Extinction of Superdiffusions and Semilinear Partial Differential Equations*

E. B. Dynkin
Department of Mathematics, Cornell University, Ithaca, New York 14853-7901
E-mail: ebd1@cornell.edu

and

E. Kuznetsov
Department of Mathematics, University of Colorado, Boulder, Boulder, Colorado 80309-0395
E-mail: sek@euclid.colorado.edu

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A superdiffusion is a measure-valued branching process associated with a nonlinear operator $L - \psi(u)$ where $L$ is a second order elliptic differential operator and $\psi$ is a function from $\mathbb{R}^d \times \mathbb{R}_+$ to $\mathbb{R}_+$. In the case $L 1 0$ (the so-called subcritical case), the expectation of the total mass does not increase and the mass vanishes at a finite time with probability 1. Most results on connections between the superdiffusion and differential equations involving $L - \psi(u)$ were obtained for the subcritical case. In the present paper, we assume only that $L 1$ is a bounded function. In this more general setting, the probability of extinction can be smaller than 1, and we show that this happens if and only if there exists a strictly positive solution of the equation $L - \psi(u) = 0$ in $\mathbb{R}^d$. We also establish a relation between the probability of extinction in a domain $D$ and strictly positive solutions of equation $L - \psi(u) = 0$ in $D$ that are equal to 0 on the boundary of $D$. We call such solutions special. For the equation $(A + c) u = u^p$, where $A$ is the Beltrami–Laplace operator on a complete Riemannian manifold $E$, strictly positive solutions in $E$ were studied in connection with a geometrical problem: Which two functions represent scalar curvatures of two Riemannian metrics related by a conformal mapping (see [1] and references there)?

Key Words: Special solutions; diffusions; multiplicative functionals; regular points on the boundary; extinction of superdiffusions.

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1. INTRODUCTION

1.1. Let

\[ Lu(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x) u(x) \]  

(1.1)

be a uniformly elliptic second order differential operator in \( \mathbb{R}^d \) with bounded locally Hölder continuous coefficients and let \( D \) be an open set in \( \mathbb{R}^d \). We investigate positive solutions of the equation\(^1\)

\[ Lu = \psi(u) \quad \text{in } D \]  

(1.2)

for a class \( \Psi \) of functions \( \psi \) which includes

\[ \psi(x, u) = k(x) u^\alpha, \quad \alpha > 1 \]  

(1.3)

under mild conditions on the coefficient \( k(x) \). [The class \( \Psi \) is defined in Section 1.4.] Equation (1.2) can be written in the form

\[ (L^0 + c) u = \psi(u) \quad \text{in } D, \]  

(1.4)

where \( L^0 \) is an operator without zero order term. We use as a tool a diffusion \( \xi = (\xi_t, \Pi_x) \) with generator \( L^0 \) and its multiplicative functional

\[ H_r^\ast = \exp \left[ \int_r^t c(s, \xi_s) \, ds \right]. \]  

(1.5)

We use also a more sophisticated tool—\( (L, \psi) \) superdiffusion \( X \). This is a measure valued process in \( \mathbb{R}^d \) associated with the nonlinear operator \( Lu - \psi(u) \). Such a process exists for a subclass \( \Psi_0 \) of \( \Psi \) which contains the functions (1.3) with \( 1 < \alpha \leq 2 \).\(^2\)

In a part of the paper, we assume that \( L^0 \) can be represented in the divergence form\(^3\)

\[ L^0 u(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left[ a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right]. \]  

(1.6)

\(^1\) \( \psi(u) \) is an abbreviation for \( \psi(x, u(x)) \).

\(^2\) See the definition of \( \Psi_0 \) in Section 1.4.

\(^3\) Only minor modifications are needed to cover the case of the Beltrami–Laplace operator on a Riemannian manifold.
1.2. In Section 2, we give an exposition of earlier results that we need in the present paper, including relations between diffusions and linear PDEs and the results of [6] on semilinear parabolic equation

$$u^* + Lu = \psi(u) \quad \text{in } Q,$$

(1.7)

where $Q$ is an open subset of $(0, \infty) \times \mathbb{R}^d$.

The principal new results are presented in Section 3. Suppose that $D$ is a regular domain in $\mathbb{R}^d$ (all domains with smooth boundaries are regular). We say that a strictly positive solution $u$ of (1.2) is special if $u = 0$ on $\partial D$ and we say a domain $D$ is special if there exists a special solution in $D$. In Section 3 we assume that $\psi \in \Psi_\alpha$ and we describe the class of special domains in terms of the extinction time $\sigma_D$ of $(L, \psi)$-superdiffusion $X$ in $D$: a regular domain $D$ is special if and only if $P_x\{\sigma_D < \infty\} < 1$ for all $x \in D$; moreover, the maximal special solution is given by the formula

$$u(x) = -\log P_x\{\sigma_D < \infty\}$$

(1.8)

(Theorem 3.4). We get this as an implication of more general results on solutions of a boundary value problem

$$Lu = \psi(u) \quad \text{in } D, \quad u = f \quad \text{on } O,$$

(1.9)

where $O$ is a relatively open subset of $\partial D$. Assuming that $f$ is continuous on $O$, we prove that the problem (1.9) has a minimal solution $v$ (which we express through the exit measure $X_D$ of $X$ from $D$) and a maximal solution $w$ (which we describe in terms of $X_D$, $\sigma_D$ and the range $\mathbb{R}^D$ of $X$ in $D$). Namely,

$$v(x) = -\log P_x e^{-\langle f, X_D \rangle},$$

$$w(x) = -\log P_x e^{-\langle f, X_D \rangle} 1_{(\sigma_D \in (D \cup O) \text{ and } \sigma_D < \infty)},$$

(1.10)

where $f_O = f$ on $O$, $f_O = 0$ on the complement of $O$ and $\langle F, M \rangle$ stands for $\int F dM$ (Theorem 3.3).

We deduce these results from analogous results for the parabolic Eq. (1.7) [see Theorems 3.1 and 3.2]. A substantial advantage of the parabolic setting is that the comparison principle for (1.7) (Theorem 2.8) holds in every bounded domain, which is not true for (1.2).

If $u$ is a function in $D$ and $z \in \partial D$, then writing “$u(z) = A$” means “$u(x) \to A$ as $x \in D$ tends to $z$.”
If \(L^0\) has the divergence form (1.6), then the maximal special solution in \(D\) can be described by the formula
\[ u(x) = -\log P_x\{X_0(O) < \infty\}, \tag{1.11} \]
where \(O\) is an arbitrary nonempty relatively open subset of \(\partial D\) (Theorem 4.9.)

1.3. A natural question arises:

(A) For which pairs \((L, \psi)\) is a given domain \(D\) special?

This problem was investigated by Brandolini, et al. [1] for \(\psi\) of the form (1.3) and the Beltrami-Laplace operator \(L = A\) on a complete Riemannian manifold. By adapting their arguments to our setting, we get: If \(L^0\) has the form (1.6), then the answer to the question (A) depends only on \(L\) but not on \(\psi\). More precisely, a bounded smooth\(^5\) domain \(D\) is special if and only if the principal eigenvalue \(\lambda_1(D)\) of \(-L\) in \(D\) is negative (Theorems 4.2 and 4.6).

[Recall that \(\lambda_1 = \lambda_1(D)\) is the principal eigenvalue of \(L\) in \(D\) if and only if there exists a function \(u > 0\) such that \(u = 0\) on \(\partial D\) and \(-Lu = \lambda_1 u\) in \(D\) (see, e.g., [15], p. 22).]

The proof that \(D\) is special if \(\lambda_1(D) < 0\) (Theorem 4.2) is valid for a general operator \(L\). However, the proof of the converse Theorem 4.6 is based on the divergence formula and it is applicable only if \(L^0\) has a divergence form (1.6). [The key role is played by the uniqueness Theorems 4.4 and 4.5.]

The extinction time \(\sigma_D\) is monotone increasing in \(D\) (Lemma 3.6) and, by formula (1.8), an unbounded domain is special if it contains a special bounded domain. We do not know if every special domain \(D\) contains a bounded special domain. This is proved only for operators in the divergence form (1.6) and if there is a square integrable special solution in \(D\) (Theorem 4.8).

Theorem 4.3 describes \(\lambda_1(D)\) in terms of multiplicative functionals of \(\xi\) and the first exit time \(\tau\) of \(\xi\) from \(D\). The theorem implies: \(\lambda_1(D) < 0\) if and only if \(II\exp \left\{ \int_0^\tau (c + \lambda)(\xi_s) \, ds \right\} = \infty\) for some \(\lambda < 0\), \(x \in D\).

1.4. We put \(\psi \in \mathcal{U}\) if:

1.4.A. \(\psi\) is continuously differentiable in \(x, u\).

1.4.B. \(\psi(x, u) \geq 0\) for all \(x, u\) and \(\psi(x, 0) = 0\) for all \(x\).

1.4.C. For every \(x\), \(\psi(x, u)\) is convex with respect to \(u\).

\(^5\) We call smooth domains of class \(C^{1,\alpha}\) (see the definition in [10, Section 6.2]).
1.4. D. \( \sup_{x \in D} \psi(x,u)/u \to 0 \) as \( u \to 0 \).

1.4. E. For all \( x \in \mathbb{R}^d, u \geq 0 \), \( \psi(x,u) \geq \varphi(u) \) for a function \( \varphi \) with the following properties:

1.4.1. \( \varphi \) is convex and \( \varphi(0+) = 0 \); \( \varphi(u) > 0 \) for \( u > 0 \).

1.4.2. \( \int_{\mathbb{R}} dx [ \int_0^1 \varphi(u) \, du ]^{-1/2} < \infty \) for some \( N > 0 \).

If \( \psi \) is of the form (1.3), then 1.4.A–1.4.C hold if \( x > 1, k \geq 0 \) is continuously differentiable, 1.4.E is satisfied if \( \inf_x k(x) > 0 \) and 1.4.D holds if \( \sup_x k(x) < \infty \).

An \( (L, \psi) \)-superdiffusion exists (see, e.g., [4] or [5]) for

\[
\psi(x,u) = b(x) \, u^2 + \int_0^{\infty} \left[ e^{-\lambda u} - 1 + \lambda u \right] \ell(x, d\lambda) \tag{1.12}
\]

if \( b \) is a positive bounded function and the kernel \( \ell \) satisfies the condition

\[
\sup_x \int_0^{\infty} (\lambda \wedge \lambda^2) \ell(x, d\lambda) < \infty.
\]

We denote by \( \mathcal{P}_0 \) the set of all functions of the form (1.12) that belong to class \( \mathcal{P} \).

1.5. Sheu [16], [17] studied the Eq. (1.2) for \( L = \frac{1}{2} A, c = \text{const.}, \) \( x \) independent of \( x \) and \( D = \mathbb{R}^d \). Put \( \sigma_D = \sigma, \mathcal{R}_D = \mathcal{R} \) for \( D = \mathbb{R}^d \) and let \( z_0 = \sup \{ u : \psi(u) \leq cu \} \). If \( \psi \) satisfies 1.4.2, then \( z_0 \) is finite; it is strictly positive if and only if \( c > 0 \). Clearly, \( u = z_0 \) is a solution of (1.2) in \( \mathbb{R}^d \). Sheu established that \( z_0 \) admits a representation

\[
z_0 = -\log P_x \{ D \text{ is compact} \} = -\log P_x \{ \sigma < \infty \}
\]

which is a special case of our Theorem 4.10. Sheu also considered \( \psi \) which do not satisfy condition 1.4.2; this part of his results is beyond the scope of the present paper.

