ and $B_1, \dots, B_n$ are in $Comp(\Gamma)$ and the substitution

$$A[x_1|B_1, \dots, x_n|B_n]$$

is without capture. $\Box$

**Definition 4.3.** The class $\mathcal{N}$ is the smallest set such that

1. $\mathcal{N}$ contains all literals;
2. if $F_1, \dots, F_n \in \mathcal{N}$ then $F_1 \vee \dots \vee F_n \in \mathcal{N}$;
3. if $F \in \mathcal{N}$ then $\nu x F \in \mathcal{N}$, provided $x$ is a variable not contained in any context of type $x \wedge G$;
4. if $\{F_1, \dots, F_n\} \subseteq \mathcal{N}$ then

$$\lambda \wedge \Diamond F_1 \wedge \dots \wedge \Diamond F_n \in \mathcal{N}$$

where $\lambda$ is a conjunction of literals.

For the sake of readability, in the following we shall use the abbreviation

$$\lambda \wedge \Diamond(F_1, \dots, F_m) := \lambda \wedge \Diamond F_1 \wedge \dots \wedge \Diamond F_m;$$

we also let $\lambda \wedge \Diamond(\emptyset) := \lambda$, so that we do not really need to start our construction from literals (which can be recovered using the construction 4 of the definition of $Comp(\Gamma)$).

**Lemma 4.4.** *Every $\nu^+$ formula F is equivalent to a formula in $Comp(\Gamma)$, for $\Gamma = \{\Box x\} \cup \mathcal{N}$.*

**Proof.** First, let us translate the formula $F$ into the usual language of $\Box$, $\Diamond$; we then proceed by induction on the number $n$ of boxes of $F$. If $n = 0$ then $F \in \mathcal{N}$, because in the construction of $F$ as a bidisjunctive formula we are not allowed to use neither the least fixed point operator ($F$ is $\nu^+$), nor the $Cover(\Gamma; \Delta)$ operator (even for $\Delta = \emptyset$, because $Cover(\Gamma; \emptyset)$ contains $\Box(\bot)$ as a subformula and we are supposing $n = 0$).

If the formula $F$ contains $n > 0$ boxes, then $F$ has a subformula of type $\Box B$, and it can be written in the form $F = A(x|\Box B)$, where no variable declared in $A$ can be free in $B$ (otherwise this variable would not be existential, contrary to the definition of $\nu^+$ formulas). Notice that the set $Comp(\Gamma)$ is closed under the $\Box$ operator: if $E \in Comp(\Gamma)$, then $\Box E \in Comp(\Gamma)$, being the composition of $\Box x$ and $E$. Since the formulas $A$, $B$ are $\nu^+$, and have strictly less than $n$ boxes, by induction $A(x)$ and $B$ are in $Comp(\Gamma)$, and so are $C = \Box B$ and $A(x|C) = A(x|\Box B)$, as compositions without capture of $Comp(\Gamma)$ formulas. $\Box$

As an example, consider the $\nu^+$ formula

$$F = \nu x(\Diamond(x \vee \Box(\nu y(P \wedge \Diamond y \wedge \Box P)))).$$

We prove that $F$ is in $Comp(\Gamma)$, for $\Gamma = \{\Box x\} \cup \mathcal{N}$: the formula $E(z) = \nu y(P \wedge z \wedge \Diamond y)$ is in $\mathcal{N}$, hence $C = \nu y(P \wedge \Diamond y \wedge \Box P)$ is in $Comp(\Gamma)$, being the composition without capture of $E(z)$ and $\Box P$ (which is in $Comp(\Gamma)$). Then $\Box C$ is in $Comp(\Gamma)$, and

$$F = \nu x(\Diamond(x \vee \Box(\nu y(P \wedge \Diamond y \wedge \Box P))))$$

is in $Comp(\Gamma)$, being the composition of the $\nu^+$ formula $A(z) = \nu x \Diamond(x \vee z)$ with the $Comp(\Gamma)$ formula $\Box C$.

In the following subsection, we shall concentrate on point 4 of the plan.

### *4.1. $\nu$-systems*

By the previous results, to prove the first order expressibility of the $\mu$-calculus on the class of finite transitive frames, we are left to show that $\mathcal{N}$ formulas are equivalent to first order formulas on this class.

The translation of $\mathcal{N}$ formulas in first order logic will be accomplished by introducing particular systems of equations which we call $\nu$-systems. The idea is that an $\mathcal{N}$ formula is equivalent to the solutions of the system, and the system solutions must correspond to certain structures which we call schemata. These schemata can be simplified in a way to be finite and to range over a finite set. Since, as we shall see, the solvability of a finite schema is first order expressible, we conclude that the formula is first order expressible.

