How to construct equivalent differential systems

V.I. Mironenko *, V.V. Mironenko

Mathematical Department, F. Scorina Gomel State University, Belarus

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ABSTRACT

For two differential systems, an algorithm which permits us to say whether reflecting functions of these systems coincide or not is given. It allows studying qualitative properties of these two systems simultaneously.

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1. Introduction

In this work we deal with differential systems of the form

\[
\frac{dx}{dt} = X(t, x), \quad t \in \mathbb{R}, x \in D \subset \mathbb{R}^n.
\]

We assume that all systems of this kind in this work have continuously differentiable right-hand sides. Under this assumption the system (1) has a general solution in the Cauchy form \( x = \psi(t; t_0, x_0) \) (see [1–3]). As a rule we cannot integrate system (1) by quadrature and study properties of the solution directly. In this case we have to look for other methods of studying system (1). In [4,5] there was elaborated the method of the reflecting function which gives us an opportunity to do this. The reflecting function (RF) for the system (1) is defined in some region near the hyperplane \( t = 0 \) by the formula \( F(t, x) := \psi(-t; t, x) \). If system (1) is \( 2\omega \)-periodic with respect to \( t \), then \( F(-\omega, x) \) is a period \([ -\omega, \omega \] transformation (or sometimes Poincaré map [6]). Therefore a solution that can be extended to \([ -\omega, \omega \], \phi(t; \omega, x_0)\), is \( 2\omega \)-periodic if and only if \( F(-\omega, x_0) = x_0 \).

The RF \( F(t, x) \) of system (1) can be found sometimes even for the case where the system (1) cannot be integrated by quadrature. For example, every system (1) for which \( X(-t, x) \equiv -X(t, x) \) has an RF given by the formula \( F(t, x) \equiv x \). We know this due to the following property:

A differentiable function \( F(t, x) \) is the RF of system (1) if and only if the following basic relation:

\[
\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} X(t, x) + X(-t, F) = 0, \quad F(0, x) \equiv x
\]

holds.

So if we can find the solution of the basic relation (2), then we can find the initial data for periodic solutions of (1) and investigate the character of the stability for those solutions.

* Corresponding author.
E-mail address: vmironenko@tut.by (V.I. Mironenko).

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We know also that the RFs of different systems can coincide. We call two systems whose RFs $F_1 : D_1 \to R^n$ and $F_2 : D_2 \to R^n$ coincide in $D_1 \cap D_2$ equivalent systems.

The class of all equivalent systems, and such systems only, we can write down in the form

$$\frac{dx}{dt} = -0, 5 \frac{\partial F}{\partial x}(-t, F) \frac{\partial F}{\partial t}(t, x) + \frac{\partial F}{\partial x}(-t, F)R(t, x) - R(-t, F),$$

where $R = (R_1, \ldots, R_n)^T$ is any differentiable vector function.

If the system

$$\frac{dy}{dt} = Y(t, y), \quad t \in R, y \in D_1 \subset R^n,$$

has the same RF $F(t, y)$ as the system (1), then $Y(0, x) \equiv X(0, x)$ and system

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}X(t, x) + X(-t, F) = 0,
\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}Y(t, x) + Y(-t, F) = 0,$$

$$F(0, x) = x$$

is compatible.

To check whether system (4) is compatible or not we can use the Frobenius theorem [1]. Doing this in practice, however, is a very hard task.

If we cannot solve the system (1) and cannot solve the problem (2), then it is good enough to construct any system (3) which is equivalent to (1). To do this, sometimes we can use:

**Theorem 1.** Let the vector functions $\Delta_1(t, x), \ldots, \Delta_m(t, x)$ be solutions of the equation

$$\frac{\partial \Delta}{\partial t} + \frac{\partial \Delta}{\partial x}X(t, x) - \frac{\partial X(t, x)}{\partial x} \Delta = 0$$

and $\alpha_1(t), \ldots, \alpha_m(t)$ be any scalar continuous odd functions.