2. DIFFUSIONS AND THEIR APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

In this section, we define a diffusion \( \xi \) by using the fundamental solution of the equation

\[
\hat{
\dot{u}} + Lu = 0. \tag{2.1}
\]
Then we describe connections between $\xi$ and linear equations involving the operator $L$. We conclude with some applications of $\xi$ to the semilinear Eq. (1.7). Most of the results stated in this section are proved in [6].

2.1. Notation. As usual $C(T)$ stand for the set of all continuous functions on $T$. Let $D$ be an open subset of $\mathbb{R}^d$. Writing $u \in C^k(D)$ means that all partials of $u$ of order $\leq k$ belong to $C(D)$. For an arbitrary compact subset $K$ of $\mathbb{R}^d$, we denote $C^{0,\alpha}(K)$ the set of all functions on $K$ that are Hölder continuous with exponent $\alpha$. A function $u$ on an open set $D$ belongs to class $C^{0,\alpha}(D)$ if all partials of $u$ of order $\leq k$ belong to $C^{0,\alpha}(K)$ for every compact subset $K$ of $D$.

We also consider open subsets $Q$ of the space $S = (0, \infty) \times \mathbb{R}^d$. Our primary interest is in cylinders $(0, t) \times D$ which we denote $D^{s,t}$.

Put $u \in C^1(Q)$ if $u$, $\partial u/\partial t$, and $\partial u/\partial x_i$, $i = 1, \ldots, d$ belong to $C(Q)$; and put $u \in C^2(Q)$ if $u$, $\partial u/\partial t$, and $\partial^2 u/\partial x_i \partial x_j$, $i, j = 1, \ldots, d$ belong to $C(Q)$. For every $T \subset S$, we denote by $C^1(T)$ the class of all continuous functions $f(r, x)$ on $T$ which satisfy, for every $t > 0$, Hölder's condition in $x$ uniformly in $r \in (0, t)$:

$$|f(r, x) - f(r, y)| \leq k \|x - y\|^\alpha \quad \text{for all} \quad (r, x), (r, y) \in T, \quad 0 < r < t.$$ 

Let $S_{<t} : (0, t) \times \mathbb{R}^d$. We put $B_{<t} = B \cap S_{<t}$ for every $B \subset S$. If $\mu$ is a measure on $S$, then $\mu_{<t}$ means the restriction of $\mu$ to $S_{<t}$. Notation $S_{x,t}$, $B_{x,t}$, $\mu_{x,t}$, have an analogous meaning.

2.2. Fundamental Solution. The fundamental solution of Eq. (2.1) is a continuous function $p(r, x; t, y)$ on the set $\{0 < r < t, x, y \in \mathbb{R}^d\}$ with the property: If $\varphi \in C(\mathbb{R}^d)$ is bounded, then

$$u(r, x) = \int_{\mathbb{R}^d} p(r, x; t, y) \varphi(y) \, dy$$

is a unique bounded solution of the problem

$$\begin{align*}
\dot{u} + Lu &= 0 \quad \text{in} \quad S_{<t}, \\
u &= \varphi \quad \text{on} \quad \{t\} \times \mathbb{R}^d.
\end{align*}$$

Theorems 2.1, 2.2, 2.3, and 2.4 follow, respectively, from Theorems 2.2, 2.4, 2.5, and 2.7 in [6]. Properties of $p_{D}$, including (2.8), are proved in Section 2.6 of [6], and Lemma 2.1 coincides with Lemma 2.4 there. Theorem 2.8 is identical to Theorem 4.1 in [6]; Theorem 2.9 follows from Theorem 3.3 and Corollary 3.4; Theorem 2.10 is an implication of Theorem 3.2 and Remark to this theorem; Theorem 2.12 follows from Theorem 4.2 and Theorem 2.13 follows from Theorems 4.3 and 4.4.

Writing $u = \varphi$ at $z \in \partial Q$ means $u(z) \to \varphi(z)$ as $z \in Q$ tends to $z$. 

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The existence and the uniqueness of a fundamental solution is proved under broad conditions on coefficients of $L$, for instance, in [9] and in [11]. In particular, these conditions hold if $a_{ij}, b_i, c$ do not depend on time, belong to class $C^0(\mathbb{R}^d)$ and are bounded.\(^8\) Function $p(r, x; t, y) = p_{r,t}(x, y)$ depends only on the difference $t - r$.

2.3. Diffusion. Put

$$L^0u = Lu - cu$$

and denote as $p^0$ the fundamental solution of the equation $\dot{u} + L^0u = 0$. There exists a continuous Markov process $\zeta(t, \Pi, x)$ in $\mathbb{R}^d$ with transition function $p^0(r, x; t, dy) = p^0(r, x; t, y) dy$ (see, e.g., [3]). We call it $L^0$-diffusion. To every bounded continuous function $c$ on $S$ there corresponds a multiplicative functional $H$ given by formula (1.5).

For every bounded continuous $\varphi$,

$$u(r, x) = \Pi_{r, x} H_r \varphi(\xi_r) = \int_{\mathbb{R}^d} p(r, x; t, y) \varphi(y) dy$$

is a unique bounded solution of the problem (2.2).

The first exit time from an open subset $Q$ of $S$ is defined by the formula

$$\tau = \inf\{t: (t, \xi_t) \not\in Q\}.$$

Note that $\Pi_{r, x}\{\tau \geq r\} = 1$ for all $(r, x)$ and $\Pi_{r, x}\{\tau = r\} = 1$ for all $(r, x) \notin Q$.

Put

$$K_Q f(r, x) = \Pi_{r, x} H_r f(\tau, \xi_r),$$

(2.5)

where $\xi$ is an $L^0$-diffusion and $H_r$ is given by (1.5).

**Theorem 2.1.** If $Q$ is bounded and if $f$ is a bounded Borel function on $\partial Q$, then $u = K_Q f$ is a solution of the equation

$$\dot{u} + Lu = 0 \quad \text{in } Q.$$  

(2.6)

2.4. Parts of Diffusion. For an arbitrary open subset $Q$ of $S$, we put

$$p_Q(r, x; t, y) = p(r, x; t, y) - \Pi_{r, x} H_r p(\tau, \xi_r; t, y) \quad \text{for } (r, x), (t, y) \in Q.$$  

(2.7)

\(^8\) No restriction is imposed on the sign of $c$. 

where $\tau$ is the first exit time from $Q$. \footnote{We set $p(r, x; t, y) = 0$ for $r \geq t$.} For every Borel function $f \geq 0$ on $Q$,

\[ \Pi_{\tau, x} \mathbb{1}_{\{\tau < t\}} H_t f(t, \xi_t) = \int_{Q_t} p_{Q_t}(r, x; t, y) f(t, y) \, dy \quad \text{for all } (r, x) \in Q. \tag{2.8} \]

where $Q_t = \{ x : (t, x) \in Q \}$ (see Theorem 2.3 in [6]). $p_{Q_t}(r, x; t, y) \geq 0$ for all $r < t$, $x, y \in Q$ and $p_{Q_t}(r, x; t, y) = p_{Q_t}(x, y)$ depends only on the difference $t - r$ for a cylinder $Q = (0, \infty) \times D$.

Function $p_R(r, x, t, y)$ is the transition density of the part of $\xi$ in $Q$ (obtained by the killing $\xi$ at the first exit time from $Q$).

Operator $G_{Q_t}$ is defined on positive Borel functions by the formula

\[ G_{Q_t}(r, x) = \int_{Q_t}^\infty \int_{Q_t} p_{Q_t}(r, x; t, y) \rho(t, y) \, dy \, dt. \tag{2.9} \]

It follows from (2.8) that

\[ G_{Q_t}(r, x) = \Pi_{\tau, x} \int_r^\tau H_t \rho(t, \xi_t) \, dt. \tag{2.10} \]

**Theorem 2.2.** If $Q$ and $\partial$ are bounded, then $u = G_{Q_t} \rho \in C^1(Q)$. If, in addition, $\rho \in \bar{C}(K)$ for all compact $K \subset Q$, then $u = G_{Q_t} \rho$ belongs to $C^2(Q)$ and it is a solution of the equation

\[ \dot{u} + Lu = -\rho \quad \text{in } Q. \tag{2.11} \]

2.5. **Total Subsets of $\partial Q$.** We say that a subset $\mathcal{F}$ of $\partial Q$ is **total** if, for all $(r, x) \in Q$,

\[ \Pi_{\tau, x} \{ (\tau, \xi) \notin \mathcal{F} \} = 0. \]

Note that $\partial Q$ is total if and only if $\Pi_{\tau, x} \{ \tau = \infty \} = 0$ for all $(r, x) \in Q$. This condition holds for every bounded $Q$. For a cylinder $D^{x,t} = (0, t) \times D$, the union of $(0, t) \times \partial D$ and $\{t\} \times D$ is total in $\partial D^{x,t}$.

**Theorem 2.3 (Improved Maximum Principle).** Suppose $Q$ is bounded and $\mathcal{F}$ is a total subset of $\partial Q$. If $v \in C(Q)$ is bounded above and satisfies the condition

\[ \dot{v} + Lv \geq 0 \quad \text{in } Q \tag{2.12} \]

\[ 9\text{ We set } p(r, x; t, y) = 0 \text{ for } r \geq t. \]
and if, for every \( \bar{z} \in \mathcal{F} \),
\[
\limsup_{z \to \bar{z}} v(z) \leq 0
\]
then \( v \leq 0 \) in \( Q \).

2.6. Regular Boundary Points and the first Boundary Value Problem. We say that a point \((t, a) \in \partial Q\) is regular and we write \((t, a) \in \partial_{\text{reg}} Q\) if, for every \( t' > t \),
\[
\Pi_{t, a}(s, \xi) \in Q \text{ for all } s \in (t, t') \}
= 0.
\]

**Theorem 2.4.** Suppose \( Q \) is bounded. If \((t, a) \in \partial_{\text{reg}} Q\) and if \( f \) is a bounded function on \( \partial Q \) that is continuous at \((t, a)\), then
\[
K_Q f(r, x) \to f(t, a) \quad \text{as} \quad (r, x) \to (t, a).
\]

If \( p \) is bounded, then
\[
G_Q p(r, x) \to 0 \quad \text{as} \quad (r, x) \to (t, a).
\]

We denote by \( \partial_{\text{reg}} Q \) the set of all interior (relative to \( \partial Q \)) points of \( \partial_{\text{reg}} Q \) and we say that \( Q \) is strongly regular if \( \partial_{\text{reg}} Q \) is a total subset of \( \partial Q \). Let \( \mathcal{O}(\partial Q) \) stand for the class of all relatively open subsets of \( \partial Q \). A set \( Q \) is strongly regular if and only if \( \partial_{\text{reg}} Q \) contains a total subset \( O \in \mathcal{O}(\partial Q) \).