We start with the following definition:

**Definition 4.5.** A *ν-system* on variables $z_1, \ldots, z_n$ is a finite set $S$ of equations of the form

$$z_i = z_j \cup z_k, \quad \text{or} \quad z_i = z_j \quad \text{or} \quad z_i = \lambda_i \wedge \Diamond(\mathcal{Z}_i),$$

where $\lambda_i$ is a conjunction of literals (view also as a subset of literals), $\mathcal{Z}_i$ is a subset of the variables $z_1, \ldots, z_n$, and each variable $z_i$ occurs at most once in the left-hand side of an equation. The variable $z_1$ is called the main variable of the system $S$.

Variables in the left-hand side of an equation will be called *modal* if their equation is of type $z_i = \lambda_i \wedge \Diamond(\mathcal{Z}_i)$, and *non-modal* otherwise.

A variable which does not appear on the left side of an equation is called *free*.

A system where in each modal equation $z = \lambda \wedge \Diamond(\mathcal{Z}_i)$ the conjunct $\lambda$ does not contain any variable is called *safe*.

Solutions of a $\nu$-system are to be found in Kripke models:

**Definition 4.6.** A *solution* of a system $S$ on variables $z_1, \ldots, z_n$ in a model $M = (W, R, r, L)$ is an $n$-tuple $A_1, \ldots, A_n$ of subsets of $W$ such that the root of $M$ belongs to $A_1$, and $A_1, \ldots, A_n$ satisfy the equalities of $S$ when $z_i$ is replaced by $A_i$. In particular,

1. if the equation $z_i = \lambda_i \wedge \Diamond(\mathcal{Z}_i)$, occurs in $S$, then:
   for all $w \in A_i$, $w \models \lambda_i$ where the variables $z_1, \ldots, z_n$ in $\lambda_i$ are interpreted as $A_1, \ldots, A_n$, and for all $z_j \in \mathcal{Z}_i$ there exists $v \in A_j$ with $wRv$;
   conversely, if $w \models \lambda_i$ and for all $z_j \in \mathcal{Z}_i$ there exists $v \in A_j$ with $wRv$, then $w \in A_i$.
2. if the equation $z_i = z_j$ occurs in $S$, then $A_i = A_j$;
3. if the equation $z_i = z_j \cup z_k$ occurs in $S$, then $A_i = A_j \cup A_k$.

Given a formula $F \in \mathcal{N}$ with free variables $x_1, \ldots, x_m$, we shall look for an *equivalent* system with the same free variables; given a solution of such a system, we use by convention the letters $B_1, \ldots, B_m$ to indicate the sets of the solutions corresponding to the variables $x_1, \ldots, x_m$.

**Lemma 4.7.** *Suppose the formula $F \in \mathcal{N}$ has free variables $x_1, \ldots, x_m$. For every variable $y \notin \{x_1, \ldots, x_m\}$ there is a system $S(y, F)$, on a set of variables containing $y, x_1, \ldots, x_m$, in which $y$ is the main variable and the variables $x_1, \ldots, x_m$ are free, such that, for all model $M = (W, R, r, L)$:*

$$(M \models F) \Leftrightarrow S(y, F) \text{ has a solution in } M \text{ with } B_i = L(x_i) \quad \text{for all } i = 1, \ldots, m.$$

**Proof.** We proceed by induction on the complexity of $F \in \mathcal{N}$.

If $\mathcal{F} = \{F_1, \ldots, F_n\} \subseteq \mathcal{N}$ and $F = \lambda \wedge \Diamond(\mathcal{F})$ (this case includes the base case, when $\mathcal{F} = \emptyset$), we consider the systems $S_i = S(y_i, F_i)$, where the $y_i$ are fresh extra variables and the systems $S_i$ have different variables, except for $x_1, \ldots, x_m$. Then $S(y, F)$ is given by the equation $y = \lambda \wedge \Diamond(y_1, \ldots, y_n)$ followed by all equations in $\bigcup_i S_i$.

If $F = G \vee H$, by induction we already have the two systems $S(y_1, G), S(y_2, H)$, where $y_1, y_2$ are fresh extra variables; we may suppose without loss of generality that, except for the free variable $x_1, \ldots, x_m$, the two systems have different variables. Then the system $S(y, F)$ is given by the equation $y = y_1 \cup y_2$ followed by the equations in $S(y_1, G) \cup S(y_2, H)$.

Finally, if $F = \nu x.G$, we consider the system $S(u, G)$, where $u$ is a fresh variable and the system $S(u, G)[u|x]$ obtained from $S(u, G)$ by substituting the variable $u$ with the variable $x$ corresponding to the free variable $x$ of $G$. Then the system $S(y, F)$ is given by the equation $y = x$ followed by $S(u, G)[u|x]$.