Then every system of the form

$$\frac{dx}{dt} = X(t, x) + \alpha_1(t)\Delta_1(t, x) + \cdots + \alpha_m(t)\Delta_m(t, x)$$

is equivalent to system (1) (here $m$ is any natural number or even $m = \infty$).

The proof of the theorem is given in [7] and it is contained in [5] also.

So if we find some solutions of Eq. (5), we can construct system (6), which has the same RF as system (1).

If, for example, in system (1) $X(t, x) \equiv 0$, then Eq. (5) is $\frac{\partial \Delta}{\partial t} = 0$ and any function $\Delta_k(x)$ is a solution of the equation. For this reason all systems of the form $\tilde{x} = \sum \alpha_k(t)\Delta_k(x)$ are equivalent to $\tilde{x} \equiv 0$.

Usually, however, we cannot solve Eq. (5). In this case we arrive at the following problem: How do we construct any system equivalent to system (1)?

In [4] and [5] it was proved that if system (1) is equivalent to an autonomous system, then this autonomous system is

$$\frac{dx}{dt} = X(0, x). \quad (7)$$

It is very simple to construct system (7) but, usually (except for the case where we can solve the system (7) and find its RF), it is not so simple to prove the equivalence of (1) and (7).

Other results concerning the RF and its applications can be found in works of Zhengxin Zhou, E.V. Musafirov and others.

### 2. The main result

The main result of this work is an algorithm for providing an answer to the following question: When does a given system (3) have the form (6) with functions $\alpha_i(t)$ differentiable sufficiently many times and, therefore, become equivalent to another given system (1)?

So we are given systems (1) and (3) and we choose a natural number $m$. Here is the algorithm:

1. Construct the functions

$$\Delta^{(0)}(t, x) := Y(t, x) - X(t, x), \quad \Delta^{(i+1)}(t, x) := \frac{\partial \Delta^{(i)}}{\partial t} + \frac{\partial \Delta^{(i)}}{\partial x}X - \frac{\partial X}{\partial x} \Delta^{(i)}, \quad i = 0; m.$$
2. Now we are looking for scalar functions $b_0(t), \ldots, b_{m-1}(t)$ for which the identity
\begin{equation}
 b_0(t)\Delta^{(0)}(t, x) + \cdots + b_{m-1}(t)\Delta^{(m-1)}(t, x) + \Delta^{(m)}(t, x) \equiv 0 \tag{8}
\end{equation}
holds true. For this reason we write down the system
\begin{equation}
 b_0(t)\Delta^{(0)}(t, x) + \cdots + b_{m-1}(t)\Delta^{(m-1)}(t, x) + \Delta^{(m)}(t, x) = 0, \quad r = 1; s,
\end{equation}
where $x_i = (x_1, \ldots, x_{m})$ are arbitrary points and $s$ is chosen so that $sn > m$.

3. Solve the system (8). If such functions $b_0(t), \ldots, b_{m-1}(t)$ do not exist, then system (3) does not have the form of (6) and we choose another $m$ greater than the previous one. Suppose, however, that we find functions $b_0(t), \ldots, b_{m-1}(t)$ for which the identity (8) holds.

4. Then we are looking for odd scalar functions $\alpha_1(t), \ldots, \alpha_m(t)$ for which the following identities hold:
\begin{equation}
 b_0(t)\alpha(t) + b_1(t)\dot{\alpha}(t) + \cdots + b_{m-1}(t)\alpha^{(m-1)}(t) + \alpha^{(m)}(t) \equiv 0,
\end{equation}
\begin{equation}
 -b_0(-t)\alpha(t) + b_1(-t)\dot{\alpha}(t) + \cdots + (-1)^mb_{m-1}(-t)\alpha^{(m-1)}(t) + (-1)^{m+1}\alpha^{(m)}(t) \equiv 0.
\end{equation}
Here $\alpha^{(m)}(t)$ is the derivative of order $m$.