**Lemma 2.1.** For every open set \( Q \) and every \( O \in \mathcal{O}(\partial Q) \) such that \( O \not\subseteq \partial_{\text{reg}} Q \), there exists a sequence of bounded strongly regular open sets \( Q_n \uparrow Q \) with the property
\[
Q_n \uparrow Q \cup O; \quad d(Q_n, Q \setminus Q_{n+1}) > 0.
\]

**Theorem 2.5.** If \( Q \) is bounded and if \( \mathcal{F} \) is a total subset of \( \partial Q \), then the problem
\[
\dot{u} + Lu = -p \quad \text{in } Q;
\]
\[
u = f \quad \text{on } \mathcal{F}
\]
has no more than one solution. If \( \mathcal{F} \subseteq \partial_{\text{reg}} Q \), \( p \in C^{0,\alpha}(Q) \) is bounded and if a bounded function \( f \) is continuous on \( \mathcal{F} \), then a solution of (2.17) is given by the formula
\[
u = G_Q p + K_Q f.
\]
The first part follows from Theorem 2.3 and the second part follows from Theorems 2.1, 2.2 and 2.4.

**Corollary 2.1.** Suppose that $Q$ and $\rho$ are bounded, $\rho \in C^{0,\infty}(Q)$ and $\mathcal{F} \subset \mathcal{F}_{\text{reg}}Q$ is total in $\partial Q$. If $\dot{u} + Lu = -\rho$ in $Q$ and if $u$ is continuous on $Q \cup \mathcal{F}$, then

$$u = G_Q \rho + K_Q u. \quad (2.19)$$

Indeed, the function in the right side of (2.19) satisfies (2.17) with $f = u$ and therefore it coincides with $u$.

2.7. **Operator $K^D$.** We set

$$K^D f(x) = \Pi_x H_x f(\xi_x), \quad (2.20)$$

where $\Pi_x = \Pi_{\infty,x}, H_x = H^x\tau$ and $\tau = \inf \{ t : \xi_t \notin D \}$ is the first exit time from $D$. [Note that $K^D = K_Q$ where $Q = (0, \infty) \times D$.]

**Theorem 2.6.** Suppose $f$ is a positive Borel function on $\partial D$. If $u = K^D f < \infty$ in $D$, then $u \in C^2(D)$ and

$$Lu = 0 \quad \text{in } D. \quad (2.21)$$

If $D$ is connected, then either $K^D f = \infty$ for all $x \in D$ or $K^D f < \infty$ for all $x \in D$.

(See, e.g., [2, Section 2.5.]) A point $a \in \partial D$ is regular if it can be touched from outside of $D$ by an open cone, in particular, if $\partial D$ is smooth in a neighborhood of $a$ [3, Theorem 13.8].

**Theorem 2.7.** Suppose that $a$ is a regular point of $\partial D$, $f$ is a bounded function on $\partial D$ that is continuous at $a$. If $K^D |f| < \infty$, then

$$K^D f(x) \rightarrow f(a) \quad \text{as } x \rightarrow a. \quad (2.22)$$

(This follows, e.g. from [3, Theorem 13.16.])

2.8. **Semilinear Parabolic Equations and the Corresponding Integral Equations.** In Section 2.8 we consider Eq. (1.7) in a bounded open set $Q$ and a boundary value problem

$$\dot{u} + Lu = \psi(u) \quad \text{in } Q; \quad u = f \quad \text{on } O. \quad (2.23)$$
(here \( O \) is a relatively open subset of \( \partial D \)). Under broad assumptions on \( O \) and \( f \) (see Theorem 2.9), (2.23) is equivalent to an integral equation
\[
\psi(u) = K_Q f.
\] (2.24)

**Theorem 2.8 (Comparison Principle).** Suppose that a function \( \psi(r, x; u) \) satisfies the condition
\[
\psi(r, x; u) \leq \psi(r, x; v) \quad \text{for all} \quad (r, x) \in Q, \quad 0 \leq u \leq v
\] (2.25)
and \( \mathcal{F} \) is a total subset of \( \partial Q \). Let \( u, v \in \mathcal{C}(Q) \) and let \( u - v \) be bounded above. If
\[
\dot{u} + Lu - \psi(u) \geq \dot{v} + Lv - \psi(v) \quad \text{in} \quad Q
\] (2.26)
and if, for every \( \tilde{z} \in \mathcal{F} \),
\[
\lim_{z \to \tilde{z}} \sup \left[ u(z) - v(z) \right] \leq 0
\] (2.27)
then \( u \leq v \) in \( Q \).

**Theorem 2.9.** Suppose that a positive bounded Borel function \( f \) on \( \partial Q \) is continuous on \( O \setminus \partial Q \). If \( u \geq 0 \) satisfies (2.24), then it is a solution of the boundary value problem (2.23).

The converse is true if \( O \) is a total subset of \( \partial Q \) and if \( u \) is bounded.

**Theorem 2.10.** Equation (2.24) has a unique solution \( u \) for every bounded positive Borel function \( f \). It can be obtained by the following monotone iteration scheme. Put
\[
Tv(r, x) = -\Pi_{r, x} \int_{r}^{\tau} e^{-\lambda(t-r)} \tilde{\psi}(v)(s, \xi_x) \, ds + \Pi_{r, x} e^{-\lambda(t-r)} f(\tau, \xi_x),
\] (2.28)
where
\[
\tilde{\psi}(v) = \psi(v) - (c + \lambda) v
\]
and suppose that a bounded function \( u_0 \) satisfies the condition
\[
T(u_0) \geq u_0,
\] (2.29)
If $\lambda$ is sufficiently large,\textsuperscript{10} then the sequence
\begin{equation}
\label{2.30}
u_n = T(u_{n-1}) \quad \text{for} \quad n \geq 1
\end{equation}
is bounded and monotone increasing and $u = \lim u_n$ satisfies (2.24).

**Theorem 2.11.** Suppose that $\varPhi \subset \partial \varOmega$ is total in $\partial \varOmega$ and a bounded positive function $f$ is continuous on $\varPhi$. If $u_0 \in C^2(\varOmega)$ is bounded and if
\begin{align*}
\dot{u}_0 + Lu_0 &\geq \psi(u_0) \quad \text{in} \quad \varOmega, \\
\dot{u}_0 &\leq f \quad \text{on} \quad \varPhi,
\end{align*}
then there exists a bounded solution $u$ of the problem (2.23) such that $u \geq u_0$. It can be obtained as the limit of the sequence (2.30).

**Proof.** The function defined in Theorem 2.10 is bigger than or equal to $u_0$. By Theorem 2.9, $u$ satisfies (2.23). Therefore we need only to check that (2.31) implies (2.29). Put $u_1 = T(u_0)$, $\nu = u_0 - u_1$. By applying Theorems 2.1 and 2.2 to $H_\nu = e^{-(\nu-c-\xi)}$, we get $\dot{u}_1 + (L^0 - \lambda)u_1 = \psi(u_0) - (c + \xi)u_0$. By (2.31), the right side is smaller than or equal to $u_0 + (L^0 - \lambda)u_0$. Therefore $\dot{\nu} + L^0 \nu - \lambda \nu \geq 0$. By Theorem 2.4, $u_1 = f \geq u_0$ on $\varPhi$. Hence, $\nu \leq 0$ on $\varPhi$. By Theorem 2.3, $\nu \leq 0$ in $\varOmega$.

**Remark 2.1.** Suppose $w \in C^2$ is bounded and
\begin{align*}
\dot{w} + Lw &\leq \psi(w) \quad \text{in} \quad \varOmega, \\
\dot{w} &\geq f \quad \text{on} \quad \varPhi,
\end{align*}
The arguments in the proof of Theorem 2.11 show that $Tw \leq w$. Suppose $u_0 \leq w$. If $\lambda$ is sufficiently large, then, for every $n$ $T^n u_0 \leq T^n w \leq w$ (this follows from part (a) of Theorem 3.2 in [6]). Hence, $u = \lim T^n u_0 \leq w$.

**Theorem 2.12.** Suppose that $\psi$ satisfies 1.4.E. Then, for every open set $\varOmega$, there exists a continuous function $u_\varOmega$ on $\varOmega$ that dominates all bounded solutions of (1.7).

[We call $u_\varOmega$ the absolute barrier in $\varOmega$.]

**Theorem 2.13.** Suppose that $\psi$ satisfies conditions 1.4.A–1.4.E and let $u_n \to u$ at every point of $\varOmega$. If $u_n$ are solutions of (1.7), then so is $u$. Let

\textsuperscript{10}A lower bound for $\lambda$ depends on upper bounds of $f$ and $u_0$.
$O \subset \partial \Omega Q$ be relatively open in $\partial Q$ and let $f$ be a positive continuous function on $O$. If $u_n$ satisfy the boundary condition

$$u_n = f \quad \text{on} \ O,$$

then the same condition holds for $u$.

2.9. Remarks. Condition 1.4.E is a modification of a condition introduced by Keller and Osserman in 1957. It is used in full in Theorems 2.12 and 2.13. In Theorems 2.10 and 2.11 it can be replaced by a weaker condition

$$\inf_{x \in D} \psi(x, u)/u \rightarrow +\infty \quad \text{as} \quad u \rightarrow \infty.$$  

[2.9.A follows from 1.4.E because 1.4.1–1.4.2 imply $\varphi(u)/u \rightarrow \infty$ as $u \rightarrow \infty$.]

Construction used in Theorem 2.12 is known as the monotone iteration scheme (see, e.g., [15]).

We use not only Theorem 2.12 but also the following intermediate result on which its proof is based.

**Lemma 2.2.** For every $\varphi$ subject to the conditions 1.4.1–1.4.2, there exists a function $F_\varphi(s)$, $R > 0$, $0 \leq s < R$, such that

(i) $F_\varphi(s) \rightarrow \infty$ as $s \uparrow R$;

(ii) $F_\varphi(0) \rightarrow 0$ as $R \rightarrow \infty$;

(iii) For every $R > 0$, $x_0 \in \mathbb{R}^d$, a function $v(x) = F_\varphi(|x - x_0|)$ satisfies $L^0 v \leq \varphi(v)$ for $|x - x_0| < R$.

**Proof.** By Lemma A2 in [8], for every $\alpha > 0$, $\gamma > 0$, there exists a function $f = f_{\alpha, \gamma}$ on $(0, \gamma)$ such that $f'(0) = 0$, $f(\gamma -) = \infty$ and $\alpha f'' + (d - 1)rf' = \varphi(f)$. Moreover (see [12, p. 507]).

$$f_{\alpha, \gamma}(0) \rightarrow 0 \quad \text{as} \quad \gamma \rightarrow \infty. \quad (2.33)$$

It was shown in the Appendix to [8] (see proof of Theorem 2.4)\(^{11}\) that there exist constants $\alpha, \beta$ (depending only on $L^0$) such that, for every $R$, $x_0$, the function

$$v_\varphi(x) = f_{\alpha, \gamma}(\frac{|x - x_0|}{R})$$  

satisfies the condition

$$L^0 v_\varphi \leq \varphi(v_\varphi) \quad \text{for} \quad |x - x_0| < R \quad (2.34)$$

\(^{11}\) There is a misprint in [8]: the left side of (A.9) is bounded by $(\lambda^2 + \lambda^2 r(\beta/x)) \psi(\varphi)$, not by $(\lambda^2 + \lambda r(\beta/x)) \psi(\varphi)$.
whenever \( \gamma^2(1 + R/\alpha) \leq R^2 \). We set \( \gamma = \gamma(R) = R(1 + R/\alpha)^{-1/2} \) and 
\( F_\mu(s) = f_{s, \xi}(\gamma s/R) \). Note that \( \gamma(R) \to \infty \) as \( R \to \infty \) and therefore (2.33) implies (ii). (i) and (iii) follow from (2.34) and (2.35).  