We leave to the reader the verification that $S(y, F)$ is a system satisfying the lemma. □

Notice that in $S(y, F)$ the variables which may appear as a conjunct of $\lambda$ in an equation of type $z = \lambda \wedge \Diamond(\mathcal{F})$ are only the variables $x_i$ corresponding to the free variables of $F$. This means that if $F$ is a sentence, i.e. a formula without free variables, then $S(y, F)$ is safe (see Definition 4.5).

As an example, let $F = \nu x.\, p \wedge z \wedge \Diamond(x \vee \nu y\, \Diamond(y, z))$; then

$$S(y_1, F) = \begin{cases} y_1 = x; \\ x = p \wedge z \wedge \Diamond(y_2); \\ y_2 = y_3 \vee y_4; \\ y_3 = x; \\ y_4 = y; \\ y = \Diamond(y_5, y_6); \\ y_5 = y; \\ y_6 = z. \end{cases}$$

**Remark 4.8.** Given a system $S$, consider the system $S^{\subseteq}$ obtained from $S$ by replacing all equalities in $S$ with inclusions. Since all right-hand sides of equations in $S$ are monotone operators, from the Knaster–Tarski fixed point theorem it follows that a system $S$ is solvable in a model $M$ if and only if $S^{\subseteq}$ is solvable in $M$. Hence, from now on, when looking for a solution of a system $S$ we will be actually satisfied by a solution of $S^{\subseteq}$.

*4.2. Schemata*

In order to describe the solutions of systems we will use a kind of pattern, which we call schemata, such that every graph containing a schema associated to a system contains a solution of the system itself.

**Definition 4.9.** Let $S$ be a safe $\nu$-system in the variables $z_1, \ldots, z_n$. A *schema* for $S$ is a rooted graph $G$, vertex labeled by the variables $z_i$, where

1. the root is labeled by $z_1$;
2. for each vertex $t$ of $G$ with label $z_i$, if the equation relative to $z_i$ in $S$ is $z_i = \lambda \wedge \Diamond(z_{i_1}, \ldots, z_{i_m})$, then the node $t$ has $m$ sons $s_1, \ldots, s_m$ labeled respectively by $z_{i_1}, \ldots, z_{i_m}$;
3. for each vertex $t$ of $G$ with label $z_i$, if the equation relative to $z_i$ in $S$ is $z_i = z_j \cup z_k$, then the node $t$ has a unique son $s$ labeled by $z_j$ or by $z_k$;
4. for each vertex $t$ of $G$ with label $z_i$, if the equation relative to $z_i$ in $S$ is $z_i = z_j$, then the node $t$ has a unique son $s$ labeled by $z_j$.

Notice that, in a schema for $S$, a vertex has a unique label and may have at most $n$ successors, where $n$ is the number of variables of the system $S$. A node in $G$ is called a *modal node* if it is labeled by a modal variable of $S$, and is called non-modal otherwise.

**Definition 4.10.** Given a model $M$, a safe system $S$, and a schema $G$ for $S$, an *interpretation* for $G$ in $M$ is a function $H$ from the vertexes of $G$ to the elements in $M$ with the following properties:

1. the root $r_g$ of $G$ is sent to the root $r_M$ of $M$, that is, $H(r_g) = r_M$;
2. if the vertex $t$ of the schema $G$ is labeled by a non-modal variable $z_i$, and $s$ is the only successor of $t$ in $G$, then $H(t) = H(s)$;
3. if the vertex $t$ of the schema $G$ is labeled by a modal variable $z_i$ with equation $z_i = \lambda_i \wedge \Diamond(z_{i_1}, \ldots, z_{i_m})$, then $H(t)$ satisfies $\lambda_i$ in $M$ (notice that, since the system is safe, $\lambda_i$ does not contain any variable); moreover, if $s_1, \ldots, s_m$ are the children of $t$ in $G$, then $H(s_i)$ is a successor of $H(t)$ in $M$, for all $i = 1, \ldots, m$.

If $H$ is an interpretation of $G$ in $M$, the vertex $t$ is labeled by the variable $z$ in the schema $G$, and $H(t) = v$, we also say that the vertex $t$ is labeled by the pair $(z, v)$.