5. Now we can find the required functions $\Delta_1(t, x), \ldots, \Delta_m(t, x)$ from the system
\begin{equation}
 \alpha_1(t)\Delta_1(t, x) + \cdots + \alpha_m(t)\Delta_m(t, x) = \Delta^{(0)}(t, x),
\end{equation}
\begin{equation}
 \dot{\alpha}_1(t)\Delta_1(t, x) + \cdots + \dot{\alpha}_m(t)\Delta_m(t, x) = \Delta^{(1)}(t, x),
\end{equation}
\begin{equation}
 \vdots 
\end{equation}
\begin{equation}
 \alpha^{(m)}(t)\Delta_1(t, x) + \cdots + \alpha^{(m)}(t)\Delta_m(t, x) = \Delta^{(m)}(t, x).
\end{equation}

6. If the functions obtained, $\Delta_1(t, x), \ldots, \Delta_m(t, x)$, satisfy the identity (5), then the system (3) really has the form of (6) and is equivalent to the system (1).

The proof of the main result is based on:

**Lemma.** Let $\Delta_1(t, x), \ldots, \Delta_m(t, x)$ be solutions of (5) for functions $\alpha_i(t)$ differentiable any number of times. Then the identities (9) are true.

To prove the lemma we have to calculate $\Delta^{(i)}(t, x)$.

So we have
\begin{equation}
 \Delta^{(1)} := \frac{\partial \Delta^{(0)}}{\partial t} + \frac{\partial \Delta^{(0)}}{\partial x} - \frac{\partial X}{\partial x} \Delta^{(0)}
\end{equation}
\begin{equation}
 = \frac{\partial}{\partial t} \sum_{i=1}^{m} \alpha_i(t)\Delta_i + \frac{\partial}{\partial x} \sum_{i=1}^{m} \alpha_i(t)\Delta_iX - \frac{\partial X}{\partial x} \sum_{i=1}^{m} \alpha_i(t)\Delta_i
\end{equation}
\begin{equation}
 = \dot{\alpha}_1\Delta_1 + \cdots + \dot{\alpha}_m\Delta_m + \sum_{i=1}^{m} \alpha_i(t) \left[ \frac{\partial \Delta_i}{\partial t} + \frac{\partial \Delta_i}{\partial x}X - \frac{\partial X}{\partial x} \Delta_i \right]
\end{equation}
\begin{equation}
 = \dot{\alpha}_1\Delta_1 + \cdots + \dot{\alpha}_m\Delta_m \Delta_m.
\end{equation}

So we have the second identity in (9). We prove the lemma by doing such calculations subsequently.

**Theorem 2.** Suppose that system (3) has the form (6); then for functions $\Delta^{(0)}(t, x), \ldots, \Delta^{(m)}(t, x)$ there exist scalar functions $b_0(t), \ldots, b_m(t)$ for which
\begin{equation}
 b_0(t)\Delta^{(0)}(t, x) + \cdots + b_m(t)\Delta^{(m)}(t, x) \equiv 0. \tag{10}
\end{equation}

**Proof.** Functions $\alpha_1(t), \ldots, \alpha_m(t)$ are solutions of the differential equation
\begin{equation}
 \begin{bmatrix}
 z & \dot{z} & \ldots & z^{(m)} \\
 \alpha_1 & \dot{\alpha}_1 & \ldots & \alpha^{(1)}_1 \\
 \vdots & \vdots & \ddots & \vdots \\
 \alpha_m & \dot{\alpha}_m & \ldots & \alpha^{(m)}_m 
\end{bmatrix} = 0,
\end{equation}
which we can write down in the form $b_0z + b_1\dot{z} + \cdots + b_mz^{(m)} = 0$. Then due to the lemma we shall have
\begin{equation}
 b_0\Delta^{(0)} + b_1\Delta^{(1)} + \cdots + b_m\Delta^{(m)} = b_0 \sum_{i=1}^{m} \alpha_i\Delta_i + b_1 \sum_{i=1}^{m} \dot{\alpha}_i\Delta_i + \cdots + b_m \alpha^{(m)} \Delta_i
\end{equation}
\begin{equation}
 = \sum_{i=1}^{m} (b_0\alpha_i + b_1\dot{\alpha}_i + \cdots + b_m\alpha^{(m)}_i) \Delta_i \equiv 0.
\end{equation}

The proof is completed. □
The identity (8) can be obtained from the identity (10) in Theorem 2.