3. SUPERDIFFUSIONS AND SEMILINEAR EQUATIONS

3.1. \((L, \psi)\)-Superdiffusion. In Section 3 we investigate Eqs. (1.2) and (1.7) with \( \psi \in \mathcal{W}_0 \) by applying \((L, \psi)\)-superdiffusion.

We denote by \( \mathcal{M} \) the set of all measures \( \mu \) on \( S \) such that \( \mu(S_{<t}) < \infty \) for all \( t < \infty \). A collection of random measures \( (X_Q, P_\mu) \) where \( Q \) is an open subset of \( S \) and \( \mu \in \mathcal{M} \) is called \((L, \psi)\)-superdiffusion, if:

3.1.A. For every positive Borel function \( f \) on \( S 
\[ P_\mu e^{-\langle f, X_Q \rangle} = e^{-\langle \nu, \mu \rangle}, \]  

(3.1)

where \( \nu \) satisfies the integral Eq. (2.24).

3.1.B. (Special Markov Property). For every positive \( F \)-measurable function \( Y \),
\[ P_\mu \{ Y | \mathcal{F}_Q \} = P_{X_Q} Y, \]  

(3.2)

where \( \mathcal{F}_Q \) is the \( \sigma \)-algebra generated by \( X_0 \), with \( Q_1 \subseteq Q \) and \( \mathcal{F}_Q \) is the \( \sigma \)-algebra generated by \( X_{Q_2} \) with \( Q_2 \supseteq Q \).

Remark. The right side in (3.2) makes sense because \( X_Q \in \mathcal{M} \) a.s.\(^{12}\) This follows from an important equation

\[ P_\mu \langle f, X_Q \rangle = \int_S \Pi_{r,x} H^*_r f(r, \xi_r) \mu(dr, dx) \]  

(3.3)

(see (5.3) in [6]) which is an implication of 3.1.A and 1.4.D.

The existence of a family \( (X_Q, P_\mu) \) with properties 3.1.A, 3.1.B is proved in [5] and [4] for bounded \( c \in C^1(S) \) and a class \( \mathcal{W}_0 \) of functions \( \psi \) described in Section 1.4.

Note that \( P_\mu \{ X_Q = \mu \} = 1 \) if \( \mu(Q) = 0 \) and that \( X_Q \) is concentrated on \( Q \) \( P_\mu \)-a.s. if \( \mu \) is concentrated on \( Q \). Measure \( X_Q \) is concentrated, a.s., on \( Q^c \) and, since it is defined only a.s., we can assume that this is true for all \( \omega \). We call \( X_Q \) the exit measure from \( Q \).

\(^{12}\) Writing "a.s." means "\( P_\mu\)-a.s. for all \( \mu \in \mathcal{M} \)."
We put $P_{r,x} = P_{r,\delta_x}$ where $\delta_x$ is the Dirac's measure at point $x$. By 3.1.A,

$$u(r, x) = -\log P_{r,x} e^{-\langle f, x \rangle}$$  \hspace{1cm} (3.4)$$
is a solution of (2.24). If $f$ is bounded and continuous on $O \subset \bar{\Omega} Q$, then, by Theorem 2.9, $u$ is a solution of the boundary value problem (2.23).

The graph of a superdiffusion $X$ in an open set $Q$ is a random closed set $\mathcal{F}_Q$ with the properties:

(a) For every open set $U \subset Q$, $X_U$ is concentrated, a.s., on $\mathcal{F}_Q$.

(b) If $\mathcal{F}$ is a closed random set and if $X_U$ is concentrated, $P_{\mu}$-a.s., on $\mathcal{F}$ for all open $U \subset Q$, then $\mathcal{F}_Q \subset \mathcal{F}$ $P_{\mu}$-a.s.

Conditions (a), (b) determine $\mathcal{F}_Q$ uniquely up to $P_{\mu}$-equivalence for all $\mu \in \mathfrak{M}$.

Denote by $\mathfrak{M}(E)$ the set of all finite measures on a measurable space $E$. A Markov process $(X_t, P_{r,\mu})$ in $\mathfrak{M}(\mathbb{R}^d)$ related to a superdiffusion $X$ can be constructed as follows. We put $X_t(B) = X_{S_{\mu}}(\{t\} \times B)$ and $P_{r,\mu} = P_{r,\mu}$, where $\mu$, as the image of $\mu$ under the mapping $x \to (r, x)$ from $\mathbb{R}^d$ to $\mathcal{F}$.

Note that $X_{S_{\mu}}$ is concentrated, $P_{r,\mu}$-a.s., on $\{t\} \times \mathbb{R}^d$ for all $r < t$, $\mu \in \mathfrak{M}(\mathbb{R}^d)$ and that $(X_t, P_{r,\mu})$ is a time-homogeneous Markov process.

By replacing diffusion $X$ by its part $X$ in $Q$, we get a definition of the part $X$ of $X$ in $Q$. This is a family of random measures $(X_U, P_{\mu})$ where $U$ is an open subset of $Q$ and $\mu \in \mathfrak{M}$ is concentrated on $Q$. Note that $\bar{X}_U$ is the restriction of $X_U$ to $Q$. Let $\bar{X}$ be a measure on $\mathcal{F}_Q = \{x: (t, x) \in Q\}$ defined by the formula

$$\bar{X}(B) = \bar{X}_{Q \cap \{t\} \times B} = X_{Q \cap \{t\}} \times B \quad \text{for} \quad B \subset Q.$$  \hspace{1cm} (3.5)$$

If $\mu$ is a measure on $Q$, then its image $\mu$ under the mapping $x \to (r, x)$ is concentrated on $\{r\} \times Q \subset Q$, and we set $\bar{P}_{r,\mu} = P_{r,\mu}$. Random measures $(\bar{X}_t, \bar{P}_{r,\mu})$ with $\mu \in \mathfrak{M}(Q)$ determine a time inhomogeneous Markov process. (It is time homogeneous if $Q$ is a cylinder $(0, \infty) \times D$.) We will drop $\bar{P}$ and write $P_{r,\mu}$ instead of $\bar{P}_{r,\mu}$.

A random variable

$$\sigma_Q = \sup\{t: \bar{X}_t \neq 0\}$$  \hspace{1cm} (3.5)$$
is called the extinction time of $X$ in $Q$.

\footnote{We have $\bar{X}_t = \delta_x P_{r,\mu}$-a.s. for all $(r, x) \in Q$, and $\bar{X}_t = 0 P_{r,\mu}$-a.s. for all $r > t$, $(r, x) \in Q$ and therefore $P_{r,\mu}(\sigma_Q \geq t) = 1$ for all $(r, x) \in Q$.}
By [14, p. 223], there exists a right version of \( \tilde{X} \) which has the property: for every \((r, \mu)\), \( \langle 1, \tilde{X}_r \rangle \) is right continuous on \([r, \infty)\) \(P_{r, \mu}\)-a.s. This implies:

\[
\{ \sigma_Q \leq t \} = \{ \tilde{X}_t = 0 \} \quad P_{r, \mu}\text{-a.s. for all } r < t.
\]

For all \( r < t \), \( P_{r, \mu}\)-a.s., \( \langle 1, \tilde{X}_r \rangle = X_{Q_r} \langle Q_{\geq r} \rangle \) and therefore

\[
\{ \sigma_Q \leq t \} = \{ X_{Q_r} \langle Q_{\geq r} \rangle = 0 \} \quad P_{r, \mu}\text{-a.s.} \tag{3.6}
\]

3.2. Minimal solution of the Problem (2.23).

**Theorem 3.1.** Let \( Q \) be a strongly regular open set and let \( O \subset \partial_{ex} Q \) be a relatively open subset of \( \partial Q \). Suppose that \( f : \partial Q \to [0, \infty) \) is continuous on \( O \) and vanishes on \( \partial Q \setminus O \). The function (3.4) is a minimal solution of the boundary value problem (2.23).

**Remark 3.1.** A particular case of this theorem for a bounded \( Q \) and \( f = \infty \) on \( O \) follows from Theorems 4.6 and 5.1 in [6].

**Remark 3.2.** If \( O \) is a total subset of \( \partial Q \), then the condition \( f = 0 \) on \( \partial Q \setminus O \) can be dropped because \( X_Q(O') = 0 \) a.s. by (3.3).

First, we prove two lemmas.

**Lemma 3.1.** Suppose \( Q \subset Q \). If \( \Gamma \cap Q' \), then \( X_Q(\Gamma) \leq X_Q(\Gamma') \). If \( 0 \leq \phi \leq \varphi \) and \( \varphi = 0 \) on \( Q \), then \( \langle \phi, X_Q \rangle \leq \langle \varphi, X_Q \rangle \) a.s.

**Proof.** The first part is proved in [7, 5.4.C] (cf. Lemma 5.1 in [6]). The second part follows from the first one if \( \phi = \varphi = 1 \). In an obvious way, we get it, first, for functions \( \varphi = \phi \) of the form \( \sum_i \lambda_i 1_i \) with \( \lambda_i \geq 0 \) and then for limits of such functions. Extension to an arbitrary pair \( \phi \leq \varphi \) is trivial.

**Remark 3.3.** The arguments in 5.4.C, [7] demonstrate that \( X_Q(\Gamma) = X_Q(\Gamma') \) a.s. if, for all \( (r, x) \in Q \setminus \tilde{Q} \),

\[
\Pi_{r, x}\{ (t, x) \notin \Gamma \text{ for all } t \} = 1.
\]

**Lemma 3.2.** Let \( Q, O \) be as in Theorem 3.1 and let \( Q_n \) be a sequence described in Lemma 2.1. If \( f : S \to [0, \infty) \) is a Borel function vanishing on \( O' \), then

\[
\langle f, X_{Q_n} \rangle \uparrow \langle f, X_Q \rangle \quad a.s. \tag{3.7}
\]

**Proof.** First, we assume that \( f \) is bounded and there exists \( t > 0 \) such that \( f = 0 \) on \( S_{\geq t} \).
By Lemma 3.1, \( \langle f, X_{Q}\rangle \uparrow Y \leq \langle f, X_{Q}\rangle \) a.s. By (3.3),
\[ P_{\tau_{n}} \langle f, X_{Q} \rangle = \Pi_{\tau_{n}} H_{f}(\tau_{n}, \xi_{n}), \quad P_{\tau_{n}} \langle f, X_{Q} \rangle = \Pi_{\tau_{n}} H_{f}(\tau_{n}, \xi_{n}), \]  
where \( \tau_{n} \) and \( \tau \) are the first exit times from \( Q_{n} \) and from \( Q \). Condition (2.16) implies \( \tau_{n} \uparrow \tau \) and \( \{ (\tau_{n}, \xi_{n}) \in O \} \uparrow \{ (\tau, \xi) \in O \} \). By passing to the limit in (3.8), we get
\[ P_{\tau_{n}} Y = P_{\tau_{n}} \langle f, X_{Q} \rangle, \]
and \( Y = \langle f, X_{Q} \rangle \) \( P_{\tau_{n}} \)-a.s. because \( \Pi_{\tau_{n}} H_{f}(\tau_{n}, \xi_{n}) < \infty \).

To cover the general case, we consider a sequence of bounded functions \( \varphi_{m} \uparrow f \) supported by finite time intervals (for instance, \( \varphi_{m} = f \land m_{S_{c_{n}}} \)).