Schemata and their interpretations in a model are connected to solutions of the system in $M$ by the following:

**Lemma 4.11.** *A safe system $S$ is solvable in $M$ if and only if there exists a schema $G$ for $S$ having an interpretation in $M$. Moreover, we can restrict to schemata whose underlying graph is a tree.*

**Proof.** Given a solution $A_1, \ldots, A_n$ of $S$ in $M$, we construct the tree $G$ and the interpretation $H$, by stages. We start with the root $r$ of the tree, labeled by $z_1$, and with $H(r) = r_M$, the root of $M$. We then add new nodes, maintaining the following invariant: if a node $t$ is labeled by $z_i$ then $H(t) \in A_i$. Hence, when we arrive at stage $n + 1$ and $s$ is a node constructed at stage $m \leq n$ and labeled by $(z_i, v)$, then $v$ belongs to $A_i$. We consider a node $s$ constructed at stage $n$. If $s$ is labeled by the pair $(z_i, v)$, we add new nodes according to the following cases:

1. if the equation relative to $z_i$ is $z_i = \lambda_i \wedge \Diamond(z_{i_1}, \ldots, z_{i_m})$, we add $m$ children $s_1, \ldots, s_m$ of $s$ and we label $s_i$ by the pair $(z_{i_j}, w_j)$, where $w_j$ is a successor of $v$ belonging to $A_{i_j}$. The existence of such a $w_j$ is guaranteed by the invariant and the equation $A_i = \lambda_i \wedge \Diamond(A_{i_1}, \ldots, A_{i_m})$.
2. if the equation relative to $z_i$ is $z_i = z_j$ or $z_i = z_j \cup z_k$, we add a unique son $t$ to $s$ and label it by a pair $(z_j, v)$ with $v \in A_j$ in the first case, and by $(z_j, v)$ or $(z_k, v)$ in the second case, depending on whether $v$ belongs to $A_j$ or to $A_k$ in $M$.

Conversely, if $G$ is a schema for $S$ with an interpretation $H$ in $M$, then we find a solution $A_1, \ldots, A_n$ of $S^{\subseteq}$ in $M$ by defining:

$$A_i = \{H(t) : t \text{ is labeled by } z_i\}.$$

By Remark 4.8 we conclude that $S$ is solvable in $M$. □

We are particularly interested in finite schemata:

**Lemma 4.12.** *For any safe system $S$ without free variables consider the language $\mathcal{L} = \{r, P_1, \ldots, P_n\}$ having a unary predicate $P_i$ for each proposition $p_i$ appearing in the equations of $S$. Given a finite schema $G$ for $S$, there exists a first order formula $F_G$ in the language $\mathcal{L}$, such that for every model $M$ it holds:*

$$(M \models F_G) \Leftrightarrow \text{ there exists an interpretation for } G \text{ in } M.$$

**Proof.** Let $V(G)$, $E(G)$ be the set of vertexes and set of edges of $G$, respectively. Let $g_1, \ldots, g_n$ be any enumeration without repetition of the vertexes of $G$, where $g_1$ is the root of $G$. Let $z_g$ be the variable labeling $g$, and let $MOD$ be the set of all modal nodes in $G$. For all $g \in MOD$, let $\lambda_g$ be such that the equation

$$z_g = \lambda_g \wedge \Diamond(\mathcal{Z}_g)$$

belongs to the system $S$. We define:

$$F_G := \exists x_{g_1} \ldots \exists x_{g_n} (x_{g_1} = r \wedge G_1 \wedge G_2),$$

where

$$G_1 := \left( \bigwedge_{g \in MOD} \lambda_g(x_g) \wedge \bigwedge_{(g,h) \in E(G)} x_g R x_h \right), \qquad G_2 := \left( \bigwedge_{g \notin MOD} \bigwedge_{(g,h) \in E(G)} x_g = x_h \right)$$

where

$$\lambda_g(x_g) := \bigwedge_{P_i \in \lambda_g} P_i(x_g) \wedge \bigwedge_{\neg P_i \in \lambda_g} \neg P_i(x_g).$$

We leave to the reader the verification that $F_G$ satisfies the lemma. □

In order to apply Lemma 4.12, given a system $S$, we want to restrict the class of schemata for $S$ having interpretations in finite transitive models to a finite number of finite schemata. For this purpose we consider the usual notion of a tree with back edges:

**Definition 4.13.** A *tree with back edges* is a graph $(V, E)$ where $E = F \cup B$, $(V, F)$ is a tree, and $B$ is a set of pairs $(t, t')$ of elements in $V$, such that $t$ is a descendant of $t'$ in $(V, F)$. The elements of $B$ are called *back edges*.

**Definition 4.14.** Let $S$ be a safe system in $n$ variables. A schema $G$ for $S$ is *reduced* if it is a tree with back edges, where, in the tree underlying $G$, we have

1. in any path there are no more than $n$ consecutive non-modal nodes different from the root;
2. in any path there are no more than $n + 1$ modal nodes.