**Remark.** All statements of the work are true also for any restriction of the systems (1) and (3) to any integral manifold common to the systems (1) and (3). In this case (8) and (9) are true on these common integral manifolds. This means that from (9) we can get an equation for the manifold.

**Example.** For the system
\[
\begin{align*}
\dot{x} &= (y - x)(1 - \sin(z - t) \sin t), \\
\dot{y} &= (x + y)(-1 + \sin(z - t) \sin t), \\
\dot{z} &= 1 + e^{2t}(x^2 + y^2) \sin 3t
\end{align*}
\]
we have (if we put \( t = 0 \))
\[
\begin{align*}
X &:= Y(0, x) = \begin{pmatrix} y - x \\ -x - y \\ 1 \end{pmatrix}, \\
\Delta^{(0)} &= \begin{pmatrix} (x - y) \sin(z - t) \sin t \\ (x + y) \sin(z - t) \sin t \\ e^{2t}(x^2 + y^2) \sin 3t \end{pmatrix}, \\
\Delta^{(1)} &= \begin{pmatrix} (x - y) \sin(z - t) \cos t \\ (x + y) \sin(z - t) \cos t \\ 3e^{2t}(x^2 + y^2) \sin 3t \end{pmatrix}, \\
\Delta^{(2)} &= \begin{pmatrix} -(x - y) \sin(z - t) \sin t \\ -(x + y) \sin(z - t) \sin t \\ -9e^{2t}(x^2 + y^2) \cos 3t \end{pmatrix}.
\end{align*}
\]
Therefore
\[
b_0(t)\Delta^{(0)} + b_1(t)\Delta^{(1)} + \Delta^{(2)} = \begin{pmatrix} (x - y) \sin(z - t)(b_0 \sin t + b_1 \cos t - \sin t) \\ (x + y) \sin(z - t)(b_0 \sin t + b_1 \cos t - \sin t) \\ e^{2t}(x^2 + y^2)(b_0 \sin 3t + 3b_1 \cos 3t - 9 \sin 3t) \end{pmatrix}.
\]
If we hope that the vector function obtained will be equal to zero, we have to put
\[
b_0 \sin t + b_1 \cos t = \sin t; \quad b_0 \sin 3t + 3b_1 \cos 3t = 9 \sin 3t.
\]
From this we get
\[
b_1 = \frac{-8 \sin t \sin 3t}{\cos t \sin 3t + 3 \sin t \cos 3t}, \quad b_0 = 1 + \frac{8 \cos t \sin 3t}{\cos t \sin 3t + 3 \sin t \cos 3t}.
\]
Therefore for the calculation of \( \alpha \) we have an equation
\[
\left[ 1 + \frac{8 \cos t \sin 3t}{\cos t \sin 3t + 3 \sin t \cos 3t} \right] \alpha - \frac{8 \sin t \sin 3t}{\cos t \sin 3t + \sin t \cos 3t} \dot{\alpha} + \ddot{\alpha} = 0
\]
from which we get \( \alpha_1 = \sin t \) and \( \alpha_2 = \sin 3t. \) Then from the system
\[
\Delta_1 \sin t + \Delta_2 \cos 3t = \Delta^{(0)},
\]
\[
\Delta_1 \cos t + \Delta_2 3 \cos 3t = \Delta^{(1)}
\]
we obtain
\[
\Delta_1 = \begin{pmatrix} (x - y) \sin(z - t) \\ (x + y) \sin(z - t) \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

**References**