For every \( m \),
\[ \langle \varphi_{m}, X_{Q} \rangle = \sup_{n} \langle \varphi_{m}, X_{Q} \rangle \text{ a.s.} \]  
(3.9)

On the other hand, \( \langle f, X_{Q} \rangle = \sup_{m} \langle \varphi_{m}, X_{Q} \rangle \) and \( \langle f, X_{Q} \rangle = \sup_{n} \langle f, X_{Q} \rangle \) a.s.

which implies (3.7).

Proof of Theorem 3.1. Consider a sequence \( Q_{n} \) mentioned in Lemma 3.2.

For every point \( z \in O \), there exists a neighborhood \( U \) in \( O \) such that \( U \subseteq \partial Q_{n} \) for all sufficiently large \( n \).

Consider an increasing sequence of bounded functions \( \varphi_{n} \) with the properties:

(i) \( \varphi_{n} \geq 0 \) and \( \varphi_{n} = 0 \) on the complement of \( O_{n} = O \cap \partial Q_{n} \);

(ii) \( \varphi_{n} \) is continuous on \( \partial Q_{n} \);

(iii) \( \varphi_{n} \uparrow f \); moreover, if \( f(z) < \infty \) for some \( z \in O \), then there exists a neighborhood \( U \subset O \) of \( z \) such that \( \varphi_{n} = f \) on \( U \) for all sufficiently large \( n \).

[These conditions hold, for instance, for \( \varphi_{n}(z) = \min(f(z), m_{d}(z, Q_{n}^{c})) \) on \( O_{n} \) and \( \varphi_{n} = 0 \) on \( O_{n}^{c} \).] Function
\[ u_{n}(\tau, x) = -\log P_{\tau_{n}} e^{-\langle \varphi_{n}, X_{Q} \rangle} \]
is a solution of the boundary value problem
\[ u_{n} + Lu_{n} = \psi(u_{n}) \text{ in } Q_{n}; \]
\[ u_{n} = \varphi_{n} \quad \text{on } \partial_{r} Q_{n}. \]  
(3.10)
It follows from Lemma 3.1 that
\[ \langle \varphi_n, X_{Q_n} \rangle \uparrow Y \leq \langle f, X_Q \rangle \] a.s.

By Lemma 3.2,
\[ \langle \varphi_m, X_Q \rangle = \lim_{n \to \infty} \langle \varphi_m, X_{Q_n} \rangle \]
for every \( m \), and, since \( \langle \varphi_n, X_{Q_n} \rangle \geq \langle \varphi_m, X_{Q_n} \rangle \) for \( n \geq m \), we get
\[ Y \geq \lim_{n \to \infty} \langle \varphi_m, X_{Q_n} \rangle = \langle \varphi_m, X_Q \rangle. \]

Since \( \langle \varphi_m, X_Q \rangle \uparrow \langle f, X_Q \rangle \), we conclude that \( Y \geq \langle f, X_Q \rangle \) and therefore \( Y = \langle f, X_Q \rangle \) a.s. By the dominated convergence theorem, \( u_n \) converge to \( u \) given by (3.4). By Theorem 2.13, \( u \) is a solution of the problem (2.23).

To establish that \( u \) is a minimal solution of (2.23), it is sufficient to show that an arbitrary solution \( v \geq u_n \) in \( Q_n \) for all \( n \). This follows from Theorem 2.8. Indeed, \( u_n \) satisfies (2.24) with \( f \) replaced by \( \varphi_n \) and therefore \( u_n \leq K_{Q_n, \sigma} \) is bounded, and we need only to prove that \( h = \lim \sup (u_n - v) \leq 0 \) on a total subset \( \partial_n Q_n \). By (3.10), \( u_n = \varphi_n \) on \( \partial_n Q_n \) and \( h = \varphi_n - \lim \inf v \leq 0 \) because \( \varphi_n \leq f \) on \( O_n \) and \( \varphi_n = 0 \) on \( \partial_n Q_n \setminus O_n \).  

### 3.3. Maximal Solution of the Problem (2.23).

**Theorem 3.2.** Let \( Q, O, f \) be as in Theorem 3.1 and, in addition, let \( f \) be finite. Then the events \( A_u = \{ \| \varphi \|_Q \text{ is compact} \} \) and \( A_v = \{ \sigma < \infty \} \) coincide \( P_r \)-a.s. for all \( (r, x) \in Q \). The maximal solution of (2.23) can be expressed by the formula
\[ w(r, x) = -\log P_{r, x} \exp^{-\langle f, X_Q \rangle} \mathbf{1}_{\{\| \varphi \|_Q \leq (Q \cup O) \text{ and } \| \varphi \|_Q \text{ is compact} \}} \] (3.11)
or by the formula
\[ w(r, x) = -\log P_{r, x} \exp^{-\langle f, X_Q \rangle} \mathbf{1}_{\{\| \varphi \|_Q \leq (Q \cup O) \text{ and } \sigma < \infty \}}. \] (3.12)

The proof is based on five lemmas.

**Lemma 3.3.** Let \( Q, O, f \) be as in Theorem 3.2 and let \( Q' \subset Q \) be a strongly regular bounded open set such that \( \partial Q' \cap \partial Q \subset O \). Denote by \( A \) the interior of \( \partial Q' \cap \partial Q \) and put \( B = \partial Q' \setminus Q \). Function
\[ u(r, x) = -\log P_{r, x} \exp^{-\langle f, X_Q \rangle} \mathbf{1}_{\partial Q \cap B} = 0 \] (3.13)
is a solution of the boundary value problem

\[ u + Lu = \psi(u) \quad \text{in } Q', \]

\[ u = f \quad \text{on } A. \]  

(3.14)

If \( v \) is any solution of (2.23), then \( v \leq u \) in \( Q' \).

**Proof.** Consider an increasing sequence of positive continuous functions \( \varphi_n \) on \( \partial Q' \) such that \( \varphi_n = f \) on \( \partial Q' \cap \partial Q \) and \( \varphi_n \to \infty \) on \( B \). \(^{14}\)

Put

\[ u_n(r, x) = -\log P_{r,x} e^{-\langle \varphi_n, x \rangle} \]

Clearly, \( u_n \uparrow u \) where \( u \) is given by (3.13). Theorem 3.1 and Remark 3.2 imply that

\[ u_n + Lu_n = \psi(u_n) \quad \text{in } Q', \]

\[ u_n = \varphi_n \quad \text{on } \partial, Q'. \]

(3.15)

and therefore \( u_n \) is a solution of (3.14). By Theorem 2.13, \( u \) is also a solution of (3.14).

Suppose \( v \) is a solution of (2.23). Note that \( v \) is continuous and therefore bounded on \( \bar{Q}' \) since \( f \) is finite and continuous on \( O \). Let \( z \in \partial Q' \) and let \( z \rightarrow \tilde{z}, \tilde{z} \in \partial Q' \). By (3.15),

\[ \limsup [v - u] \leq \limsup [v - u_n] = \begin{cases} 0 & \text{if } \tilde{z} \in O \\ v - \varphi_n & \text{if } \tilde{z} \in Q. \end{cases} \]

Hence \( \limsup [v - u] \leq 0 \) on \( \partial, Q' \) and, by Theorem 2.8, \( v \leq u \) in \( Q' \).

**Lemma 3.4.** Let \( Q, O \) be as in Theorem 3.2 and let \( Q_n \) be as in Lemma 2.1. Put \( B_n = \partial Q_n \cap Q \). We have

\[ \{ X_Q(B_n) = 0 \} \uparrow \{ \mathcal{Q} \text{ is compact and } \mathcal{Q} \subset Q \cup O \} \quad P_{r,x}\text{-a.s.} \]

for every \((r, x) \in Q\).

This follows from the bounds (5.15) and (5.16) in [6] [established there as a part of the proof of Theorem 5.1].

\(^{14}\)To construct such a sequence, consider an arbitrary continuous function \( \varphi_n \) on \( \partial Q' \) which coincides with \( f \) on \( \partial Q \cap \partial Q' \) (see, e.g., [13], Ch. 1, IV) and put \( \varphi_n = \varphi_0 + nd_{\partial Q} \).
Lemma 3.5. If
\[ \dot{u} + Lu = \psi(u) \quad \text{in } S_{<t}, \]
\[ u = 0 \quad \text{on } \partial S_{<t}, \]
then \( u = 0 \) in \( S_{<t} \).

Proof. For every constant \( \kappa, \dot{u} + Lu - \psi(u) = e^{-\kappa t}[\dot{v} + \tilde{L}v - \tilde{\psi}(v)] \) where
\[ \tilde{L} = L - \kappa, \quad v(r, x) = e^{\kappa t}u(r, x), \quad \tilde{\psi}(r, x; v) = e^{\kappa t}\psi(x, e^{-\kappa t}v). \]
If \( u \) satisfies (3.16), then
\[ \dot{v} + \tilde{L}v = \tilde{\psi}(v) \quad \text{in } S_{<t}, \]
\[ v = 0 \quad \text{on } \partial S_{<t}. \] (3.17)

If \( \varphi \) has properties 1.4.1–1.4.2, then \( \tilde{\psi}(v) = \varphi(e^{-\kappa t}v) \) also has these properties. Let \( F_{\varphi}(s) \) be the function corresponding to \( \tilde{\psi} \) by Lemma 2.2.

Suppose that \( \kappa \) is positive. By condition 1.4.1, \( \varphi \) is monotone increasing and therefore
\[ \tilde{\psi}(v) \leq e^{\kappa t}\varphi(e^{-\kappa t}v) \leq \tilde{\psi}(r, x; v) \quad \text{for all } 0 < r < t. \] (3.18)
Put \( u_R(x) = F_{\varphi}(|x - x_0|) \). By (3.18) and by Lemma 2.2(iii), for sufficiently large \( \kappa \),
\[ \tilde{L}u_R(x) \leq L^0u_R(x) \leq \tilde{\psi}(u_R(x)) \leq \tilde{\psi}(r, x; u_R(x)) \]
\[ \text{in } Q_R = \{0 < r < t, |x - x_0| < R\}. \] (3.19)

By (3.17) and (3.19), \( \dot{v} + \tilde{L}v = \tilde{\psi}(v) \geq u_R + \tilde{L}u_R - \tilde{\psi}(u_R) \) in \( Q_R \). Clearly, \( v \) is bounded on \( Q_R \). Besides \( v(r, x) \leq \infty = u_R(x) \) if \( |x - x_0| = R \) and \( v(t, x) = 0 \leq u_R(x) \) if \( |x - x_0| < R \). By Theorem 2.8, \( v \leq u_R \) in \( Q_R \). In particular, \( v(r, x) \leq u_R(x) = F_{\varphi}(0) \) and therefore \( u(x_0) = 0 \) by Lemma 2.2(ii).

Lemma 3.6. If \( \tilde{Q} \subset Q \) then \( \sigma_{\tilde{Q}} \leq \sigma_Q \) and \( \{X_{\tilde{Q}}(Q) = 0\} \subset \{X_{\tilde{Q}} = X_Q\} \) \( P_{r,x} \)-a.s. for all \( (r, x) \in \tilde{Q} \).