**Remark 4.15.** Notice that given a system $S$ in $n$ variables, if $G$ is a reduced schema for $S$ then in the underlying tree all paths have length limited by

$$h(n) = (n + 2)n + (n + 1) + 1 = n^2 + 3n + 2.$$

It follows that every reduced schema is finite and also that there are only finitely many reduced schemata for $S$.

**Theorem 4.16.** *Given a finite transitive model $M$ and a safe system $S$, if there exists a schema $G$ for $S$ having an interpretation in $M$, then there exists a reduced one.*

**Proof.** By Lemma 4.11, given $S, M, G$ as above, we may suppose that the schema is a tree $T$. Let $H$ be the interpretation of $T$ in $M$. We show how to transform $T, H$ to a reduced schema $G^*$ for $S$ with an interpretation $H^*$ in $M$. This can be done as follows. We say that a vertex $t \in T$ labeled by $(z, v)$ is *maximal* if for every descendant $s$ of $t$ in $T$ labeled by $(z, w)$, the vertex $w$ is in the same strongly connected component of $v$ in $M$ (that is, either $v = w$ or there is a path in $M$ from $v$ to $w$ and from $w$ to $v$).

First of all, we reduce to the case in which all sons of modal nodes are maximal. This can be done as follows. We define a sequence $(T_0, H_0) = (T, H), (T_1, H_1) \dots, (T_i, H_i) \dots$ of schemata for $S$ with interpretations in $M$ with the following properties:

1. all sons of modal nodes in $T_i$, whose distance from the root is less than or equal to $i$, are maximal in $T_i$;
2. $(T_i, H_i)$ coincides with $(T_{i+1}, H_{i+1})$ on all nodes whose distance from the root is less than or equal to $i$.

Once this is done, we obtain a tree $T'$ which is a schema for $S$ with an interpretation $H'$ in $M$, and where all sons of modal nodes are maximal, by defining:

$$T' = \bigcup_i T_{i,i}, \qquad H' = \bigcup_i H_{i,i},$$

where $T_{i,i}$ is the subtree of $T_i$ consisting of all nodes whose distance from the root is less than or equal to $i$, and $H_{i,i}$ is $H_i$ restricted to $T_{i,i}$.

Let $(T_0, H_0) = (T, H)$. The tree $T_{i+1}$ is defined as follows. Consider the schema $(T_i, H_i)$ and a modal node $t$ in it having a non-maximal son $s$, whose distance from the root of $T_i$ is $i + 1$; let $s'$ be a descendant of $s$ labeled by the same variable and maximal (the existence of $s'$ is guaranteed by the finiteness of $M$): erase from the tree $T_i$ the node $s$ and the subtree starting from $s$, and add an edge from $t$ to the subtree starting from $s'$; finally, consider the interpretation in $M$ induced on this new tree by $H_i$. If we repeat this procedure for every non-maximal son of modal nodes whose distance from the root is $i+1$, after a finite number of transformations we get a tree $T_{i+1}$ and an interpretation $H_{i+1}$ satisfying the above properties.

This proves the existence of a tree schema $T'$ for $S$ with an interpretation $H'$ in $M$ where all sons of modal nodes are maximal.

From $T'$, we obtain a schema $G^*$ with an interpretation $H^*$ satisfying Definition 4.14 as follows. Consider all paths of length $h(n) = n^2 + 3n + 2$ starting from the root in $T'$, and in particular the ones containing two nodes, different from the root, labeled by the same pair $(z, v)$ where $z$ is a non-modal variable. Let $\pi$ be such a path, and let $t_1, t_2$ be the first such nodes in $\pi$: erase the subtree starting from $t_2$ and add a back edge from the predecessor of $t_2$ in $\pi$ to $t_1$. After a finite number of these transformations, we get a tree with back edges $G_1$ with an interpretation in $M$ in which all paths of length

$h(n)$ do not contain two nodes different from the root labeled by the same pair $(z, v)$, where $z$ is a non-modal variable. Since consecutive non-modal nodes must be labeled by the same vertex of $M$, it follows that in the tree $T_1$ underlying $G_1$ all paths of length $h(n)$ have at most $n$ consecutive non-modal nodes (without considering the root). Hence, a path $\pi$ in $T_1$ of length $h(n)$ must have more than $n$ modal nodes: otherwise, if $k \leq n$ is the number of modal nodes in $\pi$, $\pi$ can have at most $(k+1)n + k + 1 < h(n)$ nodes. Consider the first $n+1$ modal nodes in $\pi$ different from the root, and let $s_1, \ldots, s_{n+1}$ be their sons in $\pi$. Since we have $n$ variables, there exist $i < j$ and a variable $z$ such that $s_i, s_j$ are labeled by $(z, v_i)$, $(z, v_j)$, respectively. Moreover, since $s_i$ is maximal, the vertex $v_i, v_j$ belong to the same strongly connected component in $M$, and we can erase the subtree of $T_1$ starting from $s_j$ and add a back edge from the predecessor of $s_j$ in $\pi$ to $s_i$.