Proof. Since \( \tilde{Q}_{<t} \subset Q_{<t} \) and \( \tilde{Q}_{\geq t} \subset Q_{\geq t} \), we have, by Lemma 3.1,
\[ X_{Q_{<t}}(\tilde{Q}_{<t}) \leq X_{Q_{<t}}(Q_{<t}) \leq X_{Q_{<t}}(Q_{\geq t}) \quad \text{\( P_{r,x} \)-a.s.} \]
By (3.6), this implies \( \{\sigma_{\tilde{Q}} \leq t\} \subset \{\sigma_Q \leq t\} \) \( P_{r,x} \)-a.s. for all \( t \). The second statement follows from 3.1.B and the formula \( P_{r,x} \{X_{\tilde{Q}} = v\} = 1 \) if \( v(\hat{Q}) = 0 \) (cf. 5.4.B in [7]).
Lemma 3.7. For all $Q$ and $t$, and all $(r, x) \in Q_{e,t}$,
\[ \{ \sigma_Q \leq t \} \subset \{ X_Q = X_{Q_{e,t}} \} \quad P_{r,x}\text{-a.s.} \tag{3.20} \]
and
\[ \{ \sigma_Q \leq t \} \subset \{ \mathcal{G}_Q \subset \mathcal{G}_{e,t} \} \quad P_{r,x}\text{-a.s.} \tag{3.21} \]

Proof. Since $X_{Q_{e,t}}(Q_{e,t}) = 0$, we get from (3.6) and Lemma 3.6
\[ \{ \sigma_Q \leq t \} \subset \{ X_{Q_{e,t}}(Q_{e,t}) = 0 \} = \{ X_Q(0) = 0 \} \]
\[ \subset \{ X_Q = X_0 \} \quad P_{r,x}\text{-a.s.} \]
To prove (3.21), we consider an arbitrary open set $U \subset Q$. By Lemma 3.6 and (3.20),
\[ \{ \sigma_Q \leq t \} \subset \{ \sigma_U \leq t \} \subset \{ X_U = X_U \} \quad P_{r,x}\text{-a.s.} \]
Since $U_{e,t} \subset S_{e,t}$, measure $X_{U_{e,t}}$ is concentrated, a.s., on $\mathcal{G}_{e,t}$, and therefore
\[ \{ \sigma_Q \leq t \} \subset \{ X_U \text{ is concentrated on } \mathcal{G}_{e,t} \} \quad P_{r,x}\text{-a.s.} \]
which implies (3.21).

Proof of Theorem 3.2. 1°. To prove (3.11), we consider the sets $Q_n$, $B_n$ introduced in Lemma 3.4 and we denote $A_n$ the interior of $\partial Q_n \cap \partial Q$. By Lemma 3.3,
\[ w_n(r, x) = -\log P_{r,x} e^{-\langle f, X_0 \rangle} 1_{X_0(0) = 0} \]
satisfy the conditions
\[ w_n' + Lw_n = \psi(w_n) \quad \text{in } Q_n, \]
\[ w_n = f \quad \text{on } A_n. \]
By applying Lemma 3.3 to $Q = Q_{e,t}$ and $Q' = Q_n$, we get $w_n \leq w_{n+1}$ in $Q_n$.
By Lemmas 3.2, 3.4 and the dominated convergence theorem, $w_n$ tends to $w$ given by (3.11). By Theorem 2.13, $w$ is a solution of (2.23). If $v$ is an arbitrary solution of (2.23), then $v \leq w$ in $Q_n$ by Lemma 3.2 and therefore $v \leq w$. Hence formula (3.11) represents the maximal solution of (2.23).

2°. By applying (3.11) to $Q = S_{e,t}$, $O = \partial S_{e,t}$, we conclude that
\[ u(r, x) = -\log P_{r,x} \{ \mathcal{G}_{e,t} \text{ is compact} \} \]
is a solution of the boundary value problem (3.16). By Lemma 3.5, \( u = 0 \) and therefore \( \{ \sigma_Q \leq t \} \subset \{ \tau_Q \in \mathcal{G}_Q \} \subset A_x \) \( P_{r,x} \)-a.s. Hence, \( P_{r,x} \)-a.s., \( A_x \subset A_x \).

On the other hand, if \( \tau_Q \) is compact, then, for every \( r \) there exists \( t > r \) such that \( \tau_Q \subset S_{<t} \). Since \( Q_{<t} \subset Q \), \( X_Q \) is concentrated on \( \tau_Q \subset S_{<t} \). Hence \( X_Q \subset Q_{<t} = 0 \) and \( \sigma_Q \leq t \) by (3.6). We conclude that, \( P_{r,x} \)-a.s., \( A_x \subset A_x \) which implies the first statement of our theorem. Now (3.11) implies (3.12).

### 3.4. Exit Measures From \( D \subset \mathbb{R}^d \) and the Range \( \mathcal{R}_D \)

For every Borel set \( B \subset \mathbb{R}^d \), we put

\[
B_{<t} = (0, t) \times B, \quad B_{t} = \{ t \} \times B, \quad \hat{B} = B_{<\infty} = (0, \infty) \times B
\]

(cf. the notation \( D_{<t}, \hat{D} \) in Section 2.2). We note that \( X_D(B_{<t}) = X_D(B_{<\infty}) \) for all \( D, B, t \). The exit measure from \( D \) is a measure on \( D' \) defined by the formula

\[
X_D(B) = X_D(\hat{B}) = \lim_{t \to \infty} X_D(B_{<t}).
\]

If \( f \) does not depend on \( t \) and vanishes on \( D \), then

\[
\langle f, X_D \rangle = \langle f, X_{\hat{D}} \rangle. \tag{3.22}
\]

Note that

\[
P_{r,\mu} \langle f_1, X_{D_1} \rangle \in C_1, \ldots, \langle f_n, X_{D_n} \rangle \in C_n
\]
does not depend on \( r \) and therefore it is sufficient to deal with measures \( P_{\mu} = P_{0,\mu} \).

We define a part of \( X \) in \( D \) as a part of \( X \) in \( Q = \hat{D} \) and we denote it by \( \hat{X} \). Starting from \( \hat{X} \) we define an \( \mathcal{G}(D) \)-valued time-homogeneous Markov process \( \hat{X} \) in the same way as the \( \mathcal{G}(\mathbb{R}^d) \)-valued process \( X \) was defined starting from \( X \). We write \( \sigma_D \) instead of \( \sigma_D \) and we call it the extinction time of \( X \) in \( D \). Note that

\[
\{ \sigma_D \leq t \} = \{ X_D(B') = 0 \}. \tag{3.23}
\]

The range \( \mathcal{R}_D \) of \( X \) in \( D \) is a minimal random closed subset of \( \mathbb{R}^d \) on which all \( X_u, U \subset D \) are concentrated a.s. For every Borel subset \( B \) of \( \mathbb{R}^d \),

\[
\{ \mathcal{R}_D \cap B = \emptyset \} = \{ \tau_Q \cap \hat{B} = \emptyset \} \quad \text{a.s.} \tag{3.24}
\]

Theorems 3.1 and 3.2 imply
Theorem 3.3. Suppose that $D$ is a regular open subset of $\mathbb{R}^d$ and $O$ is a relatively open subset of $\partial D$. Let $f \colon \partial D \to [0, \infty]$ be continuous on $O$ and vanish on $\partial D \setminus O$. The function

$$v(x) = -\log P_x e^{-\langle f, x \rangle}$$

(3.25)

is the minimal solution of the boundary value problem (1.9).

If $f$ is finite, then the function

$$w(x) = -\log P_x e^{-\langle f, x \rangle} \mathbb{1}_{\{\#_x \in (D \cup O) \text{ and } \sigma_D < \infty\}}$$

(3.26)

is the maximal solution of (1.9).

3.5. Special Domains. Recall that a regular domain $D$ is called special if there exists a special solution in $D$, i.e., a solution of (1.9) with $f = 0$ and $0 = \partial D$.

By applying Theorem 3.3 to $0 = \partial D$ and $f = 0$, we get:

Theorem 3.4. A regular domain $D$ is special if and only if $P_x \{\sigma_D < \infty\} < 1$ for some (and therefore for all) $x \in D$. Moreover,

$$u(x) = -\log P_x \{\sigma_D < \infty\}$$

(3.27)

is the maximal special solution.

Corollary 3.1. If $D_0$ is special, then all domains $D \supset D_0$ are special. Moreover, if $u_0, u$ are the maximal special solutions in $D_0$ and $D$, then $u_0 \leq u$ in $D_0$.

This follows from Theorem 3.4 and Lemma 3.6.

4. MORE RESULTS ON EQUATION $Lu = \psi(u)$

In Sections 4.1–4.7, we do not assume the existence of $(L, \psi)$-superdiffusion. In other words, we do not assume that $\psi \in \Psi$ belongs to $\Psi_0$. In Section 4.8, we return to the case $\psi \in \Psi_0$ and we deduce from Theorems 3.3, 3.4, and 4.5 new results on special solutions of (1.2).

4.1. Strict Positiveness of Nontrivial Solutions.

Lemma 4.1. If $D$ is connected, then every nonzero solution of (1.4) is strictly positive.

Proof. It is sufficient to show that if $u(x_0) = 0$ for $x_0 \in D$ and if the closure of $U = \{x : |x - x_0| < \epsilon\} \subset D$, then $u = 0$ in $\bar{U}$. Indeed, $N = \sup_{\bar{U}} u < \infty$.

It follows from (1.4) that, for every constant $\kappa > 0$, $(L^0 - \kappa) u = \psi(u) - (\epsilon + \kappa) u$. If $\kappa$ is sufficiently big, then the right side is negative on $\bar{U}$ because...
\( c \) is bounded and \( \psi(u)/u \) is bounded on \( \bar{U} \times (0, N) \). By the strong maximum principle (see, e.g. [10, Theorem 3.5]), conditions \((L^\kappa - \kappa)u \leq 0, u \geq 0\) in \( U \) and \( u(x_0) = 0 \) imply that \( u = 0 \) in \( U \).

4.2. Minimal Solutions \( u \geq u_0 \) of the Dirichlet Problem for (1.2). We investigate the Dirichlet problem

\[
Lu = \psi(u) \quad \text{in } D
\]
\[
u = f \quad \text{on } \partial D.
\]

[which is a particular case of (1.9) with \( O = \partial D \)] by applying Theorem 2.11 to the cylinders \( D < t \) and by passing to the limit as \( t \to \infty \).

For every cylinder \( D < t = (0, t) \times D \), we denote by \( D < t \) the union of \((0, t) \times \partial D \) and \( \{t\} \times D \).

**Theorem 4.1.** Suppose \( D \) is a bounded regular domain and \( f \) is a continuous function on \( \partial D \). Let \( u_0 \in C^2(D) \) satisfy the conditions

\[
Lu_0 \geq \psi(u_0) \quad \text{in } D,
\]
\[
u_0 = f \quad \text{on } \partial D.
\]

Then there exists a minimal solution \( u \) of the problem (1.9) with \( O = \partial D \) such that \( u \geq u_0 \).