After a finite number of these transformations, we get a tree with back edges $G^*$ with an interpretation $H^*$ in $M$, which is reduced (and whose underlying tree is of height at most $h(n)$). □

**Corollary 4.17.** *Every formula $F \in \mathcal{N}$ is expressible in first order logic over finite transitive frames.*

**Proof.** Let $S = S(y, F)$ be as in Lemma 4.7 , $G_1, \ldots, G_N$ be a list of all reduced schemata for $S$, and $F_{G_1}, \ldots, F_{G_N}$ be the corresponding first order formulas as in Lemma 4.12. By the result of this section we easily get

$$F = \bigvee_i F_{G_i}. \quad \square$$

This proves the last step of the Plan of Theorem 4.1, concluding its proof.

## 5. The $\mu$-calculus over simple graphs

In this section we consider a class of graphs, which we call *simple* graphs, containing the class of finite trees as well as, modulo bisimulation, the class of finite transitive graphs. Using automata theoretic techniques, we are able to prove the collapse of the $\mu$-calculus over simple graphs to the class $\Sigma_2 \cap \Pi_2$.

In this section we find convenient to consider Kripke models over the set of propositions $PROP = \{P_1, \ldots, P_n\}$ as pointed graphs labeled by subsets of $PROP$: $G = (V, R, v_0, \lambda)$, where $\lambda : V \to Powerset(PROP)$ and $\lambda(v)$ (the *colour* of $v$) represents the set of propositions which are true in $v$.

**Definition 5.1.** A graph $G = (V, R, \lambda)$ is *simple* if whenever two vertexes $v, w$ belong to the same strongly connected component of $G$ and have the same colour (that is $\lambda(v) = \lambda(w)$) then $v = w$.

Notice that the quotient under the maximal bisimulation of a transitive graph is simple.

We prove that the $\mu$-calculus collapses over finite simple graphs by showing that any Büchi automaton is equivalent to a co-Büchi automaton over this class. First of all we recall the definition of these types of automata and their connection with the $\mu$-calculus.

### *5.1. Büchi and co-Büchi automata*

**Definition 5.2.** A (non-deterministic) *Büchi automaton* is a tuple

$$B = (Q, \Lambda, q_0, \delta, F)$$

such that

1. $Q$ is a finite set of states;
2. $\Lambda$, the set of *colours*, is equal to $Powerset(Prop)$ where $Prop$ is a finite set of propositions;
3. $q_0 \in Q$ is the initial state;
4. $\delta : Q \times \Lambda \to Powerset(Powerset(Q))$ is the transition function;
5. $F \subseteq Q$ is the set of final states.

The formal definition of acceptance of a $\Lambda$-graph $G = (V, R, v_0, \lambda)$ by the automaton $A$ is given by means of two players, Duplicator and Spoiler, which play on the structure $G$ following the rules given by $B$. A *play* of $B$ on $G$ is defined as follows.

1. The starting position is $(q_0, v_0)$;
2. If we are in a position $(q, v)$ then Duplicator has to make a move. A legal move for Duplicator consists of a $Q$-*marking* of the set of successors of $v$, that is, a function $m : Q \to Powerset(\{v' : vRv'\})$, such that there exists a $D \in \delta(q, \lambda(v))$ with
   (a) For all $v'$ with $vRv'$ there is a $q' \in D$ with $v' \in m(q')$;
   (b) If $q' \in D$ then there is a $v'$ with $vRv'$ and $v' \in m(q')$.
   If we interpret $\delta(q, \lambda(v))$ as the formula
   $$\bigvee_{D \in \delta(q,\lambda(v))} \left[ \bigwedge_{q \in D} \exists x q(x) \wedge \forall x \left( \bigvee_{q \in D} q(x) \right) \right]$$
   we see that a $Q$-marking (seen as an interpretation over the set $\{v' : vRv'\}$ of the unary predicates $q(x)$ for all $q \in Q$) is a legal move for Duplicator if and only if $m \models \delta(q, \lambda(v))$.

3. If Duplicator has just made a move, namely a marking $m$, Spoiler picks a pair $(q', v') \in m$ (that is, $v' \in m(q')$), and the new position becomes $(q', v')$.

Spoiler, or Duplicator, wins a play if the other player cannot make a move. An infinite play $(q_0, v_0), m_0, (q_1, v_1), m_1, (q_2, v_2), m_2 \dots$ is won by Duplicator if $\{i : q_i \in F\}$ is an infinite set, otherwise the play is won by Spoiler.

We say that $G$ is *accepted* by $B$ iff there exists a strategy for Duplicator (that is, a function from partial plays to markings suggesting the next move) which allows Duplicator to win every play. It can be proved that if a player has a winning strategy, it has a positional winning strategy, that is, a strategy that only depends on the last move of the opponent. Notice that a positional strategy for Duplicator can be seen as a function $\Delta$ from $Q \times V$ to markings, with $\Delta(q, v) \models \delta(q, \lambda(v))$. If $\Delta$ is a positional strategy for Duplicator, we say that a (finite or infinite) play $\pi$ is a $\Delta$-play if it is played by Duplicator following $\Delta$, that is, if $\pi$ has the form:

$$\pi = (q_0, v_0), \Delta(q_0, v_0), (q_1, v_1), \Delta(q_1, v_1), \dots.$$

The connection between Büchi automata and the modal $\mu$-calculus is given by the following Lemma.

**Lemma 5.3.** *For every Büchi automaton $B$ there exists a $\nu\mu$-formula $F_B$ such that, for all $\Lambda$-graph $G$ it holds*

$$G \text{ is accepted by } B \Leftrightarrow G \models F_B,$$

*and, conversely, for every $\nu\mu$-formula $F$ there exists a Büchi automaton $B_F$ such that, for all $\Lambda$-graph $G$ it holds*

$$G \text{ is accepted by } B_F \Leftrightarrow G \models F.$$

In other words, Büchi automata correspond over the class of $\Lambda$-graphs to $\nu\mu$-formulas.

A *co-Büchi automaton* is defined as a Büchi automaton, except for the acceptance condition over infinite plays: in this case we require that, from a certain point on, all the states we meet are final. Co-Büchi automata correspond over the class of $\Lambda$-graphs to $\mu\nu$-formulas. This implies that the complement of the class of graphs accepted by a Büchi automaton is accepted by a co-Büchi automaton, and vice versa.

### *5.2. Büchi automata on simple graphs*

We show that the $\mu$-calculus collapses to Büchi $\cap$ co-Büchi on finite simple graphs. First we prove:

**Lemma 5.4.** *For every Büchi automaton $B = (Q, q_0, \delta, F)$ over the set of colours $\Lambda$, there exists a co-Büchi automaton $B'$ over the same set of colours which is equivalent to $B$ on finite simple graphs.*

**Proof.** We first claim that any positional winning strategy $\Delta$ for Duplicator on a finite simple graph $G$ is such that in any play of $\Delta$, after a finite number of moves, final states are encountered every $|Q| \times |\Lambda|$ Spoiler steps. To prove the claim we first notice: if, during any play of $\Delta$, a pair $(q, v)$ is chosen twice by Spoiler, then one of the states we meet between these two occurrences of $(q, v)$ must be a final state. In other words, if a segment of the game is of the following form:

$$m_i(q_i, v_i)m_{i+1}(q_{i+1}, v_{i+1})m_{i+2}(q_{i+2}, v_{i+2}) \dots (q_{i+j}, v_{i+j}),$$

with $(q_i, v_i) = (q_{i+j}, v_{i+j}) = (q, v)$, then $\{q_i, q_{i+1}, \dots, q_{i+j-1}\} \cap F \neq \emptyset$. Otherwise, since Duplicator's strategy is positional, after the second occurrence of $(q, v) = (q_{i+j}, v_{i+j})$, Duplicator must choose $m_{i+1}$ again, and Spoiler could reply with $(q_{i+1}, v_{i+1})$. Then Duplicator must answer with $m_{i+2}$ and so on: in this way Spoiler will be able to create a loop. If no final states were on this loop, we would have an infinite $\Delta$-game containing only a finite number of final states, a contradiction.

To prove the claim it is then sufficient to remark that in any infinite game on a finite graph, after a finite number of moves the game always stays inside a fixed strongly connected component of the graph. Since the graph is simple, once a play is in this component forever, we know that two vertexes with the same colour coincide. Hence we will have a repetition of a pair state-vertex every $k = |Q| \times |\Lambda|$ states, and between these two occurrences, as we showed before, there will always be a final state. This proves the claim.

We next define the co-Büchi automaton $B' = (Q', q'_0, \delta', F')$ which is equivalent to $B$ on simple graphs. The states $Q'$ of $B'$ are the pairs $(q, L)$, where $q \in Q$ and $L$ is a list of at most $k$-states in $Q$. We let $q'_0 = (q_0, \epsilon)$.