**Proof.** Conditions (4.2) imply that, for every \( t \), (2.31) holds for \( Q = D < t \), \( \tilde{\mathcal{F}} = \partial^0 D < t \) and \( f = u_0 \) and therefore, by Theorem 2.11, there exists a bounded \( u' \geq u_0 \) such that

\[
Lu' = \psi(u') \quad \text{in } D < t,
\]
\[
u' = u_0 \quad \text{on } \partial^0 D < t.
\]

If \( 0 < t_1 < t_2 \), then \( u'(t_1, x) \geq u_0(x) = u'_0(t_1, x) \) for \( x \in \bar{D} \) and \( u'(r, x) = u'_0(r, x) = u_0(x) \) for \( x \in \partial D \), \( 0 < r < t_1 \), By Theorem 2.8, \( u'(r, x) \geq u_0(r, x) \) for all \( x \in D \), \( 0 < r < t_1 \). Hence, there exists

\[
\lim_{t \to \infty} u'(r, x) = u(r, x).
\]

This limit does not depend on \( r \). Indeed, for every \( h > 0 \), \( v(r, x) = u'(r + h, x) \) is a solution of (1.7) in \( D < (t - h) \) equal to \( u_0 \) on \( \partial^0 D < (t - h) \) and, by Theorem 2.8, \( v = u_0 \). By Theorem 2.13, \( u \) is a solution of (1.7) equal to \( f \) on \((0, \infty) \times \partial D \) and therefore \( u \) satisfies (1.9).

If \( v \geq u_0 \) satisfies (1.9), then \( v \geq u \) on \( \partial^0 D < t \) by (4.3) and \( v \geq u' \) in \( D < t \) by Theorem 2.8. Therefore \( v \geq u \).
4.3. Principal Eigenvalue of $-L$ and Existence of Special Solutions. In Sections 4.3–4.8, we investigate Eq. (1.4) assuming that operator $L^0$ has the divergence form (1.6).

Denote $C^2_0(D)$ the class of functions $u \in C^4(D)$ equal to 0 on $\partial D$. If $D$ is a bounded regular domain, then there exists a strictly positive function $\varphi \in C^2_0(D)$ and a real constant $\lambda_1$ such that

$$-L\varphi = \lambda_1 \varphi. \tag{4.5}$$

All other eigenvalues of $-L$ are strictly bigger than $\lambda_1$. $\lambda_1 = \lambda_1(D)$ is called the principal eigenvalue of $-L$ in $D$.

**Theorem 4.2.** A bounded regular domain $D$ is special if $\lambda_1(D) < 0$.

**Proof.** By condition 1.4.D, $\psi(x, t) \leq -\lambda_1 t$ for all $t \leq \varepsilon$ and all $x$ if $\varepsilon$ is sufficiently small. If $k \leq \varepsilon/\|\varphi\|$, then $\psi(k\varphi) \leq -\lambda_1 k\varphi$ and condition (4.5) implies $L(k\varphi) = -k\lambda_1 \varphi \geq \psi(k\varphi)$. Hence (4.2) holds for $u_0 = k\varphi$ and $f = 0$. By Theorem 4.1, there exists a solution $u \geq u_0$ of problem (1.9) with $f = 0$ and $0 = \partial D$. Clearly, $u$ is a special solution of (1.2).

4.4. Probabilistic Description of Principal Eigenvalue. Fix a domain $D$ and put

$$H^f = \exp \left\{ \int_0^t f(\xi_s) \, ds \right\}. \tag{4.6}$$

[Note that $H^f$ coincides with $H^0$ defined by (1.5).]

**Theorem 4.3.** We have

$$\Pi_x H^{\varepsilon + \lambda_1} = \infty \quad \text{for all} \quad x \in D \quad \tag{4.7}$$

and

$$\Pi_x H^{\varepsilon + \lambda} < \infty \quad \text{for all} \quad \lambda < \lambda_1 \quad \text{and for all} \quad x \in D. \quad \tag{4.8}$$

**Proof.** 1°. Suppose that (4.7) is false. Then, by Theorems 2.6 and 2.7, $u(x) = \Pi_x H^{\varepsilon + \lambda_1}$ belongs to $C^{\lambda_1}(D)$ and satisfies the equations

$$(L + \lambda_1) u = 0 \quad \text{in} \quad D, \quad \tag{4.9}$$

$$u = 1 \quad \text{on} \quad \partial D.$$
There exists a constant $\beta$ such that $v = \beta q - u \leq 0$ and $v(x_0) = 0$ at some point $x_0 \in D$. Choose a constant $\kappa > 0$ such that $c + \lambda_1 + \kappa > 0$ in $D$. We have
\[
(L^0 - \kappa) v = -(c + \lambda_1 + \kappa) v \geq 0
\]
and $v = -1$ on $\partial D$. This contradicts the maximum principle for the operator $L - \kappa$ (see, e.g., [10, p. 32]). Hence (4.7) is true.

2°. If $\lambda < \lambda_1$, then $\lambda$ is not an eigenvalue of $-L$ and, by Theorem 2.2.4 in [15] (based on the Fredholm alternative), there exists a unique $u \in C^2_0$ such that
\[
(L + \lambda) u = -1 \quad \text{in } D. \quad (4.10)
\]
If $\varphi > 0$ satisfies (4.5) and if $\varepsilon$ and $N$ are constants, then $w = u + N \varphi + \varepsilon$ satisfies equation
\[
(L + \lambda) w = -\rho, \quad (4.11)
\]
where
\[
\rho = 1 - (c + \lambda) \varepsilon + N(\lambda_1 - \lambda) \varphi.
\]
If $\varepsilon > 0$ is sufficiently small, then, for all $\lambda < \lambda_1$ and all $N > 0$, $\rho > 1 - (c + \lambda) \varepsilon > 1/2$. On the other hand, $u = 0$ on $\partial D$ and therefore $u + \varepsilon > 0$ in a neighborhood of $\partial D$. On the complement of this neighborhood $w/\varphi = (u + \varepsilon)/\varphi + N > 0$ for sufficiently large $N$ which implies that $w > 0$ in $D$. By Corollary 2.1 [applied to $L + \lambda$, $D = t$, $T = 0$, $D = t$ and $w$],
\[
w(x) = \Pi_x \int_0^{\tau \wedge t} H_s \rho(\xi_s) ds + \Pi_x H_{\tau \wedge t} w(\xi_{\tau \wedge t}) \geq \frac{1}{2} \Pi_x \int_0^{\tau \wedge t} H_s ds,
\]
where
\[
H_s = \exp \left\{ \int_0^s (c + \lambda)(\xi_r) dr \right\}.
\]
By passing to the limit as $t \to \infty$, we get
\[
\frac{1}{2} \Pi_x \int_0^{\tau \wedge t} H_s ds \leq w(x) < \infty. \quad (4.12)
\]
Note that $d[H_s]/dt = (c + \lambda)(\xi_t) H_s$ and therefore $H^{c + \lambda} = H_s = 1 + \int_0^t (c + \lambda)(\xi_r) H_r ds \leq 1 + \text{const.} \int_0^t H_r ds$. In combination with (4.12), this yields (4.8).
4.5. **Uniqueness Results.** Formula (1.6) is equivalent to an expression

\[ L^0 u = \text{div}(\nabla u), \]  

where

\[ (\nabla u)_i = \sum_j a_{ij} \frac{\partial u}{\partial x_j}, \]

and

\[ \text{div } w = \sum_i \frac{\partial w_i}{\partial x_i}. \]

We denote by \((a^t)\) the inverse of matrix \((a)\) and we put

\[ (w, \tilde{w}) = \sum_\theta a^t_{ij} w_i \tilde{w}_j; \quad |w|^2 = (w, w). \]

Note that

\[ (\nabla u, \nabla a) = \sum_\theta a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial a}{\partial x_j}. \]  

By the divergence formula,

\[ \int_D \text{div } w \, dx = 0, \]

if \(D\) is a bounded smooth domain, \(w = (w_1, ..., w_d)\) with \(w_i\) and \(\partial w_i/\partial x_j \in C(D)\), and \(w = 0\) on \(\partial D\).

**Theorem 4.4.** Suppose \(D\) is a bounded smooth domain and \(\psi\) satisfies conditions 1.4.B–1.4.C. Suppose \(u, v > 0\) are solutions of (1.2) and \(\partial u/\partial x_i, \partial v/\partial x_i\) are bounded. If \(u = v\) on \(\partial D\), then \(v = ku\) in \(D\) where \(k\) is a constant. Moreover, \(k = 1\) if \(\psi\) satisfies 1.4.D or if \(u(x_0) > 0\) for some \(x_0 \in \partial D\).

**Proof.** If

\[ w^\delta = (v^2 - u^2)^\psi \log \frac{v + \epsilon}{u + \epsilon}, \]

15 This is a slight modification of the proof of Lemma 2.2 in [1].
then

\[ v^2 \left( V \log \frac{u + \varepsilon}{v} \right)^2 + u^2 \left( V \log \frac{v + \varepsilon}{u} \right)^2 \]

\[ = \text{div } w + \left( \frac{\varepsilon}{v + \varepsilon} \right)^2 |Vv|^2 + \left( \frac{\varepsilon}{u + \varepsilon} \right)^2 |Vu|^2 + I^*, \]

where

\[ I^* = (v^2 - u^2) \left[ \frac{L^0 u}{u + \varepsilon} - \frac{L^0 v}{v + \varepsilon} \right]. \]

Therefore (4.15) implies

\[ \int_D \left( v \left( V \log \frac{u + \varepsilon}{v} \right) \right)^2 \, dx + \int_D \left( u \left( V \log \frac{v + \varepsilon}{u} \right) \right)^2 \, dx \]

\[ = \int_D \left[ I^* + \left( \frac{\varepsilon}{v + \varepsilon} \right)^2 |Vv|^2 + \left( \frac{\varepsilon}{u + \varepsilon} \right)^2 |Vu|^2 \right] \, dx. \]

Since \( u, v \) are solutions of (1.2), we have

\[ I^* = (v^2 - u^2) \left[ (\psi(v)/(v + \varepsilon) - \psi(u)/(u + \varepsilon) - cv/(v + \varepsilon) + cu/(u + \varepsilon) \right]. \]

By passing to the limit in (4.16) as \( \varepsilon \to 0 \) and by applying Fatou's lemma to the left side and the dominated convergence theorem to the right side, we get

\[ \int_D \left( v^2 + u^2 \right) \left( V \log \frac{u}{v} \right)^2 \, dx \leq - \int_D \left( v^2 - u^2 \right) \left[ \psi(v)/v - \psi(u)/u \right] \, dx. \quad (4.17) \]

It follows from 1.4.B–1.4.C that the integrand in the right side is \( \geq 0 \). Hence, both sides of (4.17) are zeros and therefore

\[ \nabla \log \frac{u}{v} = 0, \quad (v^2 - u^2) \left[ \psi(v)/v - \psi(u)/u \right] = 0. \]

The first equation implies \( v = ku \) and, after that, the second one implies that \( \psi(x, ku(x)) = k\psi(x, u(x)) \) for all \( x \in D \). Clearly, \( k = 1 \) unless \( u = v \) on \( \partial D \). Since the ratio \( \psi(x, t)/t \) is monotone increasing in \( t \), \( \psi(x, t)/t = \psi(x, u(x))/u(x) \) for all \( t \) between \( u(x) \) and \( ku(x) \). In combination with 1.4.B, this implies: if \( k \neq 1 \), then, \( \psi(x, \cdot) \) is linear, with a positive slope, on some interval \((0, \kappa)\) which is impossible if 1.4.D holds.
Theorem 4.5. Suppose that $D$ is a bounded smooth domain and $\psi$ satisfies conditions 1.4.A–1.4.D. If $f \in C^{1,\alpha}(\partial D)$ is not identically equal to 0, then there exists no more than one solution of the problem (4.1). There exists no more than one special solution in $D$.

Proof. A solution $u$ of (4.1) is bounded in $\bar{D}$ and it satisfies equation $L^0 u = g$ where $g = \psi(u) - cu$. Clearly, $g$ is bounded on $\bar{D}$. By the maximum principle, there exists only one solution of the equation $L^0 u = g$ that is equal to $u$ on $\partial D$, and, by Theorem 8.34 in [10], partials $\partial u/\partial x_i$ are bounded. Therefore Theorem 4.5 follows from Theorem 4.4.

4.6. Converse to Theorem 4.2. We use the following well-known facts [see, e.g., [10, Section 8.12]]:

**Lemma 4.2.** If $D$ is a bounded smooth domain and if $L = L^0 + c$ with $L^0$ given by (4.13), then

$$
\lambda_1(D) = \inf \left\{ \int_D \left[ \frac{1}{2} |\nabla u|^2 - cu^2 \right] \, dx \, \int_D u^2 \, dx \right\},
$$

(4.18)

where the infimum is taken over $u \in C^2_0(D)$. If $\bar{D} \supset D$, then $\lambda_1(\bar{D}) \leq \lambda_1(D)$.

Theorems 4.6 and 4.8 are an adaptation of Lemma 3.1 in [1] to our setting.

**Theorem 4.6.** Suppose that $\psi$ satisfies conditions 1.4.A–1.4.D. If a bounded smooth domain $D$ is special, then $\lambda_1(D) < 0$.

Proof. Let $u$ be a special solution in $D$. If $w = u \psi u$, then $\text{div} w = |\nabla u|^2 + uL^0 u$. It follows from Theorem 8.34 in [10] that $w = 0$ on $\partial D$ and, by (4.15),

$$
\int_D (|\nabla u|^2 + uL^0 u) \, dx = 0.
$$

By (1.4), $L^0 u = \psi(u) - cu$ and therefore

$$
\int_D (|\nabla u|^2 - cu^2) \, dx = -\int_D u\psi(u) \, dx < 0.
$$

In combination with (4.18), this implies $\lambda_1(D) < 0$.

4.7. Class of Special Domains. The next theorem follows from Corollary 2.1 if $\psi \in \Psi_\alpha$. Now we get it for a wider class of $\psi$ but under

16 Classes $C^{\alpha,\gamma}(\partial D)$ are defined in Section 6.2 of [10].
stronger conditions on $D$ and $D_0$ and only for differential operators in the divergence form.

**Theorem 4.7.** Suppose that $\psi$ satisfies the conditions 1.4.A–1.4.D. If $D_0$ is a bounded special domain and if $D \supseteq D_0$, then $D$ is special. Moreover, if $u_0, u$ are special solutions in $D_0$, $D$, then $u_0 \leq u$ in $D_0$.

**Proof.** Domain $D$ is special by Theorems 4.2 and 4.6 because $\lambda_1(D) \leq \lambda_1(D_0) < 0$ by Lemma 4.2.

It follows from 1.4.B–1.4.C that, for $0 < \delta < 1$,

$$L(\delta u_0) \geq \psi(\delta u_0).$$

There exists a constant $\kappa > 0$ such that $u \leq \kappa$ in $\bar{D}_0$. Therefore $\delta u_0 \geq u$ on $\bar{D}_0$ for sufficiently small $\delta$. By Theorem 2.11 and Remark 2.1, there exists a solution $v$ in $D_0$ such that $\delta u_0 \leq v \leq u$. Clearly, $v$ is a special solution in $D_0$.

By Theorem 4.5, $v = u_0$.

**Theorem 4.8.** Let $\psi$ satisfy conditions 1.4.A–1.4.D. If there exists a square integrable special solution $u$ in a smooth domain $D$, then there is a special bounded smooth domain $\bar{D} \subset D$.

**Proof.** Consider a sequence of bounded smooth domains $D_n \uparrow D$ such that $d(D_n, D \setminus D_{n+1}) \geq 1$. Fix $n$ and choose a function $\varphi$ on $D$ with the properties

$$0 \leq \varphi \leq 1, \quad \varphi = 1 \quad \text{on } D_n, \quad \varphi = 0 \quad \text{on } D_{n+1}, \quad |\nabla \varphi| \leq 2. \quad (4.19)$$

If $w = \varphi^2 u \nabla u$, then

$$\text{div } w = |\nabla (u \varphi)|^2 - cu^2 \varphi^2 - u^2 |\nabla \varphi|^2 + u \psi(u) \varphi^2.$$

Note that $w = 0$ on $\partial D_{n+1}$ and therefore, by (4.15),

$$\int_{D_{n+1}} (|\nabla (u \varphi)|^2 - cu^2 \varphi^2) \, dx = A_n - B_n,$$

where

$$A_n = \int_{D_n} u^2 |\nabla \varphi|^2 \, dx = \int_{D_n \setminus D_0} u^2 |\nabla \varphi|^2 \, dx \leq 4 \int_{D_n \setminus D_0} u^2 \, dx \leq 4$$

$\text{Cf. proof of Theorem 2.1 in [1].}$
and

\[ B_n = \int_{D_{n+1}} u\varphi^2 \, dx. \]

By (4.18) and (4.19), \( \lambda_1(D_{n+1}) \int_{D_{n+1}} u^2 \, dx \leq A_n - B_n. \) Clearly, \( \lim \sup A_n = 0, \) \( \lim \inf B_n \geq \int_D \varphi^2 \, du > 0 \) and therefore \( \lim \sup \lambda_1(D_{n+1}) < 0. \) Hence, \( \lambda_1(D_{n+1}) < 0 \) for some \( n. \) The domain \( \tilde{D} = D_{n+1} \) is special by Theorem 4.6.

4.8. Exit measures and Special Solutions. Denote by \( O(\partial D) \) the class of all relatively open subsets of \( \partial D. \) By combining Theorems 3.3 and 3.4 with the uniqueness property established in Theorem 4.5, we get

**Theorem 4.9.** Let \( D \) be a bounded smooth domain and let \( \psi \) satisfy conditions 1.4.A–1.4.E. For every \( O \in O(\partial D) \) and every \( \mu \in \mathcal{M}(D), \)

\[ \{ X_D(O) < \infty \} = \{ \sigma_D < \infty \} \quad P_\mu -\text{a.s.} \quad (4.20) \]

and therefore the condition

\[ P_x\{ X_D(O) = \infty \} > 0 \quad \text{for some} \quad x \in D, \quad O \in O(\partial D) \quad (4.21) \]

is sufficient and the condition

\[ P_x\{ X_D(O) = \infty \} = 0 \quad \text{for all} \quad x \in D, \quad O \in O(\partial D) \quad (4.22) \]

is necessary for the existence of a special solution in \( D. \) This solution\(^\text{18}\) can be expressed by the formula

\[ \nu(x) = \log P_x\{ \sigma_D < \infty \} = -\log P_x\{ X_D(O) < \infty \}. \quad (4.23) \]

**Proof.** Let \( f \geq 0 \) belong to \( C^1(\partial D) \) and let \( f(z) > 0 \) at some \( z \in \partial D. \) By Theorem 4.1 (with \( u_0 = 0 \)), there exists a positive solution \( u \) of the problem (4.1). By Theorem 4.5, it is unique. Therefore the maximal solution of (4.1) given by (3.26) with \( O = \partial D, \) coincides with the minimal solution given by (3.25). This implies \( 0 = P_x e^{-\langle f, X_D \rangle 1_{\sigma_D < \infty}} \) and therefore

\[ \{ \langle f, X_D \rangle < \infty \} \subseteq \{ \sigma_D < \infty \} \quad \text{a.s.} \]

On the other hand, by (3.20) applied to \( Q = \tilde{D}, \)

\[ \{ \sigma_D < t \} \subseteq \{ \langle f, X_D \rangle = \langle f, X_{D^t} \rangle = \langle f, X_{D^t} \rangle \} \quad \text{a.s.} \]

\(^{18}\) It is unique by Theorem 4.5.
because \( \sigma_D = \sigma_D \) and \( \tilde{D} = D \). If \( \sigma_D < \infty \), then \( \sigma_D < t \) for some \( t \) and 
\[ \langle f, X_D \rangle = \langle f, X_{D^*} \rangle < \infty \ a.s. \]
Therefore, 
\[ \{ \sigma_D < \infty \} = \{ \langle f, X_D \rangle < \infty \} \ a.s. \]
If \( O \) is a relatively open subset of \( \partial D \), then \( 1_{O} \geq f \) for some nonzero positive function \( f \in C^{1,\alpha}(\partial D) \). Therefore, 
\[ \{ \sigma_D < \infty \} \subset \{ \langle 1, X_D \rangle < \infty \} \subset \{ X_D(O) < \infty \} \]
\[ \subset \{ \langle f, X_D \rangle < \infty \} \subset \{ \sigma_D < \infty \}, \]
which implies the first statement of Theorem 4.9. The rest of the theorem follows from Theorem 3.4.

**Theorem 4.10.** The maximal solution of \((1.2)\) in \( \mathbb{R}^d \) is given by the formula
\[ \tau(x) = -\log P_x\{ \sigma < \infty \} = -\log P_x\{ \mathbb{A} \text{ is compact} \}. \quad (4.24) \]

**Proof.** The first part of \((4.24)\) follows from Theorem 3.4 because the maximal solution in \( \mathbb{R}^d \) is special, if a special solution exists, and it is trivial otherwise. We use notation \( A_\sigma, A_\mathbb{A} \) introduced in Theorem 3.2 and we put \( A_\mathbb{A} = \{ \mathbb{A} \text{ is compact} \} \).

By Theorem 3.2, \( A_\mathbb{A} = A_\mathbb{A} P_{r,x} \)-a.s. for all \( r, x \). Since \( \mathbb{A} \) is the image of \( \mathbb{A} \) under the mapping \((r, x) \rightarrow \tilde{x}\), we have \( A_\mathbb{A} \subset A_\mathbb{A} P_{r,x} \)-a.s. Therefore the second part of \((4.24)\) will be proved if we show that \( A_\mathbb{A} \subset A_\mathbb{A} P_{r,x} \)-a.s.

It is sufficient to demonstrate that
\[ \{ \mathbb{A} \subset D \} \subset A_\mathbb{A} \quad P_{r,x} \text{-a.s.} \quad (4.25) \]
for every bounded smooth domain \( D \) and for all \( x \in \mathbb{R}^d \). If \( x \notin D \), then \( P_x\{ \mathbb{A} \subset D \} = 0 \) and \((4.25)\) is trivial. If \( x \in D \), then, \( P_x \)-a.s., \( \{ \mathbb{A} \subset D \} \subset \{ X_D = 0 \} \), (because \( X_D \) is concentrated on \( \mathbb{A} \)) and \( \{ X_D = 0 \} \subset \{ \sigma_D < \infty \} \) by \((4.20)\). Let \( \tilde{X} \) be the part of \( X \) in \( D \). By Lemma 3.6, \( X_t = \tilde{X}_t \) on \( \{ X_D = 0 \} \) and therefore \( \sigma = \sigma_D \) on \( \{ X_D = 0 \} \), \( P_x \)-a.s. This implies \((4.25)\). \( \square \)

**REFERENCES**


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\(^{19}\) We put \( f(t, x) = f(x) \) for \( x \in \partial D \) and \( f(t, x) = 0 \) on the complement of \( \partial D \).