The transition function $\delta'$ is defined as follows: for all $\sigma \in \Lambda$ and $(q, L) \in Q'$ we let

$$\begin{gathered}\{(q_1, L_1), \dots, (q_n, L_n)\} \in \delta'((q, L), \sigma) \\ \Updownarrow \\ \{q_1, \dots, q_n\} \in \delta(q, \sigma) \wedge L_1 = L_2 = \cdots = L_n = Update(q, L),\end{gathered}$$

where

$$Update(q, L) = \begin{cases} (q, q_1, \dots, q_r), & \text{if } L = (q_1, \dots, q_r) \text{ and } r < k; \\ (q, q_1, \dots, q_{k-1}), & \text{if } L = (q_1, \dots, q_k). \end{cases}$$

Finally, we let

$$F' =: \{(q, L) : L \cap F \neq \emptyset\}.$$

We prove that $B$ (with a Büchi acceptance) is equivalent to $B'$ (with a co-Büchi acceptance) over simple graphs. First, suppose $B$ has a winning strategy $\Delta$ over a simple graph $G$. We define a strategy $\Delta'$ for Duplicator on the game of $B'$ over $G$ in such a way that any play of $\Delta'$, ending with a move of Spoiler,

$$((q_0, L_0), v_0)m_1((q_1, L_1), v_1) \dots m_n((q_n, L_n), v_n),$$

leaves a $Q \times V$-*trace* which is a play of $\Delta$,

$$(q_0, v_0)m_1'(q_1, v_1) \dots m_n'(q_n, v_n),$$

(where $m_i' = \{(q, v) : \exists L((q, L), v) \in m_i\}$).

The move of $\Delta'$ from the position $((q_n, L_n), v_n)$ is defined as follows: we consider the strategy $\Delta$ and the marking $m \in \delta(q_n, \lambda(v_n))$ it suggests in answer to the pair $(q_n, v_n)$; then $\Delta'$ chooses the marking

$$m' = \{(q', Update(q_n, L_n)), v') : (q', v') \in m\} \in \delta'(((q_n, L_n), \lambda(v))).$$

Following the above claim we know that from a certain point the play of $\Delta$ will meet a final step every $k$ Spoiler's steps. This in turn means that the play of $\Delta'$, after a finite number of steps, always meets final points. Hence $\Delta'$ is winning for Duplicator over $G$ (with a co-Büchi condition).

Vice versa, if $\Delta'$ is a winning strategy for Duplicator over the game of $A'$ over $G$, we define a strategy $\Delta$ in such a way that to any play of $\Delta$, ending with a move of Spoiler,

$$(q_0, v_0)m_1(q_1, v_1) \dots m_n(q_n, v_n),$$

corresponds a play of $\Delta'$

$$((q_0, \epsilon), v_0)m_1'((q_1, L_1), v_1) \dots m_n'((q_n, L_n), v_n).$$

The move of $\Delta$ from the position $(q_n, v_n)$ is defined as follows: we consider the strategy $\Delta'$ and the marking $m_n' \in \delta'((q_n, L_n), \lambda(v_n))$ it suggests in answer to the pair $((q_n, L_n), v_n)$; then $\Delta$ chooses the marking

$$m = \{(q', v') : \exists L'((q', L')v') \in m'\} \in \delta'((q_n, \lambda(v_n)).$$

Since the moves of $\Delta'$ are correct and $\Delta'$ is winning with a co-Büchi condition, we know that $L_{i+1} = Update(q_i, L_i)$, and there exists $j$ such that $L_i \cap F \neq \emptyset$ for all $i > j$. This implies that any infinite play of $\Delta$ will contain infinitely many final states and hence $\Delta$ is winning for Duplicator with a Büchi condition. □

**Corollary 5.5.** *The $\mu$-calculus collapses to the level $\Sigma_2 \cap \Pi_2$ of the alternation hierarchy over the class of finite simple graphs.*

**Proof.** Using Lemma 5.4 and the correspondence between Büchi and co-Büchi automata and the class of $\Pi_2$, $\Sigma_2$ formulas, respectively, we see that the class of $\Pi_2$-formulas is contained in the class $\Sigma_2$ over simple graphs. It follows that the class $\Sigma_2 \cap \Pi_2$ is closed under negation, propositional connectives, modal operators, and fixed points, and hence it coincides with the whole $\mu$-calculus. □

## 6. Conclusions

In this paper we are concerned with the expressive power of the modal $\mu$-calculus over transitive, finite transitive, and finite and simple frames. Over this last class we prove the collapse of the $\mu$-calculus to the level $\Sigma_2 \cap \Pi_2$ of the alternation hierarchy. This result is strengthened over transitive frames to the alternation free fragment, while over finite transitive frames we prove that the $\mu$-calculus is first order definable. The investigation carried out in this paper can be extended to other interesting classes of frames. In particular, we mention the class of frames of bounded tree width, which will be considered in a future work.

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